# Hash Tables With Finite Buckets Are Less Resistant to Deletions

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***Abstract*— We show that when memory is bounded, i.e. buckets are finite, dynamic hash tables that allow insertions and deletions behave significantly worse than their static counterparts that only allow insertions. This behavior differs from previous results in which, when memory is unbounded, the two models behave similarly.**

**We show the decrease in performance in dynamic hash tables using several hash-table schemes. We also provide tight upper and lower bounds on the achievable overflow fractions in these schemes. Finally, we propose an architecture with content-addressable memory (CAM), which mitigates this decrease in performance.**

## I. Introduction

### A. Background

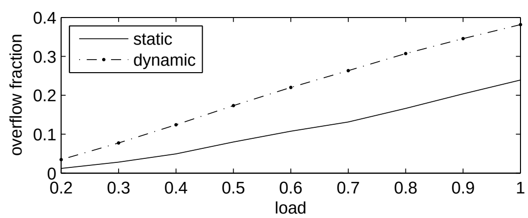
Networking devices often use *dynamic* hash tables, in which elements keep arriving and departing, and not *static* ones that are built only once. However, for simplicity, device designers typically model the performance of the dynamic hash tables using models of the static hash tables. This paper shows that these static models can lead to a significant *under-estimation of the drop rate* in the dynamic case.

This under-estimation of the drop rate can potentially affect the performance of networking devices. Hash tables form the core building block of many networking device operations, such as flow counter management, flow state keeping, elephant traps, virus signature scanning, and IP address lookup algorithms. If memory is allocated to the dynamic hash tables according to the static model, many more elements might need to be dropped from the hash tables than initially estimated.

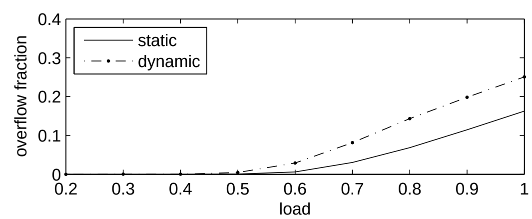
Using the static model seems natural. In fact, dynamic hash tables are known for being *typically harder to model* than static ones, sometimes even lacking any mathematical analysis [1]. Therefore, the static model appears to be a simpler and more accessible option to the network designer.

More significantly, past studies have also found the *same asymptotic behavior* in dynamic and in static hash tables, in at least three cases:
(a) In the static case in which $n$ elements are uniformly hashed into $n$ infinite buckets, the maximum bucket size is known to be approximately $\log n / \log \log n$ with high probability [2], [3]. The dynamic case yields the same result, assuming alternate departures and arrivals of random elements while keeping $n$ elements in the hash table after each arrival.
(b) Likewise, when inserting each element in the least-loaded of two random buckets ($d$-random algorithm with $d = 2$), the maximum bucket size is $\log \log n / \log 2 + O(1)$ in the static case; and again, the dynamic case yields the same result [3], [4].
(c) Similarly, using the asymmetric $d$-left algorithm, the static case and the dynamic case yield again the same bound on the maximum bucket size [5].
Therefore, as illustrated in these three cases, given a large number of elements, it appears that the network designer could use the simpler static model for the dynamic case.

In this paper, we focus on the realistic scenario in which buckets are finite, as used in networking devices, contrarily to the infinite-bucket case assumed above. We show that the dynamic hash table can exhibit a *significantly worse* drop rate than its static counterpart.




(a) $d$-random with a stash




(b) Cuckoo hashing with a stash

Fig. 1. Overflow fraction with 2 hash functions and bucket size 1, using both the static and the dynamic model.

### B. Intuitive Example

Fig. 1 plots the system overflow fraction as a function of the load, i.e. the fraction of elements not placed in the buckets as a function of the average number of elements per bucket. It shows the overflow fraction for both a static system, where there are only insertions, and a dynamic system, where we alternate between deletions and insertions while a fixed load is maintained [4], [6]. To measure the overflow fraction, it relies on an overflow list, called *stash*, to which new elements are moved when they cannot be inserted in

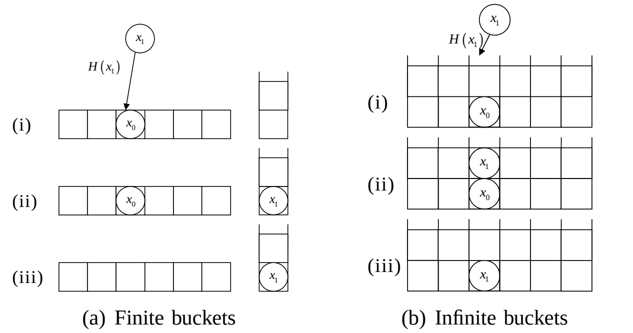



Fig. 2. An example demonstrating the degradation of performance in dynamic hash tables.

the hash table. Fig. 1(a) and 1(b) show the overflow fraction of the $d$-random algorithm with a stash [4] and the cuckoo hashing with a stash [7], [8]. The overflow fractions are obtained in simulations using 2048 buckets, $10^6$ rounds with one random element deletion and one element insertion in each round, and a standard pseudorandom number generator to obtain hash values[1].

Both figures clearly show a non-negligible degradation in the overflow fraction of the dynamic system. For instance, the cuckoo hashing scheme with load of 0.6 yields an overflow fraction of 0.52% and 2.97% in the static and dynamic models, respectively. Moreover, while for cuckoo hashing scheme with load of 0.5 the overflow fraction in the static model goes to 0 [9], it does so more slowly in the dynamic case. For instance, for $m = 1024$ we got an overflow fraction in the static and dynamic models of 0.05% and 0.44%, where for $m = 16384$ we got 0.0012% and 0.0606%, respectively.

The intuition behind this difference in behavior is that if the bucket size is bounded, once an element is placed in the overflow list it stays there regardless of whether the corresponding bucket become available later upon deletion. Therefore, the order of the insertion and deletion operations directly affects the performance. This is typically not the case in the unbounded bucket case, and the difference can cause a drastic degradation in the scheme performance.

Fig. 2 illustrates this degradation in performance, using the same scenario both for the finite and the infinite bucket sizes. For the case of finite buckets, we assume bucket sizes of 1, an overflow list, and an insertion algorithm that uses only one hash function. We consider the following scenario: Let $t$ be the time when a new element $x_1$ is hashed to a full bucket $j$ that already stores element $x_0$ (step (i) in both Fig. 2(a) and Fig. 2(b)). If a finite bucket is used, then $x_1$ is moved to the overflow list (step (ii) in Fig. 2(a)), while in the infinite-bucket case, $x_1$ is simply stored in bucket $j$ (step (ii) in Fig. 2(b)). Let $t' > t$ be the time when element $x_0$ is deleted. Assuming that element $x_1$ is not deleted before $t'$, it stays in the overflow list in the finite-bucket case, while in the infinite-bucket case it is stored in bucket $j$ (step (iii)).

Therefore, in the dynamic case with finite bucket sizes, element $x_1$ is in the overflow list, even though its corresponding bucket $j$ is empty. This could never happen in the static case (elements are stored in the overflow list only after their corresponding buckets are full, and full buckets cannot become empty). It also could never happen in the dynamic case with infinite buckets (there is no overflow list).

[1]Simulations with ten times more buckets or rounds yielded near-identical results.

### C. Our Contributions

In this paper, we show that dynamic hash tables with finite buckets behave worse than static ones.

We start by considering a simplistic dynamic scheme with a single hash function. We model this hashing scheme analytically using three different models: a discrete-time Markov chain, a continuous model with a birth-death chain, and a fluid model with a continuous-time Markov process. We find that this simplistic dynamic scheme performs notably worse than its corresponding static scheme.

Then, we derive a lower bound on the overflow fraction in the dynamic model of *any* hash-table scheme that uses uniform hash functions and does not move back elements once they were placed in the overflow list. We prove that when the *average* number of memory accesses per insertion $a$ increases, the overflow fraction can decrease as slowly as $\Omega(1/a)$. This indicates that the bad performance of dynamic schemes is fundamental, and is hard to solve by simply using additional memory accesses.

Next, we introduce an online multiple-choice scheme. We demonstrate that this scheme reaches that lower bound and therefore is optimal up to a certain rate of memory access, which depends on the system parameters.

However, due to the slow decrease of the lower bound, optimality may be insufficient for certain applications. Therefore, we suggest changing the assumptions and moving back elements from the overflow list when a bucket becomes available upon deletion. We propose the M-B (Moving-Back) scheme that uses a CAM (content-addressable memory) device that stores the elements along with their hash values. A parallel lookup operation is used once an element is deleted and its bucket becomes non-full. This operation, supported by the CAM, finds an element in the overflow list that can be moved back to the bucket. This scheme is shown to beat the initial lower bound without a CAM.

Finally, we evaluate in this paper all proposed schemes using simulations as well as experiments with real hash functions applied on real-life traces.

*Paper Organization:* We start with preliminary definitions in Section II. Section III presents and analyzes the single-choice SINGLE scheme, while Sections IV and V provide a lower bound on the overflow fraction. Then, in Section VI we present and analyze the multiple-choice MULTIPLE scheme, and in Section VII we present the CAM-based M-B scheme which, upon deletion, moves back elements from the overflow list. Finally, we evaluate all the analytical results in Section VIII.

Due to space limits, most proofs are only outlined in this paper, and presented in full in the online technical report [10].

## II. Problem Statement

### A. Terminology and Notations

This paper considers *single- and multiple-choice hash schemes with a stash* [11], [12]. Such schemes consist of two data structures: *(i)* A *hash table* of total memory size $m \cdot h$, partitioned into $m$ buckets of size $h$; *(ii)* An *overflow list*, usually stored in an expensive CAM. Note that the overflow list can also be absent, in which case overflow elements are simply dropped.

As in traditional hash tables, the schemes should support three basic operations: element insertions, element deletions, and lookups. We call the (infinitely long) sequence of these operations the *arrival sequence* of the scheme. In the paper, we focus mostly on a specific arrival sequence, alternating between departures of a random element (picked uniformly at random) and insertions of a new element [4], [6].

Multiple-choice hashing schemes employ up to $d$ probability distributions over the set of buckets; these distributions are then used to generate a *hash-function set* $\mathcal{H} = \{H_1, \ldots, H_d\}$ of $d$ independent hash functions. For each element $x$ and each operation, the scheme can consider only the buckets $\{H_1(x), \ldots, H_d(x)\}$ (and the overflow list). In addition, we assume that the scheme must access a bucket to obtain any information on it (thus, if the hashing scheme tries to insert an element in a full bucket, it must access the bucket first).

Our goal is to minimize the *expected overflow fraction* of the scheme, i.e. the fraction of elements that are placed in the overflow list, subject to the (total and average) number of *memory accesses*. We count as one memory access reading and updating all the elements of a single bucket (this corresponds to the common practice of sizing the bucket size by the width of the memory word) and we do not count accesses to the overflow list. We further assume that up to $d$ buckets can be read in parallel before deciding which one to update, requiring a total of $d$ memory accesses.

Formally, the hashing scheme and the optimization problem are captured by the following two definitions, where the load $c$ is the ratio of the total number of elements $n$ by the total memory size $mh$: $c = \frac{n}{mh}$.

*Definition 1:* When the load is $c$ and the bucket size is $h$, an $\langle a, d, c, h\rangle$ *hashing scheme* is a scheme with an expected (respectively, maximum) number of memory accesses per element of at most $a$ (respectively, $d$).

*Definition 2:* The OPTIMAL DYNAMIC HASH TABLE PROBLEM is to find an $\langle a, d, c, h\rangle$ hashing scheme that minimizes the expected overflow fraction $\gamma$ as the number of elements $n$ goes to infinity. Whenever defined, let $\gamma_{\text{OPT}}$ denote this optimal expected limit overflow fraction.

### B. Arrival Models

Throughout the paper, we will use three different models for the arrivals and departures of elements: a *discrete model* with a finite number of elements; a *continuous model* with a finite number of elements; and a *fluid model* based on differential equations with an infinite number of elements. Our objective is to model a constant load, i.e. a constant number of elements in the system, so that departing elements are replaced by arriving elements.

**Discrete Model —** In the *discrete model*, we assume that time is divided into time-slots of unit duration, and start at time $t = 0$ with $n$ elements in the overflow list. At the start of each time-slot $t > 0$, an element is chosen uniformly at random among all $n$ elements in the system to depart. Next, at the end of time-slot $t$, a new element arrives and is inserted according to the hashing scheme into either a non-full bucket or the overflow list. Therefore, by the end of each time-slot $t$, there are always $n$ elements in the system, either in the hash table or in the overflow list.

**Continuous Model —** The second model is a *continuous-time model*, starting again at time $t = 0$ with $n$ elements in the overflow list. In this model, each element stays in the system for an exponentially-distributed duration of average 1. Therefore, at each infinitesimal time-interval $[t, t + \delta t]$, the probability that a given element departs is $n \cdot \delta t + o(\delta t)$. For each element departure, another element is automatically generated and inserted in the system according to the hashing algorithm into either a non-full bucket or the overflow list. Again, there are always $n$ elements in the system at each time $t$, ensuring a constant load.

Since there are $n$ departures per time-unit on average instead of a single one, the continuous system can be seen as a speeded-up version of the discrete system. In fact, when only looking at the system during the discrete element departure times, which follow exponentially-distributed inter-departure times, *we obtain the discrete model again*.

Incidentally, although each element departure triggers the arrival of another element with different hashed buckets, we will sometimes refer by simplicity to the departed element as if it was reinserted.

**Fluid Model —** The last model is the *fluid model*, which attempts to model the behavior of the continuous system as the number of elements $n$ and the number of buckets $m$ go to infinity with a constant limit ratio $ch = \lim_{n\to\infty} \frac{n}{m}$. In the fluid model, we will often analyze the system using differential equations, and will be mainly interested in their fixed-point solutions. Again, we will assume that at $t = 0$, all elements are in the overflow list.

In the fluid model, as in the finite continuous-time model, elements stay in the system for an exponentially-distributed duration of average 1, and therefore the departure rate from each bucket is proportional to the bucket size.

In addition, as in the other models, element departures trigger element arrivals. Note that in the continuous model, the average arrival rate per bucket is $\frac{n}{m}$, since the arrival rate is $n$ and there are $m$ buckets. Therefore, in the fluid model, we model a constant average arrival rate per bucket of $ch = \lim_{n\to\infty} \frac{n}{m}$. Likewise, in the continuous model, when arriving elements use a uniformly-distributed hash function, they hash into each bucket at a rate equal to the average rate of $\frac{n}{m}$. In the fluid model, since we consider an infinite number of buckets, a *uniformly-distributed hash function* is not well defined. By extension, and for simplicity, we will define such a function as one that enables the same arrival rate of $ch$ to all buckets.

Furthermore, we will define the average hashing rate per element $a$ such that it is valid at any time $t$. We will also assume that elements may not use hash functions that pick with higher probability buckets with lower occupancy, i.e. that the average hashing rate limit of $a$ is valid given any bucket size. Thus, if one tenth of the buckets are empty, a uniform hash will find one tenth of its buckets empty as well. Of course, an element might still decide to enter a bucket with lower occupancy with higher probability.

**Model Alternatives —** In general, to model system scaling, we would be interested in using either the discrete or continuous finite models, and then in studying how their solution scales with $n$. However, given the complex interactions between the $n$ elements, these models often prove intractable. Therefore, we will use the fluid model in these cases, and will most often *not be able* to prove convergence of the discrete or continuous models to the fluid model. Likewise, we will not always prove convergence of the differential equations to the fixed-point solutions. This is, of course, a limit of our analysis.

On the other hand, for the single-choice hashing scheme (Section III), we provide a full analysis with the three models, and prove that the limit of the discrete and continuous finite models behaves indeed like in the fluid model. In simulations, we will also show that the scaled systems converge fast to their fluid model. We refer to [13] for a more complete discussion of the sufficient conditions for the convergence to the fluid-limit fixed-point solution.

## III. A Single-Choice Hashing Scheme

We start by analyzing a simplistic hashing scheme, which uses only a single uniformly-distributed hash function $H$ to insert elements in the hash table. Each element $x$ is stored in bucket $H(x)$, if it is not full, and in the overflow list otherwise. Since an element uses exactly one hash function, its average number of memory accesses per element is $a = 1$. Of course, this simplistic scheme would probably not be implemented in advanced networking devices. However, it provides a better intuition on the reasons behind the performance degradation in dynamic hash-table schemes.

**Discrete Model —** We first develop an analytical model for the scheme within the discrete framework presented in Section II. Let $p_k(t)$ denote the fraction of buckets that have $k$ elements at the end of time-slot $t$, and $p(t) = (p_1(t), \ldots, p_h(t))$. Using this discrete model, we obtain the following result on the limits of the distribution of $p$ and of the overflow fraction.

*Theorem 1:* Let $C = \sum_{\ell=0}^{h} \binom{n}{\ell} (\frac{1}{m-1})^{\ell}$. In the discrete model,
*(i)* the distribution of $p(t)$ converges to the Engset distribution $\pi^n$ [14], [15]; namely,

$$\pi_k^n = \frac{1}{C} \cdot \binom{n}{k} \cdot \left(\frac{1}{m-1}\right)^k . \quad (1)$$

*(ii)* the overflow fraction converges to

$$\frac{1}{C} \cdot \binom{n}{h} \cdot \left(\frac{1}{m-1}\right)^h \cdot \left(1 - \frac{h}{n}\right) . \quad (2)$$

*Proof:* [Proof Outline] As mentioned, the full proof appears in [10]. We build a birth-death Markov chain to model the occupancy $X_t^i$ of an arbitrary bucket $i$ at time $t$. We then show that it is is irreducible, positive recurrent and aperiodic, and therefore converges to its stationary distribution $\pi^n$. Last, we compute $\pi^n$, and deduce the stationary overflow fraction $\gamma_{\text{SINGLE}}^n$. ■

It is interesting to note that Equation (1) can be rewritten as a truncated binomial expression

$$\pi_k^n = \frac{\binom{n}{k} \left(\frac{1}{m}\right)^k \left(1 - \frac{1}{m}\right)^{n-k}}{\sum_{l=0}^{h} \binom{n}{l} \left(\frac{1}{m}\right)^l \left(1 - \frac{1}{m}\right)^{n-l}}, \quad (3)$$

which hints at the following interesting equivalent system: the bucket occupancy is distributed as if the $n$ elements were assigned uniformly at random among the $m$ buckets, and then the buckets with more than $h$ elements were completely cleared out and had *all* their elements put in the overflow list. This is in contrast with the static system in which only elements exceeding the bucket capacity of $h$ are placed in the overflow list. Therefore, it nicely illustrates the difference between the static and dynamic cases.

**Continuous Model —** We now turn to the continuous model in which elements stay in the system for an exponentially-distributed duration of average 1. It turns out that the continuous model yields similar results to those of the discrete model (Theorem 1).

*Theorem 2:* In the continuous model, the single-choice hashing scheme has the same stationary distribution and overflow fraction as in the discrete model.

*Proof:* [Proof Outline] As in the discrete model, we build a birth-death chain to model the occupancy $X_t^i$ of an arbitrary bucket $i$ at time $t$. We obtain a continuous-time Markov process with rates that are equal to the product of the transition probabilities in the discrete-time Markov chain by a scaling factor $n$, and deduce the equality of the stationary distribution and overflow fraction in both models. ■

**Fluid Model —** We now analyze the *infinite* system using a fluid model. In the fluid model, as in the finite continuous-time model, elements stay in the system for an exponentially-distributed duration of average 1, and therefore the departure rate from each bucket is proportional to the bucket size. In addition, when an element departs, a new element in inserted into the hash table (or in the overflow list if the corresponding bucket is full). As explained in Section II, the arrival rate to each bucket is therefore $ch = \lim_{n\to\infty} \frac{n}{m}$.

The following theorem, which is based on the M/M/h/h continuous-time Markov process [15], shows the performance of the scheme under the fluid model.

*Theorem 3:* In the fluid model,
*(i)* the distribution of $p(t)$ converges to the stationary distri-

bution $\pi^{\infty}$, where

$$\pi_k^{\infty} = \frac{(ch)^k}{k!} \Big/ \sum_{l=0}^{h} \frac{(ch)^l}{l!}, \quad k = 0, \ldots, h. \tag{4}$$

*(ii)* the overflow fraction converges to $\pi_h^{\infty}$ and follows the Erlang-B formula.

*Proof:* [Proof Outline] As previously, we write the differential equations, and find that they correspond to the equations ruling an M/M/h/h continuous-time Markov process [15]. The drop rate is also found to follow the well-known Erlang-B formula. ■

We have seen that the discrete and continuous models with $n$ elements yield a stationary distribution $\pi^n$, while the fluid model yields a fixed-point distribution $\pi^{\infty}$. We will now show that as expected, when scaling $n$ to infinity, $\pi^n$ converges to $\pi^{\infty}$, and so does the associated overflow fraction.

*Corollary 4:* When $n \to \infty$ with $\frac{n}{m} \to ch$,
*(i)* the stationary distribution converges to the fixed-point distribution of the fluid model: $\pi^n \to \pi^{\infty}$; and
*(ii)* the overflow fraction of the discrete (continous) model converges to the overflow fraction of the fluid model.

*Proof:* [Proof Outline] We show that for each $n \in \mathbb{N}^* \cup \{\infty\}$, $\sum_{k=0}^{h} \pi_k^n = 1$ and $\pi_0^n > 0$, so for $k \in [0, h]$, $\pi_k^n / \pi_0^n$ is defined and $\pi_k^n = (\pi_k^n / \pi_0^n) / \left(\sum_{l=0}^{h} \pi_l^n / \pi_0^n\right)$. Then, we prove the convergence of $\pi_k^n / \pi_0^n$ to $\pi_k^{\infty} / \pi_0^{\infty}$, which concludes both the convergence of $\pi^n$ to $\pi^{\infty}$ and, since $\gamma_{\text{SINGLE}}^n = \pi_h^n \cdot \left(1 - \frac{h}{n}\right)$, the convergence of $\gamma_{\text{SINGLE}}^n$ to $\gamma_{\text{SINGLE}}^{\infty}$. ■

Finally, we generalize the scheme to deal with *probabilistic insertions*. Namely, there exists some $\alpha \in [0, 1]$ such that each arriving element is either hashed into a bucket as before with probability $\alpha$, or placed directly in the overflow list with probability $1 - \alpha$, yielding an average number of memory accesses $\alpha$ (or equivalently, a total number of memory accesses $\alpha n \leq n$, less than the number of elements). Using the fluid model for simplicity, we obtain the following result. While this probabilistic scheme is probably not useful in practice (since the average memory access rate is seldom less than 1), we will later demonstrate that it is *optimal* under specific conditions.

*Theorem 5:* In the fluid model, given the single-choice hashing scheme with an insertion probability $\alpha$, we obtain $a = \alpha \leq 1$, and
*(i)* the distribution of $p(t)$ converges to the stationary distribution $\pi^{\infty}$, where

$$\pi_k^{\infty} = \frac{(\alpha ch)^k}{k!} \Big/ \sum_{\ell=0}^{h} \frac{(\alpha ch)^{\ell}}{\ell!}, \quad k = 0, \ldots, h. \tag{5}$$

*(ii)* the overflow fraction converges to $(1 - \alpha) + \alpha \cdot \pi_h^{\infty}$.

*Proof:* The differential equations are the same as in the proof of Theorem 3 when replacing $ch$ by $\alpha ch$, since $\alpha$ simply changes the arrival rate. The distribution results are then immediate. In addition, in the fixed-point equations, an arriving element either overflows immediately with probability $1 - \alpha$, or checks with probability $\alpha$ a bucket that can be full with probability $\pi_h^{\infty}$, hence the overflow equation follows as well. ■

## IV. OVERFLOW LOWER BOUND

Our objective is to find a lower bound on the optimal expected limit overflow fraction $\gamma_{\text{OPT}}$ in the OPTIMAL DYNAMIC HASH TABLE PROBLEM, and therefore on the expected overflow fraction $\gamma$ of any $\langle a, d, c, h \rangle$ hashing scheme, when assuming a fluid model. We will study the simpler case with a single uniformly-distributed hash function, as defined in Section II. The more general case with several hash functions using different subtable-based distributions appears in Section V.

The proof relies on the following result from [16]. Consider an Erlang blocking model with $N$ servers, and suppose that the arrival rate depends on the system. Let $\lambda_k$ be the arrival rate when there are $k$ transmissions in progress, $k = 0, 1, \ldots, N - 1$. Then we have:

*Lemma 1 (Theorem 4.2 in [16]):* For all increasing mappings $f : \mathbb{R} \to \mathbb{R}$ and for all $t > 0$, $\mathbb{E}f(X)$ is concave increasing as a function of $\lambda_k$, for $k = 0, 1, \ldots, N - 1$.

We use this lemma to prove the lower-bound result.

*Theorem 6:* In the fluid model, under the assumptions above where all buckets have the same probability of being hashed into, the optimal expected fixed-point overflow fraction $\gamma_{\text{OPT}}$ in the OPTIMAL DYNAMIC HASH TABLE PROBLEM is lower-bounded by

$$\gamma_{\text{LB}}^{\infty}(a) = 1 - a + a \cdot \frac{r^h}{h!} \Big/ \sum_{l=0}^{h} \frac{r^l}{l!}, \tag{6}$$

where $r = ach$.

*Proof:* [Proof Outline] The proof relies on Lemma 1 which demonstrates that the worst-case average bucket occupancy is at most the average bucket occupancy that occurs when all memory accesses are used to insert an element. We then use a standard result from Erlang theory linking the drop rate to the average occupancy [14], [15] to find a lower bound on the drop rate. ■

Note again that the Erlang-B formula appears in the lower-bound on the overflow. This yields the following optimality result:

*Theorem 7:* In the fluid model, the single-choice hashing scheme is optimal for every average number of memory accesses $a$ in $[0, 1]$ (and in particular for $a = 1$).

*Proof:* For the SINGLE scheme, there is a single hashed bucket per element, and it is accessed with probability $\alpha$, therefore $a = \alpha$. For $a \leq 1$, we get

$$\gamma_{\text{LB}}^{\infty}(a) \stackrel{(a)}{=} (1 - \alpha) + \alpha \cdot \frac{(\alpha ch)^h}{h!} \Big/ \sum_{l=0}^{h} \frac{(\alpha ch)^l}{l!} \stackrel{(b)}{=} \gamma_{\text{SINGLE}}^{\infty}$$

where (a) comes from Equation (6), $r = ach$ and $a = \alpha$, and (b) from Theorem 5. ■

*Example 1:* We illustrate the significance of the lower bound by considering a simple system with buckets of size $h = 1$, implying $\gamma_{\text{LB}}^{\infty}(a) = 1 - a + a \cdot \frac{c \cdot a}{1 + c \cdot a} = 1 - \frac{a}{1 + c \cdot a}$. In particular, for a load $c = 1$, corresponding to the scaling case where the number of buckets is kept equal to the number of elements and therefore $\lim_{n\to\infty} \frac{n}{m} = 1$, we get $\gamma_{\text{LB}}^{\infty}(a) = 1 - \frac{a}{1+a} = \frac{1}{1+a}$, which shows that the lower-bound decreases slowly as $\Theta(1/a)$ when the average number of memory accesses per insertion $a$ increases.

For instance, to get a $1\%$ drop rate we need each element to access an average of at least $a = 99$ buckets. Of course, this is impossible to implement in high-speed networking devices. Thus, this lower bound is essentially an *impossibility result*, which shows that it is not easy to obtain efficient hash tables with deletions.

## V. Lower Bound with Multiple Hash-Function Distributions

We now consider a setting with a set $\mathcal{I}$ of $I = |\mathcal{I}|$ subtables, where subtable $i \in \mathcal{I}$ uses a fraction $\alpha^i$ of all buckets. We will allow for the $d$ hash functions to use up to $d$ different distributions $\{f_j\}_{1\le j\le d}$ over the $I$ subtables, where each distribution $f_j$ assigns a probability $f_j^i$ to subtable $i \in \mathcal{I}$, with $\sum_{i\in\mathcal{I}} f_j^i = 1$, and then uniformly picks buckets within each subtable (as defined in Section II). We also assume that each distribution $f_j$ is used by a fraction $\kappa_j$ of the total memory accesses. Therefore, subtable $i$ is accessed with a total probability of $\beta^i = \sum_{j=1}^{d} \kappa_j \cdot f_j^i$, with $\sum_{i\in\mathcal{I}} \beta^i = 1$. The following result establishes that the lower-bound is reached when the hash table is used in a uniform way, i.e. the probability $\beta^i$ of accessing a subtable is equal to its fraction $\alpha^i$ in the table, and therefore the lower-bound is the same as established previously in Theorem 6.

*Theorem 8:* In the fluid model with multiple distributions as defined above, the lower-bound $\gamma_{\text{LB}}^{\infty}(a)$ on the fixed-point overflow fraction is the same as with a unique uniform hash function, and is reached iff for all $i \in [1, I]$, $\beta^i = \alpha^i$, i.e. the weighted average of all distributions is uniform.

*Proof:* [Proof Outline] The proof, which is fully presented in [10], first shows that in the worst-case, each subtable follows the worst-case Markov chain established in the bound above. Then, it computes the average occupancy, and shows that it is maximized iff $\alpha^i = \beta^i$ for all $i$. ∎

## VI. A Multiple-Choice Hashing Scheme

We now introduce a natural extension to the single-choice hashing scheme that uses an ordered set of $d$ hash functions $\mathcal{H} = \{H_1, \ldots, H_d\}$, such that all the hash functions are independent and uniformly distributed. Upon inserting an element $x$, the scheme successively reads the buckets $H_1(x), H_2(x), \ldots H_d(x)$ and places $x$ in the first non-full bucket. If all these buckets are full, $x$ is placed in the overflow list. To keep an average number of memory accesses per element of at most $a$, the algorithm attempts to insert $x$ into the hash table with a probability $\alpha$, otherwise it is directly placed in the overflow list.

We evaluate the performance of this scheme analytically using the fluid model.

*Theorem 9:* Assume the multiple-choice hashing scheme with a hashing probability $\alpha$. Using the fluid-model fixed-point distribution $\pi^{\infty}$,

*(i)* $\pi^{\infty}$ satisfies $\pi_k^{\infty}(a) = \dfrac{\frac{(ach)^k}{k!}}{\sum_{l=0}^{h} \frac{(ach)^l}{l!}}$, for each $k = 0, \ldots, h$;

*(ii)* the average bucket access rate $a$ satisfies the fixed-point equation $a = \alpha \cdot \frac{1-\pi_h^{\infty}(a)^d}{1-\pi_h^{\infty}(a)}$;

*(iii)* the overflow fraction is equal to the lower-bound, and is therefore *optimal*, for $a \in [0, a^{co}]$, where $a^{co}$ satisfies the fixed-point equation $a^{co} = \frac{1-\pi_h^{\infty}(a^{co})^d}{1-\pi_h^{\infty}(a^{co})}$.

*Proof:* [Proof Outline] The proof first establishes the fixed-point distribution as a function of $a$. Then, it computes $a$ given a full-bucket probability $\pi_h^{\infty}(a)$, by analyzing the successive steps of the MULTIPLE algorithm. It concludes by using the fact that the probability $\alpha$ cannot go beyond 1. ∎

The following example illustrates our results.

*Example 2:* For the case where $h = 1$, solving the fixed-point equation yields $a^{co} = \frac{2c-1+\sqrt{1+4c^2}}{2c}$. Therefore, for a load of one element per bucket, i.e. $c = \lim_{n\to\infty} \frac{n}{m} = 1$, we get $a^{co} = \frac{1+\sqrt{5}}{2} \approx 1.62$, and the corresponding overflow fraction is $\gamma_{\text{LB}}^{\infty}(a^{co}) = 1.5 - \frac{\sqrt{5}}{2} \approx 38.2\%$. Likewise, for a load of $c = 0.1$, we get $a^{co} = \frac{-0.8+\sqrt{1+0.04}}{0.2} \approx 1.099$, with the corresponding overflow fraction $\gamma_{\text{LB}}^{\infty}(a^{co}) \approx 0.98\%$.

## VII. Moving Back Elements

So far, we have found optimal schemes for a range of values of $a$, the average number of memory accesses per element. However, although optimal, the expected overflow fraction may still be too large.

In the literature, several solutions exist to reduce the drop rate (or collision probability) in a dynamic system. One such solution uses limited hash functions in order to be able to rebalance the hash table in case of deletion [17]. However, this approach gives up randomness, and the efficiency of a similar approach appears limited [6]. Another solution, based on the *second-chance* scheme [11], moves elements from one bucket to another by storing hints at each bucket [6]. However, we found in simulations that this solution was less effective than our suggested scheme presented below for higher loads, while it was more effective for lower loads. (Due to space considerations, we present the detailed simulation results in [10].)

To reduce the overflow fraction, we suggest a scheme that allows moving elements back from the overflow list to the buckets upon a *deletion* operation[2]. This scheme can be combined with any insertion scheme.

Our scheme, called the moving-back scheme (M-B), relies on a (binary) CAM. In general, a CAM stores keys in entries. Given some key $k$, a parallel lookup is performed over all entries and the index of the first (that is, highest priority) entry that contains $k$ is returned from the CAM. In many cases, this index is later used in order to access in regular memory a direct-access array that contains the value

[2] We also consider a scheme that works upon *insertion*, however the details are omitted due to lack of space; moving back elements upon deletion performs better in general.

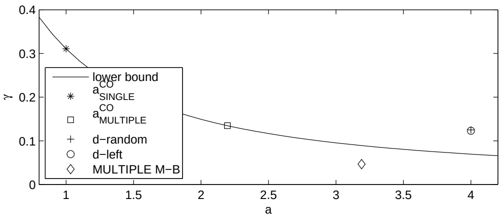
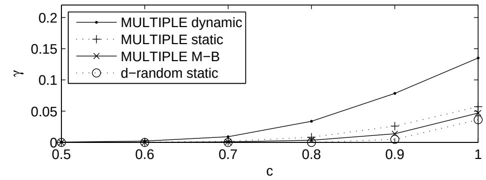
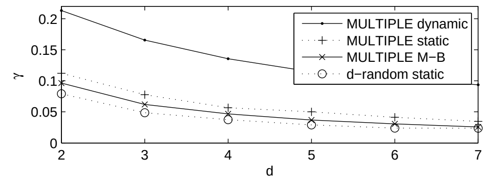



Fig. 3. Overflow fraction as a function of $a$ with $d = 4$, $h = 4$, $c = 1$.

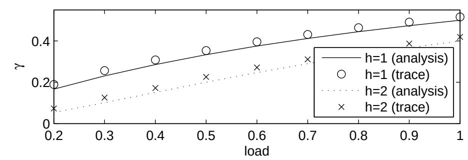



(a) $h = 4$ and $d = 4$

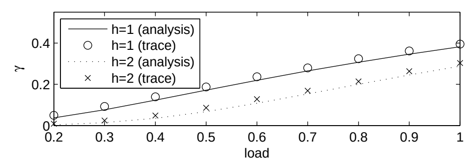



(b) $h = 4$ and $c = 1$

Fig. 4. Overflow fraction of the proposed moving-back (M-B) scheme (via simulations).




(a) SINGLE




(b) MULTIPLE

Fig. 5. Experiment using real-life traces and hash functions with SINGLE and MULTIPLE (d=2).

associated with $k$. CAMs enable constant-time operations, however they are more expensive and consume more power than regular memory. It is a common practice to implement the overflow list in a CAM [1], [11], [12], relying on the fact that the number of elements in the overflow list is small.

Our scheme uses an auxiliary CAM, besides the primary CAM used to store the element of the overflow list: For each element $x$ that is stored in the $i$-th entry of the primary CAM, we store the values $\{H_1(x), H_2(x), \ldots, H_d(x)\}$ in entries $d \cdot i, d \cdot i + 1, \ldots, d \cdot i + (d-1)$ of the auxiliary CAM.

When an element is deleted from a bucket $j$ that was previously full, we need to move an element $x$ from the overflow list to bucket $j$ such that $j$ is the result of applying at least one of the hash-functions on $x$. We can locate such an element in constant time by querying the auxiliary CAM with key $j$. Suppose the entry returned by the auxiliary CAM is $\ell$, then $x$ is located in entry $\lfloor \ell/d \rfloor$ of the primary CAM.

We note that upon moving an element back to the hash table, one should update the corresponding entries of the primary and auxiliary CAMs. An efficient way to update is to write the value $m+1$ in these entries, such that when a new element is inserted into the overflow list, one can query the auxiliary CAM with the value $m+1$ to decide in which entry (of the primary CAM) to put the new element.

In [10], we show that when $d = 1$, the proposed M-B scheme behaves as in the static case and, for $d > 1$, we present an approximate model for the overflow fraction in case MULTIPLE scheme is used.

## VIII. Experimental Results

### A. Simulations

Fig. 3 compares all the schemes. It was obtained with $d = 4$ choices, bucket size $h = 4$, $n = 4{,}096$ elements and $m = 1{,}024$ buckets, yielding a load $c = 1$.

The solid line plots the overflow fraction lower-bound $\gamma_{\text{LB}}(a)$ from Theorem 6. Simulations show that the proposed M-B scheme beats the lower bound with an overflow fraction of $4.6\%$, emphasizing the strength of this architecture. Of course, the lower bound does not apply to this case, since it moves back elements from the CAM.

As follows from Theorems 7 and 9, the overflow fractions $\gamma_{\text{SINGLE}}(a)$ and $\gamma_{\text{MULTIPLE}}(a)$ of the single-choice (SINGLE) and the multiple-choice (MULTIPLE) hashing schemes follow the lower-bound line, respectively until $a^{co}_{\text{SINGLE}} = 1$ with $\gamma_{\text{SINGLE}} = 31.1\%$, and $a^{co}_{\text{MULTIPLE}} = 2.195$ with $\gamma_{\text{MULTIPLE}} = 13.5\%$. Therefore, they are clearly optimal up to a certain point.

We further evaluate the performance of our proposed M-B scheme. Quite surprisingly, when using the MULTIPLE scheme (of Section VI), the M-B scheme outperforms the *static* case of the MULTIPLE scheme (see Fig. 4), and performs similarly to the static $d$-random scheme (in the static case, $d$-random performs better than our multiple-choice scheme, albeit consuming significantly more energy [12]). This can be explained intuitively as follows: our moving-back strategy moves back an element to the only corresponding bucket which is not full; this is equivalent to inserting the element to the least occupied bucket as in the $d$-random hashing scheme.

### B. Experiments Using Real-Life Traces

We have also conducted experiments using real-life traces recorded on a single direction of an OC192 backbone link [18]. Our goal is to compare the average overflow fraction retrieved using our models for SINGLE and MULTIPLE with the corresponding overflow fraction when using a real hash function on a real-life trace. We used a 64-bit mix function [19] to implement two 16-bit hash functions. We used $m = 10{,}000$ buckets, and set a number of elements $n$ as corresponding to various values of $h$ and $c$. To keep a

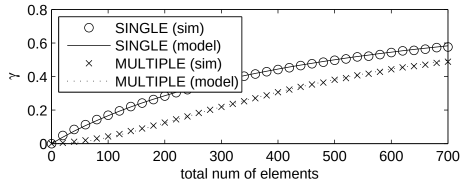



Fig. 6. Marginal overflow fraction of 100 on-off flows with $m = 500$, $h = 1$ and $d = 2$

constant desired load, we alternated $100{,}000$ times between an arrival (insertion) of a new TCP packet according to the trace, and the departure (deletion) of a random TCP packet. The hash functions were given the source and destination IP tuple as well as the sequence and acknowledgment numbers of the TCP packets. Therefore, the hash table stores the latest TCP packets, and can retrieve any needed packet based on its header. It can be used to monitor ongoing TCP flows, given a target number $n$ of packets that are stored at any time. Its objective in our experiments was mainly to test the correctness of our model.

Fig. 5 shows that the results of our experiments are relatively close to our model. The maximum gap is for the SINGLE scheme with $h = 1$ and $c = 0.3$. Our model predicts an overflow fraction of $23.08\%$, while the experiment yields $25.67\%$.

### C. Experiments Using an On-off Arrival Model

We also consider a queueing model where at each step $i$, $b_i$ elements arrive according to $k$ independent on-off bursty flows of elements [20]; then, after the arrival phase, one element is randomly deleted. Therefore, the number of elements in the system keeps changing, contrarily to the previous models with a constant load.

Fig. 6 shows the marginal overflow fraction under the above queueing model with $k = 100$ on-off flows of elements. Each flow has rate $\rho = 0.0095$ and average burst size of 10 elements. The figure shows that, given the number of elements currently in the system, the marginal overflow fraction is approximately the one we found for the constant-load case, both for SINGLE and MULTIPLE.

Moreover, by the distribution of the number of elements in the system given by the queueing model, we are able to heuristically approximate the overall expected number of elements in the overflow list. More precisely, we take the sum-product of the queue size distribution by the distribution of the overflow fraction as a function of the load. In the case of SINGLE this model gives an expected number of overflow elements of 61.63, while simulations yield 61.41. Likewise, for MULTIPLE, we obtain 40.17 and 40.26, respectively. Therefore, this heuristic model proves quite accurate.

## IX. Conclusion

In this paper we demonstrated that when the memory is bounded, dynamic schemes behave significantly worse than their static counterparts. This decrease in performance is inherent to the problem, as shown by our lower bounds.

Moreover, we considered two hashing schemes that we proved to be optimal: a single-choice hashing scheme that was used to demonstrate our approach and techniques, and a multiple-choice scheme that inserts the elements greedily.

However, due to the slow decrease of the lower bound, optimality may be insufficient for certain applications. Therefore, we suggested moving back elements from the overflow list as soon as a deletion occurs. We have shown through simulations that this strategy beats the lower bound of the dynamic case (where moving back elements is not allowed).

We also conducted an extensive experimental study to verify the accuracy of our model, the behavior of the models under realistic (rather than fully-random) hash functions, and under variable-load arrival models.

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