Control of Stochastic Processes 048913

Winter 2006 Homework Number 3

4/12/2006 Due 18/12/2006

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Consider the inventory problem of HW2 with a few small changes:

$$x_{k+1} = x_k + u_k - w_k \tag{0.1}$$

$$c_k(x_k, u_k, w_k) = cu_k + b[x_{k+1}]_- + h[x_{k+1}]_+ \tag{0.2}$$

where the state space is the integers, u is the control (how much we order), c_k the immediate cost at time k, c, b, h are positive, and $[x]_-$ is the negative part of x. The demand w_k is i.i.d. and takes value in $[0, 15]$. We add the following assumptions (and clarifications).

- u is non negative, w is non negative.
- $b > c > 0$.
- Note that the terminal cost is zero.

1. Write the immediate cost as $c(x, u)$, which is independent of k and does not depend explicitely on w.

2. Consider the finite horizon problem with horizon N.

   Prove that the optimal policy if of the following form. There is a sequence of numbers S_k so that at time k we order $u_k = [S_k - x_k]_+$: that is, we try to increase the stock size to S_k. Hints:

   (a) Show that c_k is convex

   (b) Use the facts that the expectation of a convex function is convex, and the minimum of convex functions is convex.

   (c) Use the fact that a convex function has a minimum, and if the function is increasing at some point and decreasing at some point, then then minimum is in the interior of its domain (that is, not on the boundaries). To see the last point compate the function y^2 that is defined on $\mathbb{R}$ to the same function defined on $[1, \infty)$.

(d) The final cost (being zero) is convex in u. Show by induction (using dynamic programming) that at each step (and for each x), choosing the control which minimizes the cost involves minimizing a convex function.

3. Find the optimal policy for the infinite horizon discounted cost.
Hint: show that the optimal policy is stationary. Show that the optimal policy for a very long horizon cannot be far in its value from the optimal infinite horizon cost. Consider the optimal policy as a policy for a limit of finite horizon problems.

4. (From Bertsekas, Ex. 4.19). On your way to a famous bar you look for parking. Parking is possible at discrete places, which are numbered $0, 1, \ldots$ by the distance (in car-length units) from the bar. You can see if a parking space is free only when you arrive to it. Each parking space is free with probability p, independently of other spaces.
If you arrive at the bar without finding parking you pay $C > 0$. If you park at a distance k from the bar you pay k.

(a) Show that the minimal expected cost F_k if you are k units away from the bar satisfies

$$F_0 = C \tag{0.3}$$

$$F_k = p \min\{k, F_{k-1}\} + (1-p) F_{k-1}, \quad k \geq 1. \tag{0.4}$$

(b) Define k^* as the smallest integer such that

$$(1-p)^{k-1} < (pC + (1-q))^{-1} \tag{0.5}$$

Show that the optimal policy is to contnue searching as long as $k \geq k^*$ but take the first free space once $k < k^*$.

5. Bonus question (optional): exercise 0.6 from the material about contraction operators.