Control of Stochastic Processes 048913

Winter 2006 Supplement: Contraction operators.

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Fix a complete metric space with metric d.

Definition 0.1 *T is a contraction operator with constant $K < 1$ if for any x, y*

$$d(Tx, Ty) \leq Kd(x, y). \tag{0.1}$$

Let T be a contraction operator.

Claim 0.2 *T is continuous.*

Proof. Let $x_n \to x$, that is $d(x_n, x) \to 0$. Then

$$d(Tx_n, Tx) \leq Kd(x_n, x) \to 0 \tag{0.2}$$

and the result is established. ■

Claim 0.3 *For any x, the sequence of iterates $T^n x$ converges to a fixed point.*

Proof. First we show that $T^n x$ is a Cauchy sequence. Assume $n > m$. We have

$$d(T^n x, T^m x) \leq K^m d(T^{n-m}x, x) \tag{0.3}$$

$$\leq K^m \left[d(T^{n-m}x, T^{n-m-1}x) + d(T^{n-m-1}x, x) \right] \tag{0.4}$$

by the triangle inequality

$$\leq K^m \left[d(T^{n-m}x, T^{n-m-1}x) + \ldots + d(Tx, x) \right] \tag{0.5}$$

$$\leq K^m \left[K^{n-m-1} d(Tx, x) + \ldots + d(Tx, x) \right] \tag{0.6}$$

now bound the sum of powers of K by an infinite geometric sum

$$\leq K^m \frac{1}{1-K} d(Tx, x) \to 0 \tag{0.7}$$

as $m \to \infty$. Thus by completeness $T^n x \to x_0$ for some limit x_0. That is, $d(T^n x, x_0) \to 0$. Now since T is continuous, this implies that also $d(TT^n x, Tx_0) = d(T^{n+1}x, Tx_0) \to 0$. By the triangle inequality,

$$d(Tx_0, x_0) \leq d(Tx_0, T^{n+1}x_0) + d(T^{n+1}x_0, x_0) \tag{0.8}$$

However as we just argued, the first term on the right $\to 0$ by continuity, while the second $\to 0$ since as we proved $T^m x_0 \to x_0$. Thus $d(Tx_0, x_0) = 0$ so that x_0 is a fixed point of T. ■

Claim 0.4 *The fixed point is unique, and thus independent of the starting point of the iterates.*

Proof. Suppose there are two fixed points x_0, x_1. Then

$$Kd(x_0, x_1) \geq d(Tx_0, Tx_1) \tag{0.9}$$

since T is a contraction. However, $Tx_0 = x_0, Tx_1 = x_1$ since they are fixed points. So

$$= d(x_0, x_1) \tag{0.10}$$

which, since $K < 1$, is possible only if $d(x_0, x_1) = 0$, that is $x_0 = x_1$. ■

Claim 0.5 *The iterates converge geometrically, that is,*

$$d(T^n x, x_0) \leq K^n d(x, x_0). \tag{0.11}$$

This means that the rate of convergence is at least as fast as K, with a constant that depends on the initial point.

Proof. Since x_0 is a fixed point, $x_0 = T^n x_0$ so

$$d(T^n x, x_0) = d(T^n x, T^n x_0) \tag{0.12}$$

$$\leq K^n d(x, x_0). \tag{0.13}$$

∎

Exercise 0.6 *Suppose T is not a contraction, but is continuous and there exists $K < 1$ and $J > 0$ so that T^J is a contraction with constant K^J. Show that all the results above continue to hold.*