Home Assignment 5

Submit by June 22.

Ito calculus

Part II: Stochastic Differential Equations

1. Let $X_t = W_t$ and $Y_t = e^{W_t^2}$. Show that (X_t, Y_t) solves the set of stochastic differential equations

$$dX_t = dW_t, \qquad X_0 = 0,$$

$$dY_t = 2X_t Y_t dW_t + (Y_t + 2X_t^2 Y_t)dt, \qquad Y_0 = 1.$$

2. Let $W^1, \ldots, W^n$ be independent BMs and denote $W = (W^1, \ldots, W^n)$ (W is called an n-dimensional BM). Let

$$R = |W| = \left(\sum_{i=1}^{n} (W^i)^2 \right)^{1/2}.$$

Show that R solves the stochastic Bessel equation:

$$dR = \sum_i \frac{W^i}{R} dW^i + \frac{n-1}{2R} dt.$$

3. Let W be an n-dimensional BM, for $n \geq 3$. Write $X = W + x_0$ where the point x_0 lies in the region $U = \{0 < R_1 < |x| < R_2\}$. Calculate explicitly the probability that X will hit the outer sphere $\{|x| = R_2\}$ before hitting the inner sphere $\{|x| = R_1\}$. Hint: Check that $\Phi(x) = |x|^{2-n}$ satisfies $\Delta\Phi = 0$ for $x \neq 0$. Modify Φ to build a function u which equals 0 on the inner sphere and 1 on the outer sphere.