(d) Show: If $X = c$ a.s., then $E(X|\mathcal{F}_1) = c$ a.s.; If $X \in L^1$ and $X \in \mathcal{F}_2$, then $E(X|\mathcal{F}_2) = X$ a.s.; If $X \in \mathcal{F}_3$ and $Y, XY \in L^1$, then $E(XY|\mathcal{F}_3) = XE(Y|\mathcal{F}_3)$.

(e) Let $r \in [1, \infty)$. Show that if $X_n \to X$ in L^r then $E(X_n|\mathcal{G}) \to E(X|\mathcal{G})$ in L^r.

3. Regular conditional probability distributions:

(a) Use regular conditional probability to get the conditional Hölder inequality from the unconditional one, i.e., show that if $p, q \in (1, \infty)$ with $1/p + 1/q = 1$ then

$$E(|XY||\mathcal{G}) \leq E(|X|^p|\mathcal{G})^{1/p} E(|Y|^q|\mathcal{G})^{1/q}.$$

(b) Suppose that the joint law of (X, Y, Z) has a density. Prove that if X is independent of the pair (Y, Z), then

$$E(Y|X, Z) = E(Y|Z).$$

(c) Disprove the following statement. If X, Y, Z are any random variables and X is independent of Y, then

$$E(Y|X, Z) = E(Y|Z).$$