# Fundamentals of stochastic processes 048868

## Home assignment 1—Probability and random variables,

## An example of a proof invoving sigma-fields.

The problem is that we do not have a characterization of (Borel) $\sigma$-fields. Here is an example how to work around this problem.

Suppose we want to establish the following claim. If a function $f$ has the property that the set $\{x : f(x) \in [a, b]\} = f^{-1}([a, b])$ is a Borel set for all $a, b$, then $f$ is Borel measurable.

The point is that Borel measurability is defined by looking at inverse images of all Borel sets, not just all closed intervals.

The approach: let us define a collection $D$ of sets, as follows. It is the smallest collection of sets $d$ so that $f^{-1}(d) \in B$-the Borel field. We want to show that in fact, $D$ is the Borel field.

What do we know about $D$? By definition, it contains all closed intervals. Also, note that it contains the empty set, it is closed under complements, since $\{x : f(x) \in S^c\} = \{x : f(x) \in S\}^c$ (think about it!). In a similar manner, $D$ is closed under finite and countable unions:

$$\{x : f(x) \in \cup_i S_i\} = \cup_i \{x : f(x) \in S_i\} \ .$$

So, we showed that $D$ is the smallest sigma-field that contains all closed intervals. But this means it is the Borel sigma field, by its definition!