Fixed-Rate Equilibrium in Wireless Collision Channels

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Abstract. We consider a collision channel, shared by a finite number of self-interested users with heterogenous throughput demands. It is assumed that each user transmits with a fixed probability at each time slot, and the transmission is successful if no other user transmits simultaneously. Each user is interested in adjusting its transmission rate so that its throughput demand is met. When throughput requirements are feasible, we show that there exist two equilibrium points where users satisfy their respective demands. In one equilibrium all users transmit at lower rates, compared to their transmission rates at the other equilibrium. This fact is meaningful in wireless systems, where lower transmission rates translate to power savings. Subsequently, we propose a distributed scheme that ensures convergence to the lower-rate equilibrium point. We also provide some lower bounds on the channel throughput that is obtained with self-interested users, both in the symmetric and non-symmetric case.

1 Introduction

1.1 Background and Motivation

As wireless networks become larger, it may be impractical to have a central authority (such as a base station) coordinate between wireless stations (which share the same communication medium) for better network utilization. Thus, random access, ALOHA-like protocols are often used (for example in the 802.11x standards). The incorporation of such protocols in wireless systems raises some novel challenges, as these protocols should consider additional wireless-specific items such as power control and varying channel conditions (an effect known as channel fading [1]). Hence, a major research challenge is to examine whether the distributed nature of random access protocols may lead the network to reasonable working points, thereby coping with additional complexity of wireless systems.

Since wireless nodes usually do not coordinate in establishing their transmission policies, non-cooperative game theory becomes a natural framework for analyzing their interaction. Game theoretic tools have been recently applied for analyzing selfish behavior of users (a "user" stands for a single node or station) in Aloha-like random access networks [2–6]. A common ground of most of these papers, is that users are identical (or symmetric), both in their physical parameters (such as the arrival rate of packets [4]) and also in their underlying objective (such as maximizing throughput [4], or minimizing the number of attempted transmissions before success [3]). In practice, however, network users are heterogenous in nature. For example, video, voice, ftp and e-mail applications, all have fundamentally diverse QoS requirements. A paper by Jin and Kesidis [2] does incorporate user heterogeneity, by studying an Aloha-like network with users who have fixed (and different) throughput demands. Users dynamically adapt their transmission rates in order to obtain their required throughput demands. It was shown by means of an example that the equilibrium point may not be unique. Additionally, the authors suggested a dynamic scheme that could lead to an equilibrium point.

1.2 Paper Organization and Contribution

In this paper we reconsider the model suggested in [2]. A description of the model is given in Section 2. A detailed equilibrium analysis (complementing missing analysis in [2]) is provided in Section 3. Our equilibrium analysis reveals, in particular, that when the rate requirements are within the sustainable region, there exist exactly two equilibrium points in the resulting game, and that one of them is strictly better than the other, in the sense that all users transmit at lower rates (in comparison to their transmission rates at the other equilibrium). This fact is meaningful in terms of power consumption in wireless systems. We also show that the equilibrium points can be computed in polynomial time. In the context of wireless systems, we examine in Section 4 how self-optimizing behavior affects the network performance. Specifically, we show that the performance gap (in terms of the total power consumption) between the equilibrium points is potentially unbounded, and that the better equilibrium point coincides with the socially-optimal operating point. In Section 5 we present a distributed algorithm which converges to the better equilibrium. Finally, we provide in Section 6 a simple lower bound on the channel throughput that can possibly be obtained with selfish users.

2 Model Description

We consider an ALOHA-like network, shared by a finite set of users $\mathcal{I} = \{1, \dots, n\}$ who transmit data over a shared collision channel (e.g., wireless stations who transmit to a common base station). Time is slotted, in the sense that all transmitted packets have the same length and require the same time interval (a slot) for transmission. Moreover, all transmissions start at the beginning of a slot and end before the next slot. We assume that a transmission is successful only if no other user attempts transmission simultaneously.

Each user i is characterized by a throughput ρ_i (in packets per slot) which it wishes to deliver over the network. We assume that a user always has packets to send, yet it may postpone transmission, due to the following reasons. First, a user need not transmit in every slot when its average throughput is already met, since it unnecessarily wastes additional resources, such as transmission power. Second, assume a user transmits at every slot; then other users would raise their transmission rates as well, and as a consequence packets will endlessly collide. Hence, each user i chooses a transmission probability p_i, which could be regarded as the transmission rate.

The underlying assumption of our user model is that users are selfish and do not cooperate in any manner in order to obtain their required throughput demands. Define r_i as user i's average throughput. Then

$$r_i = p_i \prod_{j \neq i} (1 - p_j). \tag{1}$$

Note that the transmission probability of each user affects the throughput of all other users through the collision channel. This situation establishes a non-cooperative game [7] between the users. We are interested in the Nash equilibrium point of that game. In our context, a Nash equilibrium point is a vector of user probabilities $\mathbf{p} = (p_1, \ldots, p_n)$, such that

$$r_i = p_i \prod_{j \neq i} (1 - p_j) = \rho_i, \quad i \in \mathcal{I}. \tag{2}$$

We shall refer to the above set of equations (2) as the *equilibrium equations*.

3 Equilibrium Analysis

In this section we analyze the Nash equilibrium point (2) of the network. We start our analysis by considering the number of equilibria.

3.1 Two Equilibria or None

Obviously, if the users' throughput requirements are too high there would not be an equilibrium point, since the network naturally has limited capacity. In case that an equilibrium point does exist, we establish that, generically, there are exactly *two equilibria* (which can be computed efficiently). In addition, we assert that the existence of an equilibrium point could be verified through a computationally efficient procedure. The main result of this section is presented below.

Theorem 1. *Consider the non-cooperative game whose Nash equilibrium point is defined in (2). There are either one, two Nash equilibrium points or none for that game. The case of a single equilibrium point is non-generic (i.e., occurs only for a set of rate vectors ρ of measure zero).*

Proof. See Appendix.

We summarize certain computability properties in the next proposition (proof is omitted due to lack of space).

Proposition 1. *The existence of an equilibrium point can be verified in polynomial time (in the number of users). Additionally, in case an equilibrium exists, both equilibria can be computed in polynomial time (in the number of users).*

3.2 Efficiency and Fairness

Besides the existence and the number of equilibrium points, we wish to characterize the equilibrium points. In particular, we are interested in the following questions:

1. How do the two equilibrium points compare: is one "better" than the other?
2. Is an equilibrium point fair in some sense?

The next theorem addresses the first question raised above. It shows that one equilibrium point is better for all users in the sense that all users transmit at lower rates.

Theorem 2. *Assume there exist two equilibria for the non-cooperative game, whose Nash equilibrium is defined in (2). Let* $\mathbf{p}$ *and* $\tilde{\mathbf{p}}$ *be these two equilibrium points. If* $p_i < \tilde{p}_i$ *for some user* i*, then* $p_j < \tilde{p}_j$ *for every* $j \in \mathcal{I}$*.*

Proof. Define $a_{ik} \triangleq \frac{\rho_i}{\rho_k}$. For every user $k \neq i$ divide the ith equation in the set (2) by the kth one. We obtain

$$a_{ik} = \frac{p_i(1-p_k)}{p_k(1-p_i)} < \frac{\tilde{p}_i(1-p_k)}{p_k(1-\tilde{p}_i)}. \tag{3}$$

Now since

$$\frac{\tilde{p}_i(1-\tilde{p}_k)}{\tilde{p}_k(1-\tilde{p}_i)} = a_{ik}, \tag{4}$$

it follows that

$$\frac{\tilde{p}_i(1-\tilde{p}_k)}{\tilde{p}_k(1-\tilde{p}_i)} < \frac{\tilde{p}_i(1-p_k)}{p_k(1-\tilde{p}_i)}. \tag{5}$$

We conclude from the last inequality that $p_k < \tilde{p}_k$. □

The last result is significant from the network point of view. Assuming that each transmission is costly (e.g., each transmission consumes a fixed power), we are interested in a network mechanism which will exclude the inferior equilibrium point. This would be our main concern in Section 5. Henceforth, we identify the better equilibrium point as the *Energy Efficient Equilibrium (EEE)*.

We now compare the user effort in a given equilibrium point. Our next result suggests that at every equilibrium, the transmission probabilities are ordered in the same order as the throughput demands ρ_i, i.e., users with a larger demand transmit more aggressively.

Theorem 3. *Let* $\mathbf{p}$ *be an equilibrium point of (2). Then if* $\rho_i \geq \rho_j$ *it follows that* $p_i \geq p_j$*.*

Proof. Follows easily from eq. (4). Details are omitted.

The above result indicates that despite user selfishness, some notion of *fairness* is maintained at equilibrium: The higher the throughput requirement, the higher the transmission rate (and consequently, the higher is the power consumption).

4 Efficiency Loss

We now turn to examine the extent to which selfish behavior affects system performance. That it, we are interested in comparing the quality of the obtained equilibrium points to the theoretical case where a central authority can set the users' transmit policies. Recently, there has been much work in quantifying the "efficiency loss" incurred by the selfishness of users in networked systems (see [8] for a comprehensive review). The two concepts which are most commonly used in this context are the *"price of anarchy"*, which is the performance ratio (of a relevant social performance measure) between the global optimum and the *worst* Nash equilibrium, and *"price of stability"*, which is the performance ratio between the global optimum and the *best* Nash equilibrium.

We focus in this section on a wireless system, where W_i represents a (fixed) energy which user i utilizes per transmission. The average power of user i is then given by $J_i(p_i) \stackrel{\triangle}{=} p_i W_i$. A natural performance criterion for evaluating the quality of an equilibrium is given by $\sum_i J_i(p_i)$, which represents the total power consumption in the network. We next show that the price of anarchy with respect to this criterion is unbounded, while the price of stability is always one.

Theorem 4. *Consider the non-cooperative game whose Nash equilibrium point is defined in (2). Define $\sum_i J_i(p_i)$ as the social performance criterion. Then (i) the price of stability is always one, i.e., the better equilibrium point coincides with the social optimum, and (ii) the price of anarchy is generally unbounded.*

Proof. (i) immediate, as a social optimum obeys the equilibrium equations (2).

(ii) we establish that the price of anarchy is unbounded by means of an example. Consider a network with two identical users with a throughput requirement of $\rho_i = \epsilon \to 0$ $(i = 1, 2)$, and an average power function given by $J(p) = Wp$, where p is the user's transmission probability (user indexes are omitted in this example, as users are identical). By symmetry, we obtain a single equilibrium equation, namely $p(1 - p) = \epsilon$. As ϵ goes to zero, the two equilibria are $p_a \to 1$ and $p_b \to 0$. Obviously, the latter point is also a social optimum; it is readily seen that the price of anarchy equals at the limit to $\frac{2J(1)}{2J(0)} = \infty$.

5 A Distributed Algorithm

5.1 The Algorithm

We have shown so far that when an equilibrium point exists, there are two equilibria, where one obtains lower transmission rates for all users. Moreover, the performance gap between the equilibria could be significantly large. This leads us to find a mechanism which will converge to the better equilibrium. We next suggest a simple distributed algorithm for that purpose.

Let $x_i = \prod_{j \neq i}(1 - p_j)$ denote the current idle probability of all users but the ith one. Each user i iteratively updates its transmission probability through the following rule:

$$p_i := p_i + \epsilon_i \left(\frac{\rho_i}{x_i} - p_i \right), \tag{6}$$

where $0 < \epsilon_i \leq 1$ is the update gain of user i. The motivation for using the above rule follows directly from the equilibrium equations (2).

A synchronized version of the above algorithm (where all users iteratively update their transmission probabilities at the same slots) is considered in [9] pp. 347–349. It was shown there that when $\epsilon_i = 1$ for every $i \in \mathcal{I}$, the algorithm asymptotically converges to the better equilibrium point.

5.2 Practical Considerations and Stability

We pause here to address certain practical implementation issues which are related to the presented algorithm. The quantity x_i required for the probability updates can be obtained by each user through dividing the overall idle probability by its own idle probability. In practice, the channel's idle probability should be estimated by each user by measuring the percentage of idle slots, where the more slots sensed, the better is the estimate. If the transmission probabilities are to be updated on a slow time-scale, users would be able to obtain good estimates of the idle probability. For the sake of our analysis, we assume that the estimation is perfect. Indeed, the frequency of the updates is an important issue. There is obviously a tradeoff between the estimation accuracy of x_i and the wish to enforce the required equilibrium as fast as possible. We leave the quantification of this tradeoff to future research.

We now consider some stability properties of the algorithm. In particular, we wish to verify whether the better equilibrium point is locally stable. Indeed, even when the network operates near a stationary working point, users continuously adjust their probabilities according to (6) (e.g., due to perturbations in their idle estimation). We then have the following stability result.

Theorem 5. *Assume users update their transmission probabilities according to (6). When each ϵ_i is small enough, the EEE is locally stable.*

Proof. (outline) Let $\epsilon_i = \lambda_i \epsilon$, with ϵ-small. The continuous-time limit of (6) as $\epsilon \to 0$ is

$$\dot{p_i} = \lambda_i \left(\frac{\rho_i}{x_i} - p_i \right). \tag{7}$$

We next apply Lyapunov's indirect method (linearization) to study the stability of (7). Consider first the case where $\lambda_i = \lambda_j$ for every $i, j \in \mathcal{I}$. Let Z be the corresponding Jacobian matrix of (7) for this symmetric case. It can be shown that the elements of Z are given by

$$z_{ij} = \begin{cases} -1 & i = j \\ \frac{p_i}{1-p_j} & i \neq j \end{cases}. \tag{8}$$

It can be further shown that the matrix Z is strictly diagonally dominant (see, e.g., [10]) for the better equilibrium point. As such, all eigenvalues of Z have a negative real part ([10], pp. 349). Consider now a general asymmetric case. Let $\lambda = diag(\lambda_1, \ldots, \lambda_n)$. Then the corresponding Jacobian matrix of (7) is given by λZ. It is known that if all eigenvalues of Z have a negative real part, then the same property holds for RZ, where R is any positive diagonal matrix (see [11], pp. 112–121). Consequently, we may conclude that the equilibrium point is asymptotically stable. The local stability of the discrete-time model follows now by continuity argument. □

6 Achievable Throughput

The aim of this section is to provide a lower bound for the maximal throughput which can be obtained in the network.

The theorem below establishes the conditions for the existence of an equilibrium point in the symmetric case.

Theorem 6 (Symmetric users). *Let $\rho_i = \rho$ for every $1 \leq i \leq n$. Then an equilibrium exists if and only if*

$$\sum_{i=1}^{n} \rho_i = n\rho \leq (1 - \frac{1}{n})^{n-1}. \tag{9}$$

Proof. In every equilibrium of the symmetric case $p_i = p_j = p$, for every i, j (immediate by Theorem 3). Thus, the equilibrium equations (2) diminish into a single (scalar) equation:

$$h(p) \stackrel{\triangle}{=} p(1-p)^{n-1} = \rho. \tag{10}$$

We next investigate the function $h(p)$. The derivative of $h(p)$ is given below.

$$h'(p) = (1-p)^{n-2}\big(1 - p - (n-1)p\big) = (1-p)^{n-2}(1 - np). \tag{11}$$

It can be seen that the maximum value of the function $h(p)$ is obtained at $p = 1/n$. An equilibrium exists if and only if the maximizer of the function obtains a value which is greater than ρ. We assign the maximizer $p = 1/n$ of $h(p)$ in (10) and the result immediately follows. □

The next corollary provides a sufficient condition for the existence of an equilibrium for any number of symmetric users.

Corollary 1. *Let $\rho_i = \rho$ for every $1 \leq i \leq n$. Then an equilibrium exists if* $\sum_{i=1}^{n} \rho_i \leq e^{-1}$.

Proof. Observe that the left hand side of (9) is the total throughput demand. It may be easily verified that the right hand side of (9) decreases with n. Since $\lim_{n\to\infty}(1 - \frac{1}{n})^{n-1} = e^{-1}$, a total throughput demand which is less or equal than this quantity guarantees the existence of an equilibrium. □

It can be shown that the simple bound obtained above holds for non-symmetric users as well, implying that the symmetric case is worst in terms of system utilization. Our result is summarized below. Due to lack of space the (somewhat lengthy) proof is omitted.

Theorem 7 (Asymmetric users). *For any set of n users, an equilibrium point exists if*

$$\sum_{i=1}^{n} \rho_i \leq (1 - \frac{1}{n})^{n-1}. \tag{12}$$

The quantity e^{-1} is also the well-known maximal throughput of a slotted Aloha system with Poisson arrivals and an infinite set of nodes [9]. In our context, if the throughput requirements do not exceed e^{-1}, an equilibrium point is guaranteed to exist. Thus, in a sense, we may conclude that user heterogeneity does not deteriorate the capacity (i.e., the maximal throughput) of the collision channel.

7 Conclusions and Model Extensions

We have investigated in this paper the interaction between heterogenous users, who adjust their transmission rates in order to obtain their individual throughput demands. We established that the network possesses either two Nash equilibria or none. In case that two equilibria exist, in one of the equilibria *all* users transmit at lower rates compared to the transmission rates at the other equilibrium. Translating this fact to power-related terms (which are relevant in wireless systems), we further demonstrated that the performance gap between the two equilibria (in terms of power consumption) could be arbitrarily large. Consequently, network users should be willing to accept a mechanism which ensures convergence to the better equilibrium. We have suggested such a mechanism, and studied some stability properties thereof.

In an ongoing work, we consider a more general scenario of a block fading channel [1], where the channel state of each user varies over time, and affects the data rate. Users, who measure their channel state, base the transmission decision at each time slot on the current measurement of their channel state. Our analysis so far indicates that many of the results of the present paper carry over to this general setup.

Several additional directions remain for future research. Among which are: (i) Further analyzing asynchronous versions of the distributed algorithm. (ii) Incorporating the transmit power as an additional decision variable (as in [5, 6]). In some models of practical interest, the transmission power not only affects the data rate, but also the reception chances of the packet. In this context, we plan to include capture models (which sometimes better represent WLAN systems) and multi-packet reception models [12] (as in CDMA systems) in our future work.

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Appendix – Proof of Theorem 1

We start with the following lemma, which relates the user probabilities in equilibrium.

Lemma 1. *In every equilibrium point the following relation holds for every $i, j \in \mathcal{I}$.*

$$p_j = \frac{a_{ji} p_i}{1 - p_i + a_{ji} p_i}, \tag{13}$$

where $a_{ji} \triangleq \rho_j / \rho_i$.

Proof. Immediate by dividing the equilibrium equation of the ith user by the equation of the jth one. □

The idea behind the proof of the theorem is to represent the equilibrium conditions through a single scalar equation. The result then follows by showing concavity of this equation.

The equilibrium point $\mathbf{p}$ (if exists) is by (2) such that

$$p_i = \rho_i / \prod_{j \neq i} (1 - p_j), \quad 1 \leq i \leq n. \tag{14}$$

Fixing the ith user and substituting (13) into (14) for every $j \neq i$ we obtain the following equation in p_i only.

$$p_i \prod_{j \neq i} \left(1 - \frac{a_{ji} p_i}{1 - p_i + a_{ji} p_i} \right) = p_i \prod_{j \neq i} \left(\frac{1 - p_i}{1 - p_i + a_{ji} p_i} \right) = \rho_i. \tag{15}$$

Taking log from both sides we have

$$\log p_i + \sum_{j \neq i} \log \left(\frac{1 - p_i}{1 - p_i + a_{ji} p_i} \right) = \log \rho_i. \tag{16}$$

Let

$$g(p_i) \stackrel{\triangle}{=} \log p_i + \sum_{j \neq i} \log \left(\frac{1 - p_i}{1 - p_i + a_{ji} p_i} \right). \tag{17}$$

We next investigate the properties of the function $g(p_i)$. Specifically, we concentrate on the user with the maximal demand. Without loss of generality, let user i be the one with the maximal demand, i.e., $\rho_i \geq \rho_j$ for every $j \neq i$. Thus, $a_{ji} \leq 1$. The derivative of $g(p_i)$ is calculated below.

$$\begin{aligned} g'(p_i) &= \frac{1}{p_i} + \sum_{j \neq i} \frac{1 - p_i + a_{ji} p_i}{1 - p_i} \cdot \frac{-(1 - p_i + a_{ji} p_i) + (1 - a_{ji})(1 - p_i)}{(1 - p_i + a_{ji} p_i)^2} \\ &= \frac{1}{p_i} - \sum_{j \neq i} \frac{a_{ji}}{(1 - p_i)(1 - p_i + a_{ji} p_i)}. \end{aligned} \tag{18}$$

We focus our analysis on $p_i \in [0, 1]$. Observe that $g(0) = g(1) = -\infty$. Additionally, $g'(0) = \infty$, $g'(1) = -\infty$. Hence, if the derivative $g'(p_i)$ is monotonously decreasing in $p_i \in [0, 1]$, then there are either two roots (which may coincide) or none for the equation (16). Indeed, deriving $g'(p_i)$ yields

$$g''(p_i) = -\frac{1}{p_i^2} + \sum_{j \neq i} \frac{a_{ji} \big[(1 - p_i)(-1 + a_{ji}) + (-1 + p_i - a_{ji} p_i) \big]}{\big((1 - p_i)(1 - p_i + a_{ji} p_i) \big)^2}. \tag{19}$$

Noting that $a_{ji} \leq 1$, it may be easily verified that $(1 - p_i)(-1 + a_{ji}) < 0$ for $p_i \in [0, 1)$; additionally $(-1 + p_i - a_{ji} p_i) = -1 + p_i(1 - a_{ji}) < 0$ for $p_i \in [0, 1)$, so overall $g''(p_i) < 0$ for $p_i \in [0, 1]$. We conclude that there are either two roots or none for the equation (16). Hence, there are either two equilibrium points or none for the game. There is a single equilibrium point in the non-generic case where the maximum of $g(p_i)$ equals $\log \rho_i$. □