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Persistency of excitation in continuous-time systems

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Abstract: For the class of linear time-invariant single-input continuous systems we find conditions on the input under which the state is persistently exciting for adaptive identification purposes. These conditions are expressed through time-domain properties of the filtered input. They are both necessary and sufficient and no prior constraints are placed on the structure of the input wave or its boundedness.

Keywords: Persistency of excitation, Parameter identification, Input richness.

1. Introduction

Persistent excitation plays a key role in establishing parameter convergence in adaptive identification and control schemes. Early results for continuous-time systems specify conditions for (exponential) convergence through some 'persistence of excitation' conditions on the signal vector used in the identification algorithm. This vector contains both the system input and output signals (see e.g. [1]). Since output signals cannot be directly controlled, some effort was devoted recently to the characterization of persistently exciting (PE) inputs. Namely, inputs which cause the outputs of the system to be PE. The results in Boyd and Sastry [2,3], and Mareels and Gevers [7] are limited to stable systems. Results for unstable systems first appeared in Dasgupta, Anderson and Tsoi [1] assuming however that system output is bounded. This constraint was removed in Nordström and Sastry [4], where sufficient conditions for PE inputs were given in the frequency and time domains for possibly unstable systems. There were, however, assumptions made on the input which include boundedness, piecewise uniform continuity and stationarity (for the frequency-domain condition) or differentiablility (for the time-domain condition).

In the work reported here conditions for PE inputs were derived. These conditions complement existing results in three respects: (i) They are both necessary and sufficient. (ii) No prior assumptions on the input are made save for local integrability. (iii) It is shown that if an input is PE, a uniform excitation period for all systems always exists. We believe that this closes the gap which existed between the discrete-time results (e.g. [5,6]) and the continuous-time results.

The basic approach taken here is inspired by Mareels [5]. We comment however that the conditions given there for the continuous case, though necessary, are not sufficient unless complemented by other assumptions as in [4].

We proceed as follows: After establishing notation in Section 2, we present and prove in Section 3 our main results. In Section 4 we derive a more general filtered version and, finally, in Section 5 we briefly consider systems with output equations.

2. Notation

The following notation will be used throughout.

$L_e^1(\mathbb{C}^n)$ – the space of functions $f:\mathbb{R}^+ \to \mathbb{C}^n$ which are Lebesgue integrable over any finite interval.

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$L_e^2(\mathbb{C}^n)$ – the space of functions as above which are also square-Lebesgue-integrable over any finite interval.

$\| \ \|_T$ – norm of a function in L_e^2 restricted to $[0, T]$:

$$\| f \|_T = \left[\int_0^T | f(t) |^2 \, dt \right]^{1/2}.$$

$| \cdot |$ – Euclidean norm of a vector in $\mathbb{C}^n$: $| v | = [v^* v]^{1/2}$.

M^{T}, $\overline{M}$, M^* – transposed, complex conjugate and Hermitian conjugate respectively of a matrix or vector M.

The following abbreviations will also be used: $(If)(t)$ denotes the definite integral of function, $(If)(t) = \int_0^t f(\tau) \, d\tau$; and $f_\tau(t)$ denotes translation of f along the real axis, $f_\tau(t) = f(t + \tau)$. Hence, we have $(If_\tau)(t) = \int_0^t f(\sigma + \tau) \, d\sigma$.

3. Persistent excitation of the state

Consider the class SC_n of linear time-invariant single-input controllable systems of order n:

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0, \quad A \in \mathbb{C}^{n \times n}, \quad b \in \mathbb{C}^n, \quad u \in L_e^1(\mathbb{C}). \tag{1}$$

The state x is assumed to be the signal vector used in identification. Hence, for parameter convergence, x is required to be PE in the following sense:

Definition 1. The function $x \in L_e^2(\mathbb{C}^n)$ is said to be *persistently exciting (PE)* iff there exist positive constants ε_1, T such that for all $\tau \geq 0$,

$$\int_\tau^{\tau + T} x(t) x(t)^* \, dt > \varepsilon_1 I_n. \tag{2}$$

T will be termed an 'excitation period' of x.

Remark 1. For the system (1), $u \in L_e^1$ implies $x \in L_e^2$ so the integral in Definition 1 is always well-defined. Moreover, since the only property of u needed in the sequel is that $\int_0^t u(\sigma) \, d\sigma \in L_e^2$, we can include impulse functions in u (provided that their total power is bounded over any finite interval).

Definition 2. The input $u \in L_e^1$ is said to be *persistently exciting for the class* SC_n ((PESC$_n$) iff for any system in SC_n it produces a PE state x, uniformly in x_0 (i.e. ε_1 and T in Definition 1 are independent of the initial condition x_0).

We proceed now to define a richness property of the input that will characterize the class of PESC$_n$ inputs. Using the notation

$$(I^n u_\tau)(t) = \int_0^t d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \cdots \int_0^{\sigma_{n-1}} d\sigma_n u(\sigma_n + \tau), \tag{3}$$

we define

$$V_\tau(t) \triangleq [Iu_\tau, \ldots, I^n u_\tau]^{\mathrm{T}}(t) \tag{4}$$

and

$$W_\tau(M, t) \triangleq V_\tau(t) + M\theta(t). \tag{5}$$

where $M \in \mathbb{C}^{n \times n}$ is some constant matrix and

$$\theta(t) \triangleq [1, t, t^2, \ldots, t^{n-1}]^{\mathrm{T}}. \tag{6}$$

Also define

$$J_\tau(M, T) = \int_0^T W_\tau(M, t) W_\tau^*(M, t) \, \mathrm{d}t. \tag{7}$$

Definition 3. The function $u \in L_e^1(\mathbb{C})$ is said to be *rich of order n* iff there exist positive constants ε_2, T such that for all $\tau \geq 0$,

$$J_\tau(M_\tau, T) = \int_0^T [V_\tau(t) + M_\tau \theta(t)][V_\tau(t) + M_\tau \theta(t)]^* \, \mathrm{d}t \geq \varepsilon_2 I \tag{8}$$

where M_τ is defined by

$$M_\tau = -\left(\int_0^T V_\tau(t) \theta^{\mathrm{T}}(t) \, \mathrm{d}t\right) N_0^{-1} \tag{9}$$

and

$$N_0 \triangleq \int_0^T \theta(t) \theta(t)^{\mathrm{T}} \, \mathrm{d}t.$$

Note that $[N_0]_{i,j} = T^{i+j-1}/(i+j-1)$.

Remark 2. Definition 3 has a simple interpretation if we view our functions (for every constant τ) as vectors in the Hilbert space L^2 of functions $f: [0, T] \to \mathbb{C}$. Lemma 1 below shows that the term $M_\tau \theta(t)$ in fact removes from the entries of V_τ (the functions $Iu_\tau, \ldots, I^n u_\tau$) all components which lie in the subspace S spanned by $(1, t, \ldots, t^{n-1})$. Hence, Definition 3 is roughly equivalent to requiring that the projections of the functions $I^i u_\tau$ onto $S^\perp$, the orthogonal complement of S, are (uniformly) linearly independent.

The following lemma summarizes some of the properties of M_τ.

Lemma 1. *For any positive τ, T the matrix M_τ (as defined in (9)) satisfies*
(a) $J_\tau(M_\tau, T) \leq J_\tau(M, T)$ *for all* $M \in \mathbb{C}^{n \times n}$;
(b) $\int_0^T t^k W_\tau(M_\tau, t) \, \mathrm{d}t = 0$, $k = 0, 1, \ldots, n-1$;
(c) $(I^k W_\tau(M_\tau, \cdot))(T) = 0$, $k = 1, 2, \ldots, n$.

Proof. Direct substitution of (5) and (9) confirms that

$$\int_0^T W_\tau(M_\tau, t) \theta^{\mathrm{T}}(t) \, \mathrm{d}t = 0 \tag{10}$$

and (b) follows directly. (a) can now be established since

$$J_\tau(M, T) - J_\tau(M_\tau, T) = \int_0^T [M - M_\tau] \theta(t) \theta(t)^{\mathrm{T}} [M - M_\tau]^* \, \mathrm{d}t \geq 0. \tag{11}$$

Finally, (c) follows from (b) by successive integration by parts. □

Definition 3 is now motivated by our main result.

Theorem 1. *An input $u \in L_e^1(\mathbb{C})$ is persistently exciting for the class* $\mathrm{SC}_n(\mathrm{PESC}_n)$ *iff it is rich of order n.*

Proof. Let

$$p(s) = p_n s^n + p_{n-1} s^{n-1} + \cdots + p_0, \quad p_n = 1, \tag{12}$$

be the characteristic polynomial of A. We form now the matrix $Q \in \mathbb{C}^{n \times n}$:

$$Q = [b,\ Ab, \ldots, A^{n-1}b]P_A, \quad P_A = \begin{bmatrix} p_n & \cdots & p_1 \\ & \ddots & \vdots \\ 0 & & p_n \end{bmatrix}, \tag{13}$$

which is nonsingular for a controllable pair $(A,\ b)$.

The state $x(t)$ of system (1) satisfies for all $\tau > 0$ the following integral equation:

$$\sum_{i=0}^{n} p_i (I^{n-1} x_\tau)(t) = QV_\tau(t) + \tilde{Q}_\tau \begin{bmatrix} 1 \\ t \\ \frac{t^2}{2} \\ \vdots \\ \frac{t^{n-1}}{(n-1)!} \end{bmatrix}, \tag{14}$$

where $\tilde{Q}_\tau$ is defined similarly to Q in (13) with b replaced by the initial condition $x(\tau)$. Eqn. (14) is in fact an integral form of the equation

$$\sum_{i=0}^{n} p_i x^{(i)}(t) = Q \begin{bmatrix} u^{(n-1)} \\ \vdots \\ u^{(1)} \\ u \end{bmatrix} \tag{15}$$

used in [5].

To prove sufficiency we note that the richness assumption is equivalent to: $\exists T > 0,\ \varepsilon_2 > 0$ such that $\forall \tau \geq 0,\ d \in \mathbb{C}^n$,

$$\| d^{\mathrm{T}} W_\tau(M_\tau,\ t) \|_T^2 = \int_0^T d^{\mathrm{T}} W_\tau(M_\tau,\ t) W_\tau^*(M_\tau,\ t) \bar{d}\ \mathrm{d}t \geq \varepsilon_2 |d|^2. \tag{16}$$

Let $v \in \mathbb{C}^n$ be an arbitrary vector and take any system $(A,\ b)$ in SC_n. For simplicity we denote

$$W(t) \triangleq W_\tau(M_\tau,\ t), \qquad d^{\mathrm{T}} = v^{\mathrm{T}} Q.$$

Then, using (5), (14) and Lemma 1 (b) we get

$$\| d^{\mathrm{T}} W \|_T^2 = \int_0^T v^{\mathrm{T}} \left[\sum_{i=1}^{n} p_i (I^{n-i} x_\tau)(t) \right] \cdot W^*(t) \bar{d}\ \mathrm{d}t. \tag{17}$$

Integrating by parts and using Lemma 1 (c) yields

$$\| d^{\mathrm{T}} W \|_T^2 = \int_0^T v^{\mathrm{T}} x_\tau(t) f(t)\ \mathrm{d}t \tag{18}$$

where

$$f(t) \triangleq \sum_{i=0}^{n} (-1)^i p_i (I^{n-i} W)^*(t) \bar{d}.$$

Since for every function $g \in L_e^2$ we have from the Schwarz inequality

$$\| Ig \|_T^2 = \int_0^T \left| \int_0^t g(\tau)\ \mathrm{d}\tau \right|^2 \mathrm{d}t \leq \int_0^T \left[\int_0^t \mathrm{d}\tau \right] \left[\int_0^t |g(\tau)|^2\ \mathrm{d}\tau \right] \mathrm{d}t \leq \int_0^T t \| g \|_T^2\ \mathrm{d}t \leq T^2 \| g \|_T^2,$$

it follows that

$$\| f \|_T \le \sum_{i=0}^{n} | p_i | \| (I^{n-i} W^*) \bar{d} \| \le \left(\sum_{i=0}^{n} | p_i | T^{n-i} \right) \| d^{\mathrm{T}} W \|_T. \tag{19}$$

Applying the Cauchy–Schwarz inequality to (18) results in

$$v^{\mathrm{T}} \left[\int_0^T x_\tau(t) x^*(t) \, \mathrm{d}t \right] \bar{v} = \| v^{\mathrm{T}} x_\tau \|_T^2 \ge \frac{\| d^{\mathrm{T}} W \|_T^4}{\| f \|_T^2} \ge \frac{\| d^{\mathrm{T}} W \|_T^2}{\left(\sum_{i=0}^{n} | p_i | T^{n-i} \right)^2}. \tag{20}$$

On the other hand, since $d^{\mathrm{T}} = v^{\mathrm{T}} Q$ we have from assumption (16),

$$\| d^{\mathrm{T}} W \|_T^2 \ge \varepsilon_2 | d |^2 \ge \varepsilon_2 \lambda_{\min}(QQ^*) | v |^2. \tag{21}$$

Combining (20) and (21) clearly establishes that x is PE according to Definition 1, with

$$\varepsilon_1 = \varepsilon_2 \lambda_{\min}(QQ^*) \Big/ \left(\sum_{i=0}^{n} | p_i | T^{n-i} \right)^2. \tag{22}$$

To prove necessity let us assume that the input is PESC_n.

Let $C_n \subset \mathrm{SC}_n$ be a set of n systems such that the characteristic polynomials of their A's are mutually coprime. Clearly, by our assumption, each system in C_n has an ε_1^i and T_i so that (2) is satisfied for that system. Define $\varepsilon_1 = \min_{C_n}(\varepsilon_1^i)$ and $T = \max_{C_n}(T_i)$; then clearly

$$\int_\tau^{\tau + T} x(t) x(t)^* \, \mathrm{d}t > \varepsilon_1 I \tag{23}$$

for every system in C_n, all $\tau \ge 0$ and all initial states $x(0)$.

Now for any $\tau \ge 0$ and $d \in \mathbb{C}^n$ with $| d | = 1$ there exists a system $(A, b) \in C_n$ such that the characteristic polynomial of A is coprime with the polynomial $[1, s, \dots, s^{n-1}]d$. (This is clearly true since $[1, s, \dots, s^{n-1}]d$ can have common factors with at most $n-1$ of the A's in C_n.) This is equivalent (see [5]) to the pair (v^{T}, A) being an observable pair, where

$$v^{\mathrm{T}} = d^{\mathrm{T}} Q^{-1}. \tag{24}$$

Since (see the definition of $\tilde{Q}$ following Eqn. (14))

$$v^{\mathrm{T}} \tilde{Q}_\tau = x(\tau)^{\mathrm{T}} \left[v, A^{\mathrm{T}} v, \dots, (A^{\mathrm{T}})^{n-1} v \right] P_A,$$

the above-mentioned observability implies that by choice of $x(\tau)$ the left-hand side can be assigned any desired value. In particular, there exists an $x(\tau)$ (hence an $x(0)$) for which

$$v^{\mathrm{T}} \sum_{i=0}^{n} p_i \left(I^{n-i} x_\tau \right)(t) = d^{\mathrm{T}} W_\tau(M_\tau, t). \tag{25}$$

This follows from (14), (24) and (5) by choosing $x(\tau)$ such that

$$v^T \tilde{Q}_\tau \left[1, t, \dots, \frac{t^{n-1}}{(n-1)!} \right]^{\mathrm{T}} = d^{\mathrm{T}} M_\tau \theta(t).$$

By solving (25) for $v^{\mathrm{T}} x_\tau(t)$ (most easily accomplished by writing (25) in state-variable form with the state vector

$$z(t)^{\mathrm{T}} = \left[v^{\mathrm{T}} I^n x_\tau, v^{\mathrm{T}} I^{n-1} x_\tau, \dots, v^{\mathrm{T}} I x_\tau \right]$$

and noting that $z(0)=0$), it can be directly shown that (25) implies

$$\| v^{\mathrm{T}} x_\tau \|_T < \alpha \| d^{\mathrm{T}} W_\tau(M_\tau, t) \|_T \tag{26}$$

where α depends on A and T only.

On the other hand, from (23) and (24) we have

$$\| v^{\mathrm{T}} x_\tau \|_T^2 > \varepsilon_1 | v \|^2 \geq \varepsilon_1 | Q^{-1} |^2. \tag{27}$$

(The matrix norm used here is $|M|^2 = \lambda_{\max}(M^*M)$.) Defining

$$\varepsilon_2 = \varepsilon_1 \min_{C_n} \frac{|Q^{-1}|^2}{\alpha^2} \tag{28}$$

and using (26) and (27) we can conclude that

$$\| d^{\mathrm{T}} W_\tau(M_\tau, t) \|_T^2 > \varepsilon_2$$

where we note that ε_2 is independent of both d and τ so that 'richness of order n' by Definition 3 is established. □

Remark 3. The sufficiency part of the theorem can be viewed locally as well as globally. Namely, if the input u satisfies the richness condition (8) over any finite interval, the state x will satisfy the PE requirement (2) over that same interval.

The following corollary is a direct consequence of Theorem 1.

Corollary 1. *Let $C_n \subset \mathrm{SC}_n$ be any set of n systems such that the characteristic polynomials of their A's are mutually coprime.*

(i) *If the input $u \in L_e^1(\mathbb{C})$ is persistently exciting for every system in C_n it is* PESC_n.

(ii) *For every* PESC_n *input a uniform (over all systems in* SC_n*) excitation period T_0 exists. Furthermore, T_0 can be chosen as the maximal excitation period over the n systems in C_n.*

Proof. (i) Recall that in the proof of Theorem 1 we have shown that if u is persistently exciting for C_n it is rich of order n. Hence, by Theorem 1, it is PESC_n.

(ii) In the proof of Theorem 1 it was shown that T_0 can be taken as the T which is used in the definition of richness of order n, as well as the maximal excitation period over the systems in C_n. □

Two important consequences of the foregoing discussion are summarized in the following two remarks.

Remark 4. Corollary 1 implies that any PE result which is limited to stable systems (a number of which appear in the literature) does in fact apply to the whole class SC_n.

Remark 5. For identification algorithms such as the 'least squares with covariance resetting', the resetting period must be larger than the excitation period of the state to ensure (exponential) parameter convergence (see e.g. [4]). Corollary 1 (ii) verifies that this period can be chosen independently of the (unknown) system parameters and is therefore of immediate practical importance in parameter identification.

4. A filtered version of the basic result

The results of Section 3 can now be generalized into a filtered version which complements the results in [1].

Define $D = \mathrm{d}/\mathrm{d}t$, and let $d(D)/n(D)$ denote a zero-initial-conditions filter which can also be interpreted as an operator from L_e^2 to L_e^2. We also assume throughout most of this section that the interval $[\tau, \tau+T]$ is fixed.

We rewrite now W in (8) as

$$W_\tau(M, t) = V_\tau(t) + M\theta(t) = \begin{bmatrix} D^{-1}u_\tau \\ \vdots \\ D^{-n}u_\tau \end{bmatrix} + M \begin{bmatrix} 1 \\ \vdots \\ t^{n-1} \end{bmatrix}. \tag{29}$$

Note that V_τ can be considered as the state vector of the filter $1/D^n$ when realized in controller form and $\theta(t)$ is associated with the modes of this filter.

Replace now the above filter with a general filter of the form $1/q(D)$, where

$$q(D) = \prod_{i=1}^{n} (D - q_i). \tag{30}$$

Assuming (for demonstrative purposes only) that the q_i's are distinct, the analogue of (29) will be

$$\tilde{W}_\tau(M, \tau) = \tilde{V}_\tau(t) + M\tilde{\theta}(t) = \begin{bmatrix} \dfrac{D^{n-1}}{q(D)} u_\tau \\ \vdots \\ \dfrac{1}{q(D)} u_\tau \end{bmatrix} + M \begin{bmatrix} \mathrm{e}^{q_1 t} \\ \vdots \\ \mathrm{e}^{q_n t} \end{bmatrix}. \tag{31}$$

The following lemma now shows that $\tilde{W}$ can replace W in the characterization of rich inputs, and thus proves Theorem 3 below.

Lemma 2. *For any $u \in L_e^1$ the following are equivalent:*

(i) $\exists \varepsilon_1 > 0$ *such that* $\forall M \in \mathbb{C}^{n\times n}$,

$$J(M) = \int_0^T W_\tau(M, t) W_\tau^*(M, t)\, \mathrm{d}t > \varepsilon_1 I.$$

(ii) $\exists \tilde{\varepsilon}_1 > 0$ *such that* $\forall M \in \mathbb{C}^{n\times n}$,

$$\tilde{J}(M) = \int_0^T \tilde{W}_\tau(M, t) \tilde{W}_t^*(M, t)\, \mathrm{d}t > \tilde{\varepsilon}_1 I.$$

Proof. Direct calculation shows that

$$\frac{D^n}{q(D)} W_\tau(M, t) = \tilde{W}_\tau(M, t) \quad \forall M.$$

Consider now the filter $D^n/q(D)$ as a linear operator H over the Hilbert space L^2 of functions $f : [0, T] \to \mathbb{C}$. H is bounded and has an inverse $H^{-1} = q(D)/D^n$ which is also a bounded operator. Therefore $\exists \alpha > 0$ such that $\forall f \in L^2$, $\| Hf \| \geq \alpha \| f \|$.

Let S and $\tilde{S}$ be the subspaces spanned by θ and $\tilde{\theta}$ respectively. By direct calculation $H\{S\} \subset \tilde{S}$ and $H^{-1}\{\tilde{S}\} \subset S$ which means that $H\{S\} = \tilde{S}$. The matrix M_τ which 'minimizes' (i) removes from W_τ all components which lie in S. Similarly, $\tilde{M}_\tau$ which 'minimizes' (ii) removes from $\tilde{W}_\tau$ all components which lie in $\tilde{S}$ (see Lemma 1 and Remark 2). Therefore $HW_\tau(M_\tau, \cdot) = \tilde{W}_t(\tilde{M}_\tau, \cdot)$ so that $\forall d \in \mathbb{C}^n$,

$$\| d^{\mathrm{T}} \tilde{W}_\tau(\tilde{M}_\tau, \cdot) \|_T \geq \alpha \| d^{\mathrm{T}} W_\tau(M_\tau, \cdot) \|_T$$

which gives (i) $\Rightarrow$ (ii). The opposite direction can be established similarly. $\square$

Theorem 2. *Let $q(D)$ be any monic polynomial of degree n. Then the input $u \in L_e^1(\mathbb{C})$ is* PESC$_n$ *iff* $\exists \varepsilon > 0$, $t > 0$ *such that* $\forall \tau > 0$,

$$\int_0^T \tilde{W}_\tau(\tilde{M}_\tau, t)\tilde{W}_\tau(\tilde{M}_\tau, t)^* \, dt > \varepsilon I,$$

where $\tilde{W}_\tau$ *is defined by* (31) *and* $\tilde{M}_\tau$ *by* (9) (*with* θ *and* V_τ *replaced by* $\tilde{\theta}$ *and* $\tilde{V}_\tau$ *respectively*).

The proof of this theorem follows directly from Theorem 1 and Lemma 2.

An interesting result can be established if we constrain the input to belong to a class $E(\mathbb{C})$ of inputs which when applied to an asymptotically stable system in SC$_n$ results in a bounded state. This class includes bounded inputs as well as stationary inputs (see [4]).

Corollary 2. *Let $u \in E(\mathbb{C}) \subset L_e^1(\mathbb{C})$. Then u is* PESC$_n$ *iff it generates a* PE *state in some asymptotically stable system in* SC$_n$ *with some fixed initial conditions.*

Proof. Necessity follows from the fact that the system belongs to SC$_n$. To prove sufficiency let (A, b) be the system for which u generates PE state and denote $q(s) = \text{char.poly.}(A)$. W.l.o.g. assume (A, b) to be in controller canonical form, hence its state is given by

$$x(t+\tau) = \begin{bmatrix} \dfrac{D^{n-1}}{q(D)} u_\tau(t) \\ \vdots \\ \dfrac{1}{q(D)} u_\tau(t) \end{bmatrix} + e^{At}x(\tau) = \tilde{V}_\tau + N(\tau)\tilde{\theta}(t) \quad \forall t \geq 0, \tag{32}$$

where $\tilde{\theta}(t)$ contains the modes of $q(D)$ in a way similar to (31) (repeated modes may induce powers of t which will not change basic desired properties), and $N(\tau) \in \mathbb{C}^{n \times n}$ is bounded since the system is stable and the state is bounded. It can also be shown that $\tilde{M}_\tau$, defined by (9) (with the replacement of θ and V_τ by $\tilde{\theta}$ and $\tilde{V}_\tau$), is uniformly bounded for every τ and $T > \delta > 0$.

Now since it is assumed that the input u is persistently exciting for the given system we have for its state

$$\int_0^T x_\tau(t)x_\tau(t)^* \, dt > \varepsilon I \quad \forall \tau > 0 \tag{33}$$

for some $\varepsilon > 0$ and $T > 0$.

On the other hand, combining (31) and (32) leads to

$$\tilde{W}_\tau(M, t) = \tilde{V}_\tau(t) + \tilde{M}_\tau\tilde{\theta}(t) = x_\tau(t) + \left[\tilde{M}_\tau - N(\tau)\right]\tilde{\theta}(t) \tag{34}$$

where $\tilde{\theta}(t) \to 0$ exponentially as $t \to \infty$. Hence, there exists a $T_1 \geq T$ such that [1]

$$\int_0^{T_1} \tilde{W}_\tau(\tilde{M}_\tau, t)\tilde{W}_\tau(\tilde{M}_\tau, t)^* \, dt \geq \int_{T_1 - T}^{T_1} \tilde{W}_\tau(\tilde{M}_\tau, t)\tilde{W}_\tau(\tilde{M}_\tau, t)^* \, dt \geq \tfrac{1}{2}\int_{T_1 - T}^{T_1} x_\tau(t)x_\tau(t)^* \, dt$$

so that using (33) we can conclude

$$\int_0^{T_1} \tilde{W}_\tau(\tilde{M}_\tau, t)\tilde{W}_\tau(\tilde{M}_\tau, t)^* \, dt > \tfrac{1}{2}\varepsilon I. \tag{35}$$

Thus, u is rich of order n and by Theorem 1 it is PESC$_n$. □

[1] We note that T_1 depends on the choice of initial state for the system but all we are concerned with is the existence of a T_1 as required.

5. Persistent excitation of the output

Here we generalize the results to the case where the signal vector of interest is the output of a system rather than its state. Thus, consider the class SOR(n, m) of output-reachable systems, specified by

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0, \qquad y(t) = Cx(t), \qquad A \in \mathbb{C}^{n \times n}, \quad b \in \mathbb{C}^n, \quad C \in \mathbb{C}^{m \times n}. \tag{36}$$

Definitions of PE for this class are the same as Definitions 1 and 2 with $y(t)$ replacing $x(t)$.

It can be readily observed that by substituting $Q_c = CQ$ for Q in the sufficiency proof for Theorem 1 the following can be established. (Note that Q_c has a full rank for an output-reachable system.)

Corollary 3. *An input $u \in L_e^1(\mathbb{C})$ is persistently exciting for the class* SOR(n, m) *if it is rich of order n.*

To argue that the above condition is not necessary, consider the class SOR(n, 1) and choose in this class a controllable system. Assume now that for a particular input the state of the system is not sufficiently rich in a 'weak' sense. Namely, $\lambda_{\min}(X_T(\tau)) \to 0$ as $\tau \to \infty$ for every finite T, where $X_T(\tau) = \int_0^T x_\tau(t) x_\tau(t)^* \, dt$. It is, however, still possible if $X_T(\tau)$ is not bounded that this 'near singularity' of $X_T(\tau)$ will not be reflected in any *fixed* direction C. Hence, $\lambda_{\min} C X_T(\tau) C^* \to 0$ is *not* implied. Thus the following adjustment is required:

Theorem 3. *The input $u \in L_e^1(\mathbb{C})$ is persistently exciting for the class* SOR(n, m) *iff $\forall C \in \mathbb{C}^{m \times n}$, $\exists \varepsilon_3 > 0$, $T > 0$ such that*

$$C J_\tau(M_\tau, T) C^* \geqslant \varepsilon_3 \lambda_{\min}(CC^*) I \quad \forall \tau \geqslant 0.$$

The proof of this theorem is similar to the proof of Theorem 1.

6. Conclusions

Necessary and sufficient conditions for an input, not necessarily bounded, to be persistently exciting for continuous systems are provided. Hence, the class of persistently exciting inputs is completely characterized. These conditions, while not always easy to apply directly, provide a basis for some fundamental results and insights for the problem.

The results presented here for a single-input system could probably be extended to the multi-input case using similar methods but that is the subject of further research.

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