Parameterization and orbital angular momentum of anisotropic dislocations

Yoav Y. Schechner

Faculty of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

Joseph Shamir

Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel

Waves that have an isotropic intensity distribution about the propagation axis may carry well-defined, quantized orbital angular momentum. The angular momentum is nonzero in the presence of screw dislocations. It is shown that these dislocations can be represented by a complete orthogonal set of functions (such as the Gauss−Laguerre set, which is valid for the paraxial approximation). Representing anisotropic dislocations by a superposition of isotropic ones provides easy derivation of various quantities related to the angular momentum, such as its expectation value and the uncertainty. Concentrating on dislocations that are solutions of the Laplace equation, we propose two natural and convenient ways of parameterization that bring forth their isotropic components as well as the geometrical properties of their phase map. © 1996 Optical Society of America

1. INTRODUCTION

More than two decades ago, Nye and Berry[1] introduced the concept of wave dislocations. In recent years the issue has regained interest. Optical dislocations were produced by use of computer-generated holograms,[2,3] and their propagation in free space was studied. Intensive research on dislocations, such as investigations of spatiotemporal phenomena in laser cavities, has been done.[4−6] At the same time, while paraxial laser beams were treated with Laguerre−Gaussian modes, it was found that each such mode has a well-defined orbital angular momentum.[7−14]

In the present paper we investigate some aspects of anisotropic dislocations with emphasis on the derivation of angular momentum. We concentrate on linear solutions of the Laplace equation. Such dislocations are dominant in speckle fields and were parameterized in previous studies.[15,16] However, those parameters are inconvenient for physical interpretation, and we propose more convenient parameterizations that are well suited for analysis and interpretation. Most of the conclusions are valid for screw dislocations that occur in any scalar wave, and are not restricted to optics.

In Section 2 a background is provided on some basic principles and characteristics of dislocations, on the Gauss−Laguerre function set, and on the relevance of the Laplace equation. In Section 3 we discuss anisotropic dislocations that obey the Laplace equation and various parameterizations for them. The orbital angular momentum of waves with anisotropic dislocations is treated in Section 4 with the derivation of their expectation values and standard deviations. Conclusions are provided in Section 5.

2. BACKGROUND ON SCREW DISLOCATIONS

A screw dislocation is a line in space, along the direction of a wave propagation, where the phase is not determined and the amplitude vanishes. The phase changes by $2\pi m$ for a revolution around the dislocation, where m can be any integer, positive or negative.[1] Viewed in space, a cophasal surface (wave front) traces a helicoid (as its name screw dislocation implies)[2] with a phase pitch of $2\pi|m|$. The handedness of the helix (dislocation) is determined by the sign of m.

A. Paraxial Basis
In free space the three-dimensional monochromatic wave field is uniquely defined by its values over one plane. Let the scalar wave be

$$F(\mathbf{r}, t) = F_0 u(\mathbf{r})\exp[i(kz - \omega t)], \tag{1}$$

where ω is the frequency and k is the wave number.

The paraxial wave equation, for propagation along the z axis (x and y are the transversal coordinates),

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(\mathbf{r}) = -2ik\frac{\partial}{\partial z}u(\mathbf{r}), \tag{2}$$

is analogous to the Schrödinger equation of a free particle in two dimensions if the z coordinate in Eq. (2) is replaced by the time variable.[10] This analogy allows us to use the formalism of quantum mechanics to analyze wave functions. As we shall see, the problem reduces to two dimensions, and the z dependence is given by a propagator **U** such that

$$|u(z)\rangle = \mathbf{U}(z)|u(z=0)\rangle. \tag{3}$$

Considering this two-dimensional function, it is preferable to look for a complete orthogonal basis that matches the present application. Expansion in a set of Gauss–Laguerre functions appears to be most appropriate because

1. These wave functions are typical laser beams and hence are frequently encountered in laboratory experiments.

2. These functions are stationary in the z coordinate (in analogy to stationary states in time, in quantum mechanics), except for a z-dependent scale change and a radial phase factor. So some properties, such as the shape and the azimuthal dependence, remain invariant.

3. The functions have a definite dislocation of order m.

The wave functions are given by

$$u_{n,m}(\mathbf{r}) = \langle \mathbf{r}\,|\,n,\ m\rangle = C_{n,m}G(\rho, z)\exp[-in\psi(z)]$$
$$\times \left[\sqrt{2}\,\rho/w(z)\right]^{|m|}L_{(n-|m|)/2}^{|m|}[2\rho^2/w^2(z)]\exp(im\phi), \tag{4}$$

where $\mathbf{r} = (\rho, \phi, z)$,

$$w^2(z) = w_0^2[1 + (z/z_0)^2], \tag{5a}$$

$$z_0 = \pi w_0^2/\lambda \tag{5b}$$

is the Rayleigh length, and

$$\psi(z) = \arctan(z/z_0) \tag{6}$$

is the Gouy phase. The function

$$G(\rho, z) = \frac{w_0}{w(z)}\exp[-\rho^2/w^2(z)]\exp[ik\rho^2/2R(z)]$$
$$\times \exp[-i\psi(z)] \tag{7}$$

is common to all modes and comprises the radial Gaussian envelope of the beam, a Gouy phase, and a radial quadratic phase factor. The radius of curvature of the wave front is given by

$$R(z) = z[1 + (z_0/z)^2], \tag{8}$$

$L_{(n-|m|)/2}^{|m|}$ are the generalized Laguerre polynomials, and the integers n and m obey the relation

$$n = |m|, |m| + 2, |m| + 4, \ldots. \tag{9}$$

$C_{n,m}$ are constants, chosen so that

$$\iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\,|u_{n,m}|^2 = 1. \tag{10}$$

Inspecting Eq. (4), we see that the only azimuthal dependence is through the term $\exp(im\phi)$. Hence it has one azimuthal degree of freedom (m). Because the phase changes linearly with the azimuth for all ϕ and the intensity pattern is isotropic, these dislocations (with $|m| \neq 0$) will be referred to as isotropic dislocations. The radial dependence is the product of $G(\rho, z)$ and a polynomial with powers $|m|, \ldots, n$ of ρ. Hence the radial dependence has its own degree of freedom (n).

B. Laplace Equation

Let us consider solutions to the case in which

$$u(\mathbf{r}) = u(x, y). \tag{11}$$

For the function of Eq. (1) to obey the wave equation, $u(x, y)$ should be a solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{12}$$

These solutions describe plane waves and stable non-diffracting beams that propagate without change along the z axis, but they need an infinite aperture. However, for a finite aperture, Eq. (12) can be satisfied approximately, for example, near the waist of Gaussian beams as in Eq. (4); that is, $|z| < z_0$ and $\rho \ll w_0$. In the case in which N dislocations appear, as in a speckle field, it was shown[15] that the approximation is valid for $|z| < (z_0/N)$. Moreover, this invariance is demonstrated at the far field of a paraxial wave, except for a z-proportional scale factor. One can see this from Eqs. (4)–(8) by taking $z \gg z_0$, $k\rho^2$, performing the transformation $(\rho/z) \to \tilde{\rho}$, and replacing the coordinates in Eq. (12) by scaled ones ($\tilde{x} = x/z$ and $\tilde{y} = y/z$).

If a dislocation is embedded in a slowly varying envelope, the field of the dislocation is dominant near its core. Using these assumptions, we can approximate the $|n, \pm 1\rangle$ modes of Eq. (4) by

$$u(x, y) \sim (x \pm iy). \tag{13}$$

3. ANISOTROPIC $|m| = 1$ DISLOCATIONS

In this section we broaden the $|m| = 1$ family of solutions of the Laplace equation discussed in Subsection 2.B to include anisotropic functions. For this purpose it is useful to parameterize these functions. We introduce a new parameterization that provides better insight and overcomes some drawbacks of an earlier one.[15,16]

A. Solutions in a Linear Approximation

Let u in Eq. (1) be written as

$$u(x, y) = u_R(x, y) + iu_I(x, y). \tag{14}$$

The functions $u_R(x, y)$ and $u_I(x, y)$ describe two-dimensional surfaces that may have complicated shapes. Constant values of these functions are projected as curves on the (x, y) plane. Some of these lines may be described by the relations

$$u_R(x, y) = 0, \tag{15a}$$

$$u_I(x, y) = 0, \tag{15b}$$

and then dislocations may appear at points of their intersections. For the linear case there will be only one intersection, and, for simplicity, we shall take this as the origin [if Eqs. (15a) and (15b) coincide, we obtain a dark fringe].

Following Refs. 15 and 16, we approximate the surfaces $u_R(x, y)$ and $u_I(x, y)$ in the close vicinity of the dislocation by their tangent planes:

$$u_R = a_{Rx}x + a_{Ry}y, \tag{16a}$$

$$u_I = a_{Ix}x + a_{Iy}y. \tag{16b}$$

We shall not deal here with higher-order dislocations ($|m| > 1$), for which this approximation is invalid. The (a_{Rx}, a_{Ry}, a_{Ix}, a_{Iy}) parameters do not explicitly give information about the properties of the dislocation. Therefore we propose two ways of parameterization that describe its geometrical and physical properties.

B. Geometrical Parameterization

The difference between the phase maps of anisotropic and isotropic dislocations (which are linear solutions of the Laplace equation) is the crowding of cophasal lines along a specific line in the (x, y) plane, as if the phase map were stretched along this line (assumed, for simplicity, to go through the origin). Thus the two most important characteristics of the anisotropic dislocation are the amount of stretching and its direction. The other two characteristics are the global phase and the mean intensity, as in any wave. Each of the above four intuitive and functionally independent characteristics will be represented by a parameter.

Let us assume a stretch along the x axis, defined by s:

$$s_x = \cos \epsilon, \qquad s_y = \sin \epsilon, \, s = s_y/s_x, \qquad (17)$$

$$u(x, y) = I^{1/2} \exp(i\gamma)\sqrt{2}\,(x \cos \epsilon + iy \sin \epsilon). \qquad (18)$$

I and γ are the mean intensity and global-phase parameters, respectively. Without losing generality, we impose $\cos \epsilon > 0$, for uniqueness.

Transformation to the coordinate system

$$X = x\sqrt{2} \cos \epsilon, \qquad Y = y\sqrt{2} \sin \epsilon \qquad (19)$$

leads to a form similar to relation (13):

$$u(x, y) = I^{1/2} \exp(i\gamma)(X + iY)$$
$$= I^{1/2} \exp(i\gamma)M(\rho, \phi)\exp[i\Theta(\rho, \phi)]. \qquad (20)$$

The local phase is

$$\Theta = \arctan(Y/X) = \arctan(\tan \epsilon \tan \phi), \qquad (21)$$

where (ρ, ϕ) are the standard cylindrical coordinates of the (x, y) plane. The quadrant of Θ is the same as the quadrant of $\phi \, \text{sgn} \, \epsilon$. The modulus M obeys

$$M^2 = X^2 + Y^2 = \rho^2(1 + \cos 2\epsilon \cos 2\phi). \qquad (22)$$

If the stretching is along a line that makes an angle ϕ' with the x axis, the above relations are retained after the transformation
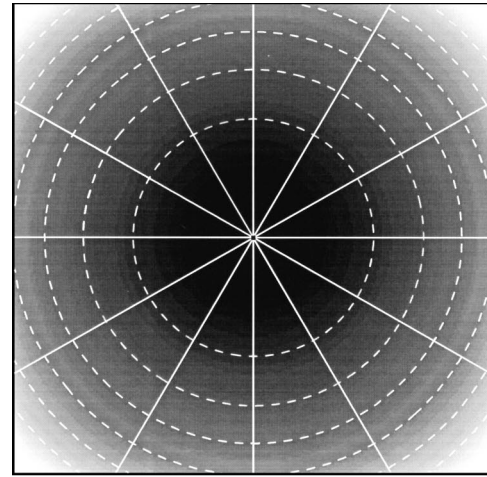
$$\phi \to \phi - \phi'. \qquad (23)$$

As can be seen from Eq. (22), constant-intensity contours form ellipses, with their major (minor) axes along (perpendicular to) the stretch for $|s| > 1$ (see Fig. 1).

For uniqueness, some of the parameters can be bounded in a finite region. For the interval
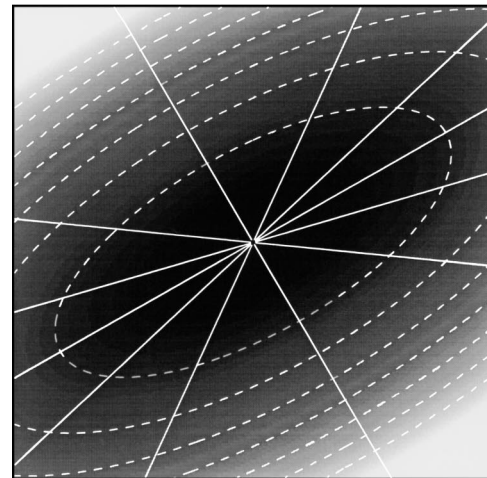
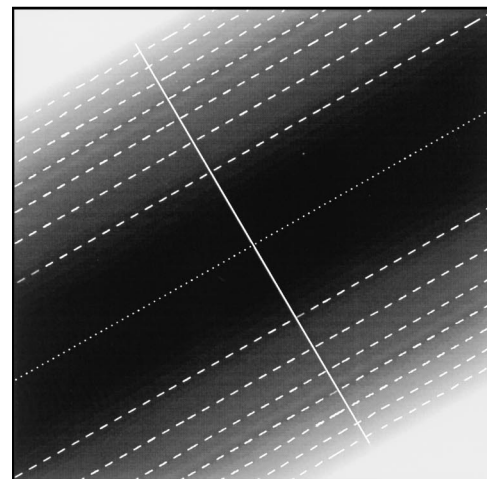$$-\pi/2 < \phi' \leq \pi/2 \qquad (24)$$

we may have

$$(\pi/4) \leq |\epsilon| < (\pi/2),$$
$$1 \leq |s| < \infty \qquad (25a)$$

(a)

(b)

(c)

Fig. 1. Intensity images and phase maps of dislocation, all with the same I and γ parameters. Solid contours indicate cophasal lines, each of which is different from its consecutive line by $\pi/6$. Dashed contours indicate isointensity lines. (a) Isotropic dislocation. (b) Anisotropic dislocation with $\epsilon = 3\pi/8$ (that is, $\eta = -\pi/8$) and $\phi' = \pi/6$. (c) Annihilation with $\epsilon = \pi/2$ ($\eta = -\pi/4$) and the same ϕ' as in (b). Note that the phase changes by π as a solid line crosses the dislocation. In the dark fringe (dotted) the cophasal lines are infinitely crowded, whereas in each bright lobe the phase is uniform.
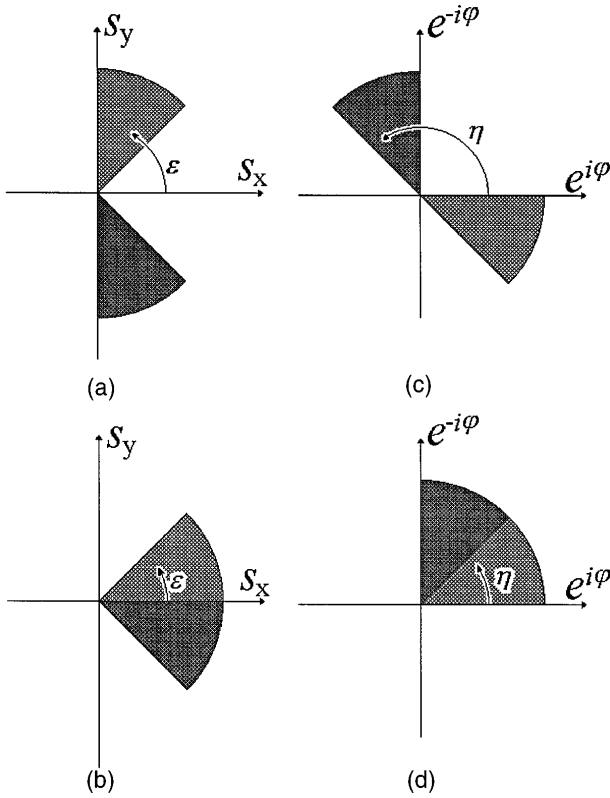
(a)   (c)

(b)   (d)

Fig. 2.   Ranges of the ϵ and η parameters.   Lighter-shaded regions indicate $m = 1$ dislocations; darker-shaded regions indicate $m = -1$ ones.   (a) and (b) correspond to the intervals defined by inequalities (25); (c) and (d) are the corresponding intervals for η, given by inequalities (33).

or, alternatively,

$$(\pi/4) \geq |\epsilon| > 0,$$
$$1 \geq |s| > 0 \tag{25b}$$

(see Fig. 2).   For an isotropic dislocation, $|\epsilon| = \pi/4$, corresponding to $|s| = 1$.

Transformation from the $(\epsilon, \phi', I, \gamma)$ parameterization to $(a_{Rx}, a_{Ry}, a_{Ix}, a_{Iy})$ is accomplished by the relations

$$a_{Rx} = \sqrt{2I}\,(\cos \phi' \cos \gamma \cos \epsilon + \sin \phi' \sin \gamma \sin \epsilon),$$
$$\tag{26a}$$
$$a_{Ry} = \sqrt{2I}\,(\sin \phi' \cos \gamma \cos \epsilon - \cos \phi' \sin \gamma \sin \epsilon),$$
$$\tag{26b}$$
$$a_{Ix} = \sqrt{2I}\,(\cos \phi' \sin \gamma \cos \epsilon - \sin \phi' \cos \gamma \sin \epsilon),$$
$$\tag{26c}$$
$$a_{Iy} = \sqrt{2I}\,(\sin \phi' \sin \gamma \cos \epsilon + \cos \phi' \cos \gamma \sin \epsilon).$$
$$\tag{26d}$$

The inverse transformation of Eqs. (26) is obtained by the following equations:

$$I = (1/2)(a_{Rx}^2 + a_{Ry}^2 + a_{Ix}^2 + a_{Iy}^2), \tag{27}$$

$$\kappa = \arcsin\left(\frac{a_{Rx}a_{Iy} - a_{Ix}a_{Ry}}{I}\right), \qquad \epsilon = \kappa/2, \tag{28}$$

where, for choice (25a), κ is in the range $(\pi/2) \leq |\kappa| \leq \pi$, and for choice (25b) $|\kappa| \leq \pi/2$.

For $|\epsilon| \neq (\pi/4)$, we define $-\pi \leq \tau \leq \pi$:

$$\tau = \arcsin\left(\frac{a_{Rx}a_{Ry} + a_{Ix}a_{Iy}}{I \cos(2\epsilon)}\right)$$
$$= \arccos\left[\frac{a_{Rx}^2 + a_{Ix}^2 - a_{Ry}^2 - a_{Iy}^2}{2I \cos(2\epsilon)}\right], \tag{29a}$$
$$\phi' = \tau/2, \tag{29b}$$

and we define $\phi' = 0$ for $|\epsilon| = (\pi/4)$.   For $\phi' \neq 0$,

$$\gamma = \arccos\left[\frac{a_{Ry} \cos \epsilon + a_{Ix} \sin \epsilon}{\sqrt{2I}\,\cos(2\epsilon)\sin \phi'}\right]$$
$$= \arcsin\left[\frac{a_{Iy} \cos \epsilon - a_{Rx} \sin \epsilon}{\sqrt{2I}\,\cos(2\epsilon)\sin \phi'}\right]. \tag{30a}$$

If $\phi' = 0$

$$\gamma = \arccos\left(\frac{a_{Rx}}{\sqrt{2I}\,\cos \epsilon}\right) = \arcsin\left(\frac{a_{Ix}}{\sqrt{2I}\,\cos \epsilon}\right)$$
$$= \arccos\left(\frac{a_{Iy}}{\sqrt{2I}\,\sin \epsilon}\right) = \arcsin\left(\frac{-a_{Ry}}{\sqrt{2I}\,\sin \epsilon}\right), \tag{30b}$$

where the last two definitions in Eq. (30b) should be used in the case $\cos \epsilon \to 0$ and the first two if $\sin \epsilon \to 0$.

C.   Decomposition into Isotropic Modes
The radial dependence of the amplitude of Eq. (20) is proportional to ρ, and in Eq. (4) it is proportional to $\rho^{|m|}$ (when the approximation assumptions of Subsection 2.B are used).   Therefore, when the close vicinity of the dislocation described by Eq. (20) is considered, the $|m| = 1$ isotropic modes are dominant.   We define the parameter η, so that

$$\cos \eta = \frac{1}{\sqrt{2}}\,(\cos \epsilon + \sin \epsilon),$$
$$\sin \eta = \frac{1}{\sqrt{2}}\,(\cos \epsilon - \sin \epsilon). \tag{31}$$

If $\phi' = 0$, Eq. (18) can be written as

$$u(x, y) = I^{1/2} \exp(i\gamma)[(x + iy)\cos \eta + (x - iy)\sin \eta]$$
$$= I^{1/2} \exp(i\gamma)\rho[\exp(i\phi)\cos \eta + \exp(-i\phi)\sin \eta]; \tag{32}$$

otherwise this relation is retained after the transformation $\phi \to \phi - \phi'$.   Hence an anisotropic dislocation that is a linear solution of Eq. (12) is merely a superposition of $|m = \pm 1\rangle$ modes, which are not fully annihilated.   For choice (25a),

$$(3\pi/4) \geq \eta \geq (\pi/2),$$
$$(-\pi/4) \leq \eta \leq 0 \tag{33a}$$

(see Fig. 2).   The left-hand equalities of relations (33a) are cases of annihilation, and the right-hand equalities are cases of isotropic dislocations.   For choice (25b),

$$(\pi/2) \geq \eta \geq 0. \tag{33b}$$

We can thus use (η, ϕ', I, γ) to parameterize the dislocations.   The isotropic intensity wave functions such as

Eq. (4) span a space in which the $|m = \pm 1\rangle$ modes span a subspace. In this subspace we represent u as

$$|u\rangle = I^{1/2} \exp(i\gamma) \, | \, \phi', \, \eta\rangle, \qquad (34)$$

$$|\phi', \, \eta\rangle = \begin{bmatrix} \exp(-i\phi')\cos \, \eta \\ \exp(i\phi')\sin \, \eta \end{bmatrix}. \qquad (35)$$

The upper (lower) component of Eq. (35) is the projection onto the $|m = 1\rangle$ ($|m = -1\rangle$) subspace.

In concluding this section it is worth noting that Eq. (35) represents a more general state than does Eq. (32) since it does not contain the radial dependence explicitly. Therefore all $|n, |m| = 1\rangle$ modes (with $n = 1, 3, 5, \ldots$) of Eq. (4) can be used. These functions, however, obey the Laplace equation only close to the dislocation. Moreover, modes with $|m| > 1$ can also be components of u, but their contribution near the z axis is negligible. If only one dislocation is involved, the parameters I and γ can be absorbed into F_0 of Eq. (1).

D. Notes on a Previous Parameterization
A different parameterization, which uses another four-dimensional vector, was proposed in Refs. 15 and 16. An adaptive coordinate system (x', y'), having its origin at the dislocation, was defined, with the unit vectors $\hat{x}'$ and $\hat{y}'$ perpendicular to the lines defined in Eqs. (15a) and (15b), respectively. In this system

$$u(x', y') = a(x' + i\alpha y'), \qquad (36)$$

where a was termed in Ref. 16 "the amplitude factor" and α the "anisotropy." Two additional parameters were "the skewness," σ, the deviation from orthogonality of $\hat{y}'$ relative to $\hat{x}'$, and ρ, the angle between $\hat{x}$ and $\hat{x}'$, both measured counterclockwise. In the (x, y) system we obtain

$$u(x, y) = a\{x \cos \, \rho + y \sin \, \rho + i\alpha[-x \sin(\rho + \sigma)$$
$$+ \, y \cos(\rho + \sigma)]\}. \qquad (37)$$

We found the $(a, \alpha, \rho, \sigma)$ parameterization inconvenient for several reasons. Consider the wave

$$u(x, y) = b(x + i\beta y). \qquad (38)$$

Comparing Eq. (38) with Eq. (36), we see that $x' = x$, $y' = y$, $a = b$, $\alpha = \beta$, $\rho = 0$, and $\sigma = 0$. If we add to Eq. (38) a global phase of $\pi/2$, that is, multiply the equation by i, we obtain

$$u_{\pi/2}(x, y) = b\beta\left(-y + \frac{i}{\beta} x\right). \qquad (39)$$

The adaptive coordinates are $x'' = -y$ and $y'' = x$, so

$$u_{\pi/2}(x'', y'') = b\beta\left(x'' + i \frac{1}{\beta} y''\right). \qquad (40)$$

Hence, the anisotropy parameter becomes $\alpha = 1/\beta$ and the amplitude factor becomes $a = b\beta$.

Alternatively, if we add a global phase of $\pi/4$, we obtain

$$|\alpha| = 1, \qquad \sigma = \arctan(\beta) - \arctan(1/\beta),$$
$$\rho = -\arctan(\beta). \qquad (41)$$

It is obvious that the α and σ parameters depend strongly on the global phase. Thus these parameters are inappropriate to describe fundamental wave characteristics. One can also see that the parameters are not functionally independent from each other.

This parameterization is also problematic from the mathematical point of view. For example, dislocations with $\alpha = \beta$ and $\alpha = 1/\beta$ have the same phase map, except for a coordinate system rotation. Therefore, in a random field, we expect to find such dislocations with equal probability. So the interval $0 < \alpha \le 1$ has the same weight as $1 < \alpha < \infty$ in $(a, \alpha, \rho, \sigma)$ space. Hence infinitesimal intervals of α in different places in parameter space do not have the same weight. The α coordinate is not homogeneously dense. Incautiously using this space in integrals or differentials may lead to improper conclusions. In Ref. 16 the probability-density function (pdf) of α was calculated and was found to be

$$\text{pdf}(\alpha) = \frac{|\alpha|}{(1 + \alpha^2)^2} . \qquad (42)$$

Since this pdf has a maximum (when differentiated with respect to α) at $|\alpha| = 3^{-0.5}$, it was implied that this is the most probable value of $|\alpha|$. This may indicate a tendency for increasing the phase gradient (crowding cophasal lines) near the imaginary axis, implying a preferred global phase (changing the global phase changes α, as noted above).

4. ORBITAL ANGULAR MOMENTUM OF DISLOCATIONS

In quantum mechanics, if we are in a state $|\psi\rangle$, a rotation of the system yields a state $|\psi'\rangle = \mathbf{R}|\psi\rangle$, where $\mathbf{R}$ is the rotation operator. If the rotation is about the z axis, through an infinitesimal angle $d\alpha$, then

$$\mathbf{R} = \mathbf{R}_z(d\alpha). \qquad (43)$$

The definition of the angular-momentum operator $\mathbf{L}_z$ is through the connection[17]

$$\mathbf{R}_z(d\alpha) = \mathbf{1} - (i/\hbar)\mathbf{L}_z d\alpha . \qquad (44)$$

Rotation through a finite angle α about the z axis is obtained by consecutive infinitesimal rotations:

$$\mathbf{R}_z(\alpha) = \exp[-(i/\hbar)\mathbf{L}_z\alpha]. \qquad (45)$$

The orbital angular momentum in the $|r\rangle$ representation is

$$\langle\mathbf{r}|\mathbf{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle\mathbf{r}|. \qquad (46)$$

A. Isotropic Dislocations
In general, isotropic dislocations have the form

$$\langle\mathbf{r} \, | \, u\rangle = f(\rho, z)\exp(im\phi). \qquad (47)$$

Rotation by an angle α of this function about the z axis yields

$$\langle \mathbf{r} \,|\, u' \rangle = f(\rho, z)\exp[im(\phi - \alpha)] = \langle \mathbf{r} \,|\exp(-im\alpha)\,|\, u \rangle. \tag{48}$$

Using Eq. (45), one can write

$$\langle \mathbf{r} \,|\, u' \rangle = \langle \mathbf{r} \,|\, \mathbf{R}_z(\alpha) \,|\, u \rangle. \tag{49}$$

Comparing Eqs. (48) and (49), we conclude that $|u\rangle$ is an eigenstate of the rotation operator and hence also of the angular-momentum operator. Thus it carries a well-defined angular momentum.

The full time dependence of the state in Eq. (1) can be written in the form

$$\langle \mathbf{r} \,|\, F \rangle = F_0 \exp[i(kz - \omega t)]\langle \mathbf{r} \,|\, u \rangle. \tag{50}$$

If $|u\rangle$ is given by Eq. (47), then, using Eq. (46), we obtain

$$\langle \mathbf{L}_z \rangle = \langle F \,|\, \mathbf{L}_z \,|\, F \rangle = |F_0|^2 \hbar m \langle u \,|\, u \rangle. \tag{51}$$

The energy of the state is

$$\langle \mathbf{E} \rangle = \iint \mathrm{d}x\mathrm{d}y F^*(\mathbf{r}, t)\left(i\hbar \frac{\partial}{\partial t} \right) F(\mathbf{r}, t) = |F_0|\,\hbar\,\omega\langle u \,|\, u \rangle. \tag{52}$$

Therefore the ratio between the orbital angular momentum carried by the wave to the energy that it carries is

$$\frac{\langle \mathbf{L}_z \rangle}{\langle \mathbf{E} \rangle} = \frac{m}{\omega}. \tag{53}$$

This simple relation is valid classically as well as quantum mechanically, since $\hbar$ does not appear in it and we did not need to assume normalization of $|u\rangle$ or $|F\rangle$. In fact, the more we enter into the classical wave (not particle) characteristics regime (ω decreases), the more the above ratio increases. In the quantum-mechanical regime, as in the case of photons, the state is normalized. Thus we see from Eq. (51) that every photon in a Gauss–Laguerre beam carries the quantized $\hbar m$ orbital angular momentum.

To conclude this subsection several remarks are in order.

(a)   The above derivation is based on fundamental physical considerations (i.e., connection between rotations and angular momentum; the symmetry of wave functions) and not on the specific relations that describe angular momentum for a specific field. This indicates that the conclusions are valid to any scalar field having a wave function satisfying Eq. (47).

(b)   We deal with orbital angular momentum that originates from rotation of the scalar wave function about the axis defined by the dislocation. The states considered are not eigenstates of rotation operators around a different axis. This is in contrast to spin angular momentum, which arises from the vector character of some wave fields (such as electromagnetism).

(c)   The operator of the total angular momentum of the electromagnetic field can be separated into spin and orbital parts, which are observables but are not proper angular-momentum operators of the photon.[12,13]   Therefore the separation of the total angular momentum into spin and orbital angular momentum is generally impossible. This separation, however, becomes possible in the paraxial approximation. Moreover, it has been shown[14] that, even for nonparaxial beams, the angular momentum is related exactly to the orbital dependence if the projection of the polarization vector onto the (x, y) plane is uniform and linear. Assuming that at least one of these conditions is satisfied, our scalar analysis is valid for electromagnetic waves.

(d)   The phenomenon of angular momentum of Gauss–Laguerre laser beams was investigated intensively by a group of researchers in the Huygens Laboratory.[7–14] They give more formal but longer proofs for some of the conclusions in this subsection.

B.   Anisotropic Dislocations

In the $||m| = 1\rangle$ subspace the orbital angular-momentum operator may be written as

$$\mathbf{L}_z = \hbar\sigma_z, \tag{54a}$$

with

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{54b}$$

$$\mathbf{L}_z{}^2 = \hbar^2\sigma_z{}^2 = \hbar^2\mathbf{1}. \tag{55}$$

For the state equations (34) and (35) the expectation value of the orbital angular momentum is

$$\begin{aligned} L_z = \langle \mathbf{L}_z \rangle &= \langle \phi', \eta \,|\mathbf{L}_z|\, \phi', \eta \rangle \\ &= \hbar[\exp(i\phi')\cos\eta \quad \exp(-i\phi')\sin\eta]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \exp(-i\phi')\cos\eta \\ \exp(i\phi')\sin\eta \end{bmatrix} = \hbar\,\cos(2\eta) \\ &= \hbar\,\sin(2\epsilon). \end{aligned} \tag{56}$$

The expectation value of $\mathbf{L}_z{}^2$ is calculated in a similar manner from Eq. (55). Hence the rms deviation is

$$\Delta L_z = (\langle \mathbf{L}_z{}^2 \rangle - \langle \mathbf{L}_z \rangle^2)^{1/2} = \hbar \,|\sin(2\eta)| = \hbar \,|\cos(2\epsilon)| \tag{57}$$

per photon. For pure $|m| = 1$ isotropic dislocations Eq. (56) agrees with Eq. (53), and the uncertainty of the orbital angular momentum is zero; hence the orbital angular momentum is well defined. At total annihilation $[|\epsilon| = \pi/2$ for choice (25a) or $|\epsilon| = 0$ for choice (25b)], the expectation value is zero but the uncertainty is maximal.

If N photons that constitute the anisotropic field are absorbed, orbital angular momentum is transferred to the detector. Although the average result of such measurements is N times the result of Eq. (56), the rms deviation is obtained from Eq. (57):

$$\Delta L_z = \hbar\sqrt{N}\,|\cos(2\epsilon)| = \hbar\sqrt{N}\,|\sin(2\eta)|. \tag{58}$$

Consequently the determination of the anisotropy from a measurement of the orbital angular momentum has an uncertainty that becomes negligible in the classical limit.

By differentiating Eq. (56) and then using Eq. (58), we obtain

$$\Delta\eta = \frac{\Delta L_z}{2N\hbar\,|\sin(2\eta)|} = \frac{1}{2\sqrt{N}} = \frac{1}{2}\sqrt{\hbar\omega/E}, \qquad (59)$$

where E is the absorbed energy. The same expression is obtained for $\Delta\epsilon$.

The directional uncertainty demonstrates the opposite behavior from the angular-momentum uncertainty (Fig. 1). For the isotropic wave function it is maximal, since there is no preferred direction. As the dislocation becomes anisotropic, the isointensity curves, which can be calculated by Eq. (22), become more elliptical, and the pattern becomes darker (brighter) along their major (minor) axes. At total annihilation [as can be seen by inspection of Eq. (32)], a dark fringe exists along the phase map stretch, and the radiation is concentrated in two lobes perpendicular to it.

5. SUMMARY

In this paper we have parameterized anisotropic wave dislocations, which are linear solutions of the Laplace equation, in ways that give better insight and convenience. The first method is geometrical, as it describes the stretching of the phase map of an anisotropic dislocation relative to an isotropic one. The second method takes advantage of the fact that anisotropic dislocations can be decomposed into isotropic dislocations. This is a special case, since any wave can be decomposed into a complete isotropic function set, of which the Gauss–Laguerre set seems to be the most appropriate in the paraxial approximation.

Isotropic scalar wave functions carry well-defined orbital angular momentum, which is nonzero for dislocations. The mentioned decomposition thus permits straightforward calculation of various quantities related to orbital angular momentum, such as the expectation value and the rms deviation. Finally we obtained the relations between these quantities, and the dislocations' parameters.

We focused our treatment on the neighborhood of the stable and most frequent dislocations[1,2,15,16] as opposed to those with $|m| > 1$. The present research can be further extended by use of the mentioned complete orthogonal set.

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