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Convergence Rates of Consensus Algorithms in Stochastic Networks

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Abstract— **We study the convergence rate of average consensus algorithms in networks with stochastic communication failures. We show how the system dynamics can be modeled by a discrete-time linear system with multiplicative random coefficients. This formulation captures many types of random networks including networks with links failures, node failures, and network partitions. With this formulation, we use first-order spectral perturbation analysis to analyze the mean-square convergence rate under various network conditions. Our analysis reveals that in large networks, the effect of communication failures on the convergence rate is similar to the effect of changing the weight assigned to the communication links. We also show that in large networks, when the probability of communication failure is small, correlation in communication failures plays a negligible role in the convergence rate of the algorithm.**

I. INTRODUCTION

In the distributed average consensus problem, each node in the network has an initial value, and the goal is for the nodes to reach consensus at the average of these values using only communication between neighbors. Average consensus algorithms have a wide variety of applications including distributed optimization [1], sensor fusion [2], load balancing [3], [4], and vehicle formation control [5], [6], [7]. A topic of recent interest is the performance of consensus algorithms in networks with unreliable communication. Messages may be lost due to wireless network interference or hardware or software failures, and it is desirable to understand the effect of these communication failures on the algorithm convergence behavior.

Several works have presented necessary and sufficient conditions for convergence in many variants of stochastic networks [8], [9], [10], [11], [12], [7], [13]. However, the study of convergence rates in such networks is a less mature area. Our aim in this paper is to analytically quantify the relationship between the probability of communication failure and the mean-square *convergence rate* of consensus algorithms in stochastic networks.

We consider a class of stochastic networks that can be represented by a discrete-time linear system with multiplicative random coefficients. This formulation can be used to model a wide variety of network scenarios including link failures, node failures, and network partitions. With this formulation, it is possible to compute the convergence rate for any network in a simulation-free manner. This formulation is also amenable to spectral perturbation analysis to produce a closed form expression for the convergence rate when the probability of link failure is small. Through this analysis, we are able to draw several interesting conclusions about the effects of communication failures on convergence rates in large networks.

- The probability of link failure plays a similar role in algorithm performance as the weights assigned to the to the links. An increase in the probability of failure causes a comparable change in convergence rate to a decrease in edge weight.
- Correlation in communication failures is of little significance in the convergence rate. In large networks, while the probability of link failure has a large effect on convergence rate, the effect is essentially the same, regardless of whether links fail independently or simultaneously.

Details on these observations are given in Section IV.

In the remainder of this section, we briefly review related work on consensus in stochastic networks and then describe our problem setting. In Section II, we formalize the dynamics of the consensus algorithm in a general stochastic network, and we illustrate how this general form can be used to model several network scenarios. In Section III, we present perturbation analysis for the general model and the specific cases of link failures, node failures, and network partitions. Section IV contains two examples that highlight the relationship between the convergence rate, the probability of link failures, and the correlation of these failures. Finally, we conclude in Section V.

A. *Related Work*

The distributed consensus has been studied in switching networks, where it has been shown that the convergence rate of the algorithm can be defined in terms of the number of bounded intervals for which the network satisfies some connectivity property [14], [15]. The recent work by Fagnani and Zampieri has addressed the convergence rate of consensus algorithms in directed stochastic networks [16], [17]. This work uses a similar method to model such networks to the one presented in this work. The authors derive lower and upper bounds for the convergence rate of the consensus algorithm, and present analytical results for these bounds for several special networks.

In our previous work [18], we addressed the problem of convergence rates in networks where links fail independently, and presented analysis for the convergence rate in tori networks. In this paper, we generalize the model presented in [18] to include both correlated and uncorrelated link failures. This generalization also encompasses some of the scenarios

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studied in [17]. With our general model, we derive tight upper bounds for the convergence rate as well as analytical expressions for the convergence rate for any undirected, stochastic network.

B. Preliminaries

We consider a network of n identical nodes with a fixed, underlying communication structure. The network is modeled by an undirected, connected communication graph $G = (V, E)$, where V is the set of nodes, with $|V| = n$, and E is the set of bi-directional communication links, with $|E| = m$. Let $A = [a_{ij}]$ be the weighted adjacency matrix of G, with $a_{ij} > 0$ if and only if $(i, j) \in E$. We also assume that for every i, $\sum_{j=1}^{n} a_{ij} < 1$. The *weighted Laplacian matrix* of G is defined as $L := D - A$, where $D = [d_{ij}]$ is a diagonal matrix with each $d_{ii} = \sum_{j=1}^{n} a_{ij}$. It is well-known that L has an an eigenvalue of 0 with eigenvector $\mathbf{1}$ (the vector of all ones). Since G is a connected graph, all other eigenvalues of L are strictly positive.

Each node has an initial state $x_i(0)$, and the objective is for all nodes to reach consensus at the average of the node states, $x_{ave} := \frac{1}{n}\sum_{i\in V} x_i(t)$, using only communication with neighboring nodes. Communication takes place in discrete rounds, and in each round, some subset of links $E(t) \subseteq E$ may fail, meaning no communication takes place across these links in either direction. A link that does not fail in a round is called *active*. Let $\mathcal{N}_i(t)$ denote the neighbor set of node i in round t, *i.e.* the set of nodes that share an active link with node i in round t. Each agent updates its state according to the following nearest neighbor averaging rule,

$$x_i(t+1) = x_i(t) + \sum_{j\in\mathcal{N}_i(t)} l_{ij}(x_j(t) - x_i(t)).$$

The dynamics of the entire system are

$$x(t+1) = (I - L(t))x(t), \tag{1}$$

where $L(t)$ is the weighted Laplacian of the graph of active links at time t.

We measure how far the system is from consensus by the deviation from average vector, with each component defined as

$$\tilde{x}_i(t) := x_i(t) - x_{ave}.$$

Let Q be the orthogonal projection operator $Q := I - \frac{1}{n}\mathbf{1}\mathbf{1}^*$. The vector $\tilde{x}(t)$ is the projection of $x(t)$ onto the subspace orthogonal to $\mathbf{span}(\mathbf{1})$,

$$\tilde{x}(t) = Qx(t).$$

In a network with a fixed topology, the dynamics of x and $\tilde{x}$ are given by

$$x(t+1) = (I - L)x(t) \tag{2}$$
$$\tilde{x}(t+1) = (Q - L)\tilde{x}(t). \tag{3}$$

The vector x converges asymptotically to x_{ave}, or equivalently, the vector $\tilde{x}(0)$ converges asymptotically to 0 at a rate that depends on the second largest eigenvalue of $I - L$ by magnitude, equivalently, the largest eigenvalue of $Q - L$ by magnitude. Let $\underline{\lambda}$ be the eigenvalue value of L that maximizes the expression

$$\max(1 - \lambda_2(L), \lambda_n(L) - 1),$$

where $\lambda_2(L)$ and $\lambda_n(L)$ are the second smallest and the largest eigenvalues of L, respectively. The eigenvalue of $Q - L$ with largest magnitude is then $1 - \underline{\lambda}$.

In the case of stochastic networks, we are interested in the second order statistics of the system (1). We say that the system *converges in mean square* if for all $i \in V$,

$$\lim_{t\to\infty} \mathbf{E}\left\{\tilde{x}_i(t)^2\right\} = 0.$$

Our aim in this work is to characterize how quickly the system converges in mean square, a concept that will be made precise the next section. In particular, we are interested in how the convergence rate scales with the size of the network. For a given set of edge weights, if, the maximum node degree of the network is fixed, $\lambda_2(L)$ will grow closer and closer to one as the network size increases. However $\lambda_n(L)$ is bounded above by twice the maximum node degree (see [19]). Therefore, the interesting case, in terms of asymptotic analysis of convergence rates, is that where $\underline{\lambda} = 1 - \lambda_2(L)$. We assume that the edge weights are such that this is true.

II. PROBLEM FORMULATION

We study the consensus algorithm in class of stochastic networks which can be modeled by a linear recursion with multiplicative noise. The general form of the dynamics of such systems is

$$x(t+1) = \left(I - \sum_j \delta_j(t)L_{E_j}\right)x(t), \tag{4}$$

where $\delta_j(t)$ are (not necessarily independent) Bernoulli random variables with

$$\delta_j(t) := \begin{cases} 0 & \text{with probability } p_j \\ 1 & \text{with probability } 1 - p_j. \end{cases}$$

Each L_{E_j} is a weighted Laplacian of the graph $G_j = (V, E_j)$, where $E_j \subseteq E$ is a subset of edges that fail simultaneously with probability p_j. For compactness, we also use L_j to mean L_{E_j}. We assume that $\bigcup_i E_i = E$ and $E_j \cap E_k = \emptyset$ for every j and k. We note that if $E_j \cap E_k \neq \emptyset$, there is always an equivalent expression with non-intersecting edge sets. When $\delta_j(t) = 0$, the links in E_j have failed; when $\delta_j(t) = 1$, the links are active. For example, if the network has one unreliable link (i, j) that fails with probability p in each round, the dynamics of the system are given by the following,

$$x(t+1) = \left(I - L_{E\setminus\{(i,j)\}} - \delta_{(i,j)}(t)L_{(i,j)}\right)x(t),$$

where $\delta_{(i,j)} = 0$ with probability p and 1 with probability $1 - p$.

This general model (4) encompasses a variety of network scenarios, including the three that we describe below.

Link Failures. In this scenario, each edge $(i, j) \in E$ fails independently with probability $p_{(i,j)}$ in each round. When a link fails, no communication takes place across that link in either direction. The recursion that describes the network dynamics is

$$x(t+1) = \left(I - \sum_{(i,j)\in E} \delta_{(i,j)}(t) L_{(i,j)} \right) x(t), \qquad (5)$$

where $\delta_{(i,j)}$ are Bernoulli random variables with $\delta_{(i,j)} = 0$ with probability $p_{(i,j)}$, meaning the link has failed, and $\delta_{(i,j)} = 1$ with probability $1 - p_{(i,j)}$, meaning the link is active. Here, $L_{(i,j)}$ is the weighted Laplacian of the graph $G_{(i,j)} = (V, \{(i,j)\})$, the graph the containing the single edge (i,j).

Node Failures. We assume that each node i fails with independent probability p_i. When a node fails, no communication takes place across the links adjacent to that node in either direction. The recursion on x is the same as given in (5). However the definition of $\delta_{(i,j)}$ is different. In the node failure case, link (i,j) fails when *either* node i or node j fails. To capture this property, we define each $\delta_{(i,j)}$ as

$$\delta_{(i,j)}(t) = \mu_i(t)\mu_j(t)$$

where μ_i, $i \in V$ are Bernoulli random variables with $\mu_i(t) = 0$ with probability p_i, meaning node i failed, and $\mu_i(t) = 1$ with probability $1-p_i$, meaning node i is active. When either μ_i or μ_j (or both) is 0, $\delta_{(i,j)}$ is 0, and link (i,j) is not used in that round of the algorithm.

Network Partitions. In a network partition, some subset of edges fail simultaneously, isolating a group of nodes from the rest of network. Communication still takes place between the nodes within the partition. Let $P \subseteq E$ be the set of edges that fail simultaneously with probability p. The dynamics of the system are given by

$$x(t+1) = \left(I - \left(L_{E\setminus P} + \delta_P(t) L_P \right)\right) x(t), \qquad (6)$$

with $\delta_P(t) = 0$ with probability p and 1 with probability $1 - p$. This formulation can easily be extended to include multiple, uncorrelated partitions by adding additional multiplicative noise terms.

In each of these scenarios, at each time step, $I - L(t)$ is a doubly stochastic matrix. It has been shown that if $|\lambda_2 (\mathbf{E}\{I - L(t)\})| < 1$, then the system converges in mean square to x_{ave} (see [13]). Our aim is to quantify the relationship between the probability of link failure and the *mean-square convergence rate* of consensus algorithms of the general form (4). As we will show, the per node variance of the deviation from average decays geometrically.

Problem Statement: *Consider a distributed consensus algorithm over a connected, undirected graph with stochastic communication failures as modeled by the system with multiplicative noise (4). For a given failure probability p, determine the worst-case rate (over all initial conditions, over all nodes) at which the variance of the deviation from average* $\boldsymbol{E}\left\{\tilde{x}_i(t)^2\right\}, i \in V$, *converges to* 0 *as* $t \to \infty$.

In the following section, we present analysis of the convergence rate of the general system (4) and give analytical forms for the convergence rate for each of the scenarios listed above.

III. CONVERGENCE RATE ANALYSIS

In order to study the evolution of $\tilde{x}$, we define the autocorrelation matrix of $\tilde{x}$,

$$\begin{aligned} M(t) &:= \mathbf{E}\{\tilde{x}(t)\tilde{x}^*(t)\} \\ &= \mathbf{E}\{Qx(t)x^*(t)Q\} \end{aligned}$$

Each diagonal entry M_{ii} is the variance at node i, $\tilde{x}_i(t)^2$. For the general system (4), the recursion of the autocorrelation matrix is

$$\begin{aligned} M(t+1) &= (Q - \mathbf{E}\{L\})\; M(t)\; (Q - \mathbf{E}\{L\}) \\ &\quad + \sum_i \sum_j \mathbf{cov}(\delta_i, \delta_j)\, L_i\; M(t)\; L_j, \end{aligned} \qquad (7)$$

where $\mathbf{E}\{L\} = \sum_i (1 - p_i) L_i$. The derivation of this recursion can be found in [20].

We define the matrix-valued operator $\mathcal{L}(\cdot)$,

$$\begin{aligned} \mathcal{L}(X) &:= (Q - \mathbf{E}\{L\})\; X\; (Q - \mathbf{E}\{L\}) \\ &\quad + \sum_i \sum_j \mathbf{cov}(\delta_i, \delta_j)\, L_i\; X\; L_j, \end{aligned} \qquad (8)$$

and note that (7) is equivalent to $M(t+1) = \mathcal{L}(M(t))$. The mean-square convergence rate of (4) depends on the spectral radius of $\mathcal{L}$ as follows (see [18], [20] for proof).

Theorem 3.1: The system (7) converges in mean square if and only if $\rho(\mathcal{L}) < 1$, and the per node variance decays, in worst case, as

$$\tilde{x}_i(t)^2 = \rho(\mathcal{L})^t \tilde{x}_i(0)^2.$$

While the $\mathcal{L}$ operator is Lyapunov-like, there is no straightforward way to derive the eigenvalues and eigenvectors of $\mathcal{L}$ from those of of the matrices $Q, \mathbf{E}\{L\}$, and L_i. One can compute these eigenvalues using an $n^2 \times n^2$ matrix representation of $\mathcal{L}$,

$$\mathcal{L} = (Q - \mathbf{E}\{L\}) \otimes (Q - \mathbf{E}\{L\}) + \sum_i \sum_j \mathbf{cov}(\delta_i, \delta_j)\, L_i \otimes L_i.$$

However, this computation may be prohibitively expensive when considering large networks, and so it is desirable to find an analytical expression for the convergence rate. For the case where links have identical probability of failure, an analytical form can be obtained using spectral perturbation analysis [21]. We first briefly review this technique and then present convergence rate analysis for the link failure, node failure, and network partition scenarios.

A. Perturbation Analysis

Let $\mathcal{L}(X, p)$ be a matrix-valued function of a scalar $p \in \mathbb{R}$ and a matrix X of the form

$$\mathcal{L}(X, p) = \mathcal{L}_0(X) + p\mathcal{L}_1(X) + p^2\mathcal{L}_2(X) + \dots. \qquad (9)$$

For $p \in (-\epsilon, \epsilon)$, each eigenvalue of $\mathcal{L}(\cdot, p)$ is also a function of p, denoted $\gamma(p)$, and this function is well-defined and analytic for $p \in (-\epsilon, \epsilon)$. $\gamma(p)$ has the power series expansion

$$\gamma(p) = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \dots$$

where γ_0 is an eigenvalue of $\mathcal{L}_0$ with eigenmatrix V. To find the value of $\gamma(p)$ up to first order in p, we must calculate the value of γ_1. This is given by

$$\gamma_1 \;=\; \frac{\langle V, \mathcal{L}_1(V)\rangle}{\langle V, V\rangle}.$$

Note that the inner product on matrices is

$$\langle X, Y\rangle \;:=\; \mathbf{tr}\left(X^* Y\right).$$

Therefore, if we can write $\mathcal{L}$ in the form (9), the value of the spectral radius of $\mathcal{L}$, up to first order in p, is

$$\rho(\mathcal{L}) \;=\; \rho(\mathcal{L}_0) + \frac{\langle V, \mathcal{L}_1(V)\rangle}{\langle V, V\rangle} p + O(p^2),$$

where V is the eigenmatrix of $\mathcal{L}_0$ corresponding to $\rho(\mathcal{L}_0)$.

B. Network Scenarios

We now show how spectral perturbation analysis can be used to obtain the mean-square convergence rate with respect to small, uniform probability of failure.

Link Failures. For the system with stochastic link failures (5), if each link has probability p of failing, the matrix-valued operator $\mathcal{L}$ is

$$\begin{aligned}\mathcal{L}(X,p) \quad &= \quad (Q-(1-p)L)X(Q-(1-p)L) \\ &\quad\; +(p-p^2)\sum_{(i,j)\in E} L_{(i,j)} X L_{(i,j)}.\end{aligned}$$

Note that $\mathbf{var}\left(\delta_i\right) = p - p^2$ for all $i \in V$ and $\mathbf{cov}\left(\delta_i, \delta_j\right) = 0$ if $i \neq j$.

An equivalent representation of this operator is

$$\mathcal{L}(X,p) \;=\; \mathcal{L}_0(X) \;+\; p\;\mathcal{L}_1(X) \;+\; p^2\;\mathcal{L}_2(X),$$

where

$$\begin{aligned}\mathcal{L}_0(X) \;&=\; (Q-L)X(Q-L) \\ \mathcal{L}_1(X) \;&=\; LX(Q-L) + (Q-L)XL \\ &\qquad + \sum_{(i,j)\in E} L_{(i,j)} X L_{(i,j)} \\ \mathcal{L}_2(X) \;&=\; LXL - \sum_{(i,j)\in E} L_{(i,j)} X L_{(i,j)}.\end{aligned}$$

The spectral radius of $\mathcal{L}_0$ is $\rho(\mathcal{L}_0) = (1-\underline{\lambda})^2$. Let v be the eigenvector of L with eigenvalue $\underline{\lambda}$ with $\|v\| = 1$. Then vv^* is an eigenmatrix of $\mathcal{L}_0$ associated with $\rho(\mathcal{L}_0)$. The value of γ_1 is then

$$\begin{aligned}\gamma_1 \;&=\; \mathbf{tr}\left(vv^*\mathcal{L}_1(vv^*)\right) \\ &=\; \mathbf{tr}\left(vv^*Lvv^*(Q-L) \;+\; vv^*(Q-L)vv^*L\right) \\ &\qquad + \mathbf{tr}\left(vv^* \sum_{(i,j)\in E} L_{(i,j)}\; vv^* L_{(i,j)}\right) \\ &=\; 2\underline{\lambda}(1-\underline{\lambda}) \;+\; \sum_{(i,j)\in E} (v^* L_{(i,j)} v)^2 \\ &=\; 2\underline{\lambda}(1-\underline{\lambda}) \;+\; \sum_{(i,j)\in E} \left(l_{ij}(v_i - v_j)^2\right)^2 \qquad (10)\end{aligned}$$

The summation term can be bounded as follows [19],

$$\sum_{(i,j)\in E} \left(l_{ij}(v_i-v_j)^2\right)^2 \leq \left(\sum_{(i,j)\in E} l_{ij}(v_i-v_j)^2\right)^2 = \underline{\lambda}^2.$$

Therefore, for a system with stochastic link failures the spectral radius of $\mathcal{L}$ is

$$\rho(\mathcal{L}) = 1 - (1-p)2\underline{\lambda} + \underline{\lambda}^2 + pO(\underline{\lambda}^2) + O(p^2). \qquad (11)$$

Node Failures. For a system with stochastic node failures, we note that

$$\mathbf{cov}\left(\delta_i, \delta_j\right) = \begin{cases} (1-p)^2 & \text{if } i = j \\ (1-p)^3 & \text{if } (i,j) \in E \\ (1-p)^4 & \text{otherwise.} \end{cases}$$

The matrix-valued operator associated with this system is

$$\begin{aligned}&\mathcal{L}(X,p) = (Q-(1-p)^2L)X(Q-(1-p)^2L) \\ &+ \left((1-p)^2 - (1-p)^4\right) \sum_{(i,j)\in E} L_{(i,j)} X L_{(i,j)} \\ &+ \left((1-p)^3 - (1-p)^4\right) \sum_{(i,j)\in E} \sum_{k\in\mathcal{N}_i\setminus\{j\}} L_{(i,j)} X L_{(i,k)} \\ &+ \left((1-p)^3 - (1-p)^4\right) \sum_{(i,j)\in E} \sum_{k\in\mathcal{N}_j\setminus\{i\}} L_{(i,j)} X L_{(j,k)}.\end{aligned}$$

In order to perform the first-order perturbation analysis, we must first rewrite this operator in the form (9). The operators of this expansion that are relevant to the first-order perturbation are $\mathcal{L}_0$ and $\mathcal{L}_1$. They are

$$\begin{aligned}\mathcal{L}_o(X) \;&=\; (Q-L)X(Q-L) \\ \mathcal{L}_1(X) \;&=\; 2LX(Q-L) + 2(Q-L)XQ \\ &\qquad + \sum_{(i,j)\in E} L_{(i,j)} X L_{(i,j)} \\ &\qquad + \sum_{(i,j)\in E} \sum_{k\in\mathcal{N}_i} L_{(i,j)} X L_{(i,k)} \\ &\qquad + \sum_{(i,j)\in E} \sum_{k\in\mathcal{N}_j} L_{(i,j)} X L_{(j,k)}.\end{aligned}$$

From this, we compute the value of γ_1, where, as in the link failure case, $(\underline{\lambda}, v)$ is the eigenvalue, eigenvector pair of L with $\|v\| = 1$.

$$\begin{aligned}\gamma_1 \;&=\; 4\underline{\lambda}(1-\underline{\lambda}) + \sum_{(i,j)\in E} \left(l_{ij}(v_i - v_j)\right)^2 \qquad (12) \\ &\qquad + \sum_{(i,j)\in E} \sum_{k\in\mathcal{N}_i} l_{ij} l_{ik} (v_i - v_j)^2 (v_i - v_k)^2 \quad (13) \\ &\qquad + \sum_{(i,j)\in E} \sum_{k\in\mathcal{N}_j} l_{ij} l_{jk} (v_i - v_j)^2 (v_j - v_k)^2 \quad (14)\end{aligned}$$

The summation term in (12) is identical to that in (10) and thus can be bounded above by $\underline{\lambda}^2$. The summation terms in (13) and (14) combined are bounded above by

$$\sum_{(i,j)\in E} \sum_{(r,s)\in E} l_{ij} l_{rs} (v_i - v_j)^2 (v_r - v_s)^2 = \underline{\lambda}^2.$$

This gives the following expression for the spectral radius of $\mathcal{L}$ in a network with stochastic node failures,

$$\rho(\mathcal{L}) = 1 - (1-2p)2\underline{\lambda} + \underline{\lambda}^2 + pO(\underline{\lambda}^2) + O(p^2). \quad (15)$$

The difference between this expression and $\rho(\mathcal{L})$ for the link failure scenario given in (11) is in the first order term in p and $\underline{\lambda}$. We reason that this term is twice as large in the node failure scenario because each link has a probability of failure that depends on its the probabilities of failure of its two adjacent nodes.

Network Partitions. For a network where a subset of edges $P \subseteq E$ fails simultaneously with probability p, the corresponding operator $\mathcal{L}$ is

$$\begin{aligned}\mathcal{L}(X,p) &= (Q-(L-pL_P))\ X\ (Q-(L-pL_P)) \\ &\quad + (p-p^2)L_P\ X\ L_P \\ &= \mathcal{L}_0(X) + p\mathcal{L}_1(X),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}_0(X) &= (Q-L)X(Q-L) \\ \mathcal{L}_1(X) &= L_P X(Q-L) + (Q-L)XL_P + L_P X L_P.\end{aligned}$$

Again, let v be an eigenvector of L with eigenvalue of $\underline{\lambda}$. The value of γ_1 for the network partition model is

$$\begin{aligned}\gamma_1 &= 2(1-\underline{\lambda})v^* L_P v + (v^* L_P v)^2 \\ &= 2(1-\underline{\lambda}) \sum_{(i,j)\in E_P} l_{ij}(v_i - v_j)^2 \\ &\quad + \sum_{(i,j)\in E_P} {l_{ij}}^2 (v_i - v_j)^4.\end{aligned}$$

The spectral radius of $\mathcal{L}$ is then

$$\begin{aligned}\rho(\mathcal{L}) &= (1-\underline{\lambda})^2 + p\left(2(1-\underline{\lambda})\sum_{(i,j)\in E_P} l_{ij}(v_i-v_j)^2\right. \\ &\quad \left. + \sum_{(i,j)\in E_P} {l_{ij}}^2(v_i-v_j)^4\right) + O(p^2). \quad (16)\end{aligned}$$

Since v is an eigenvector of L but not an eigenvector of L_P, in general, there is no straightforward simplification for the above expression. However, for certain types of networks, the entries of v have a closed form, and thus, for these networks, it is possible to find an analytical bound for $\mathcal{L}$.

As an example, we consider a d-dimensional torus network over $\mathbb{Z}_N^d$ where $N^d = n$. We assume that all edge weights l_{ij} are equal to β, with $0 < \beta < 1/(2d)$. In this case, L is a circulant operator and its eigenvalues and eigenvectors can be determined analytically using the Discrete Fourier Transform. The eigenvalue $\underline{\lambda}$ has multiplicity d with d independent eigenvectors. Without loss of generality, we consider one of these eigenvectors. The difference between each component of the eigenvector is (see [20])

$$v_j - v_k = \frac{1}{\sqrt{n}}\left(e^{-i\frac{2\pi}{N}j_1} - e^{-i\frac{2\pi}{N}k_1}\right).$$

If nodes j and k share an edge in the first dimension, then $|j_1 - k_1| = 1$. Otherwise, $|j_1 - k_1| = 0$. Therefore, for all $(j,k) \in E$,

$$v_j - v_k \leq \frac{1}{\sqrt{n}}\frac{2\pi}{n^{1/d}}|j_1 - k_1|.$$

With this fact, γ_1 can be bounded as follows,

$$\gamma_1 \leq 2\beta(1-\underline{\lambda})\left(\frac{|P|}{n}\right)\left(\frac{4\pi}{n^{2/d}}\right) + \beta^2\left(\frac{|P|}{n^2}\right)\left(\frac{2\pi^2}{3n^{4/d}}\right).$$

For a d-dimensional torus, the eigenvalue $1-\underline{\lambda}$ also has an analytical expression,

$$1 - \underline{\lambda} = 1 - \frac{2\beta\pi^2}{n^{2/d}} + O\left(\frac{1}{n^{4/d}}\right).$$

Substituting these expressions into the expression for the spectral radius of $\mathcal{L}$ given in (16), we obtain the spectral radius for $\mathcal{L}$ for a network partition in a torus,

$$\begin{aligned}\rho(\mathcal{L}) &= 1 - \left(1 - \frac{|P|}{n}p\right)2\beta\frac{4\pi^2}{n^{2/d}} \\ &\quad + \left(1 + \frac{|P|}{n^2}p\right)O\left(\frac{1}{n^{4/d}}\right) + O(p^2).\end{aligned}$$

This expression shows that, for large networks, the fraction of edges that fail in the partition is directly related to the decrease in the decay factor.

IV. SPECIAL CASES

In this section, we highlight several special formulations of the consensus algorithm in stochastic networks that provide a deeper intuition into the effects of the probability of link failure on the algorithm performance.

A. Stochastic Link Failures with Uniform Edge Weights

One well-studied version of the consensus algorithm is that in which every edge has identical weight β. In a network with no communication failures, the dynamics of this system are

$$x(t+1) = (I - \beta\bar{L})x(t),$$

where $\bar{L}$ is the unweighted Laplacian matrix. This system converges if the graph is connected and $\beta < \frac{1}{\Delta}$, where Δ is the maximum node degree of the graph. Let $\lambda_2(\bar{L})$ denote second smallest eigenvalue of $\bar{L}$. We assume that $\beta < \frac{1}{2\Delta}$ and therefore, the second largest eigenvalue of $(I - \beta\bar{L})$ is $1 - \beta\lambda_2(\bar{L})$.

In a network where links fail with independent probability p and where all edge weights are equal to β, the expression (11) for the spectral radius of the $\mathcal{L}$ operator can be simplified as follows,

$$\begin{aligned}\rho(\mathcal{L}) &= 1 - 2(1-p)\beta\lambda_2(\bar{L}) + \beta^2\lambda_2(\bar{L})^2 \\ &\quad +\ p\ O(\beta^2\lambda_2(\bar{L})^2) + O(p^2).\end{aligned}$$

Note that, if links do not fail, the spectral radius of $\mathcal{L}$ is precisely

$$\rho(\mathcal{L}) = 1 - 2\beta\lambda_2(\bar{L}) + \beta^2\lambda_2(\bar{L})^2.$$

Therefore, for large networks (networks where $\lambda_2(\bar{L})$ is small), link failures decrease $\rho(\mathcal{L})$ by a factor of $1-p$. In fact, a change in p has the same effect as a change in the edge weight β. In an informal sense, one can consider $(1-p)\beta$ to be the *expected edge weight* over time.

B. Correlated vs. Uncorrelated Failures

We consider an extreme version of the network partition scenario where all links fail simultaneously with probability p. In other words, $E_P = E$. We call this a *blinking network* because all links "blink" on and off. Note that in this scenario, $\mathbf{E}\{L(t)\} = (1-p)L$ which is precisely the same as $\mathbf{E}\{L(t)\}$ for the stochastic link failure scenario. However, one expects the convergence rate in a network where links fail simultaneously to differ from the rate in a network where links fail independently. This leads to the question of the significance of correlation in link failures on the convergence rate of the consensus algorithm.

To investigate this question further, we examine the spectral perturbation of $\mathcal{L}$ for the blinking network model,

$$\begin{aligned}\mathcal{L}(X) &= (Q-(1-p)L)X(Q-(1-p)L) \\ &\quad + (p-p^2)LXL \\ &= (1-p)(Q-L)\ X\ (Q-L) + pQ\ X\ Q.\end{aligned}$$

Since the Q and L matrices have the same eigenvectors, $\rho(\mathcal{L})$ for the blinking network can be derived as follows,

$$\begin{aligned}\rho(\mathcal{L}) &= (1-p)\rho\left((Q-L)\ X\ (Q-L)\right) + p\ \rho(Q\ X\ Q) \\ &= (1-p)(1-\underline{\lambda})^2 + p \\ &= 1-(1-p)2\underline{\lambda} + (1-p)\underline{\lambda}^2.\end{aligned}$$

Recall that, for a network with stochastic link failures,

$$\rho(\mathcal{L}) = 1-(1-p)2\underline{\lambda} + \underline{\lambda}^2 + pO(\underline{\lambda}^2) + O(p^2).$$

We observe that the difference between $\rho(\mathcal{L})$ for the blinking network and $\rho(\mathcal{L})$ for the network with link failures occurs in the term of order $\underline{\lambda}^2$. Therefore, for small p and large networks, where $\underline{\lambda}^2$ is negligible with respect to $\underline{\lambda}$, the convergence behavior for the two network models is the same. This result indicates that in such networks, the probability with which links fail plays an important role in the convergence rate, but the correlation of these failures does not.

V. CONCLUSION

We have presented a discrete-time linear system representation of consensus algorithms in stochastic networks in which communication failures are modeled by multiplicative noise terms. With this formulation, we have used first-order spectral perturbation analysis to study the convergence rate of the consensus algorithm under various network conditions including links failures, node failures, and network partitions. Our analysis has revealed that in large networks, the effect of communication failures on the convergence rate is similar to the effect of changing the weight assigned to the communication links. We have also shown that in large networks, correlation in link failures plays a negligible role in the convergence rate of the consensus algorithm. Although the analysis presented in this work applies to scenarios where the probability of failure is small, we have observed through computations that these results also hold for larger failure probabilities.

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