8 Kinematic Models for Target Tracking

Target tracking is a major application area of Kalman Filtering. Typical applications include aircraft tracking using noisy remote sensors (radar, vision, IR, etc.), and tracking moving objects in video sequences. We consider here very briefly some basic models for target tracking, in order to illustrate:

* Physical modeling and discretization.
* The basic structure of a tracking filter.

For simplicity we consider tracking in one cartesian coordinate only.

8.1 Modelling Target Motion

A. Second-order models: random acceleration

These are the simplest useful models. Here the velocity is constant except for a noise term.

Let $p(t)$ denote the object position, $V(t) = \dot{p}(t)$ its velocity, and $a(t) = \ddot{p}(t)$ its acceleration. The basic second-order model in continuous time is described by

$$\ddot{p}(t) = w(t)\,,$$

where $w(t)$ is white noise: $R_w(t) = \sigma_w^2\,\delta(t)$. The corresponding state equations are:

$$x = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \quad ; \quad \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)\,.$$

As the measurements are typically obtained in discrete times, we need to consider discretized versions of this model.

B. Discretization

Consider the state-space equation:

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t)$$

and sampling times $t_k = kT$.

As we know,

$$x(t) = e^{A(t-t_k)}\, x(t_k) + \int\limits_{t_k}^{t} e^{A(t'-t_k)} \Big(Bu(t') + Gw(t')\Big)\, dt' \,.$$

We now assume that the input u_k changes slowly relative to the sampling period, so that $u(t) \simeq u(t_k)$ on $[t_k,\, t_{k+1})$. This gives

$$x(t_{k+1}) = Fx(t_k) + \overline{B}u(t_k) + w_k \,,$$

where

$$\begin{aligned} F &= e^{AT} \\ \overline{B} &= \int_0^T e^{A(T-t')}\, B\, dt' \\ w_k &= \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t')}\, Gw(t')\, dt' \,. \end{aligned}$$

If $w(t)$ is a white noise process with $R_w(t) = Q_w(t)\,\delta(t)$, then $\{w_k\}$ is a white noise sequence with

$$Q_k = \mathrm{cov}(w_k) = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-t')}\, GQ_w(t')G^T\, e^{A^T(t_{k+1}-t')}\, dt' \,.$$

If $w(t)$ is also Gaussian, then $w_k \sim N(0, Q_k)$.

For our 2nd–order model we get

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} , \quad Q = \begin{bmatrix} \frac{1}{3}T^3 & , & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & , & T \end{bmatrix} \sigma_w^2 \, .$$

We note that the "order of magnitude" of possible velocity change over $[t_k, t_{k+1}]$ is $\Delta V \approx \sqrt{Q_{22}} = \sqrt{T\,\sigma_w^2}$. This should guide the choice of σ_w^2.

Typical measurements include position, velocity, or both. The measurement equation is of the usual form $z(t_k) = Hx(t_k) + v_k$, where v_k is the measurement noise (which depends on the sensor).

C. Simplified discretization

A slightly different model can be obtained from $\ddot{p}(t) = w(t)$ by making the simplifying assumption that the noise is constant between sampling instants, that is $w(t) \equiv w(t_k)$ for $t \in [t_k,\, t_{k+1})$. This gives the equations

$$\begin{aligned} V(t_{k+1}) &= V(t_k) + Tw(t_k) \\ p(t_{k+1}) &= p(t_k) + T\,V(t_k) + \frac{T^2}{2}\,w_k \end{aligned}$$

so that

$$x_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \tilde{w}_k \, ,$$

where

$$\tilde{w}_k = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} w(t_k) \, , \quad \tilde{Q}_k = \begin{bmatrix} \frac{1}{4}T^4 & , & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & , & T^2 \end{bmatrix} \sigma_w^2 \, .$$

After scaling σ_w^2 by T, the model is similar to the previous one except for the "$\frac{1}{4}$" coefficient in Q_{11}.

D. Direct modeling in discrete time

A simplified model can sometimes be directly constructed in discrete time. Let p_k and V_k denote the target position and velocity at time t_k. Let $x_k = (p_k, V_k)^T$, and let $T = t_{k+1} - t_k$ The approximate state equations are:

$$p_{k+1} = p_k + T\,V_k$$
$$V_{k+1} = V_k + w_k$$

where (w_k) is a white noise sequence. These equations reflect the following assumptions:

- The velocity V_k is constant on $[t_k, t_{k+1}]$.
- The velocity *increments* $(V_{k+1} - V_k)$ are white.

The last assumption can be interpreted as a white-noise acceleration. The variance σ_w^2 should reflect the possible change in velocity over period T.

The resulting state model is:

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k \,.$$

The model is similar to the previous ones except for zero noise in the position component. The previous models should be preferred unless T is very small.

8.2 Steady-State Filter for 2nd Order Models: The α-β Filter

Assuming a fixed sampling interval T, we have arrived at the stationary model:

$$x_{k+1} = Fx_k + w_k$$
$$z_k = Hx_k + v_k$$

with

$$x = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} , \quad F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} , \quad Q = \begin{bmatrix} \frac{T^4}{4} & \frac{T^3}{2} \\ \frac{T^3}{2} & T^2 \end{bmatrix} \sigma_w^2$$

Assume a noisy *position* measurement: $z_k = p(t_k) + v_k$, so that

$$H = [1, 0] , \quad R \triangleq \sigma_v^2 .$$

The steady-state filter will be of the form

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tilde{z}_k \triangleq \hat{x}_{k|k-1} + \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \tilde{z}_k$$

where, as usual, $\tilde{z}_k = z_k - H\hat{x}_{k|k-1}$. Note that this filter gives both position and velocity estimates.

We wish to compute the Kalman gain K, namely the coefficients α and β, and the error covariance P.

Recall the Ricatti equation for $P \equiv P^-$:

$$P = F[P - PH^T S^{-1} HP] F^T + Q$$

with $S = HPH^T + R$, and $K = PH^T S^{-1}$. Denote

$$P = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix} .$$

After some algebra, we obtain 3 quadratic equations in $(m_{11},\, m_{21},\, m_{22})$, which can be solved (excercise). The Kalman gain elements may be expressed as:

$$\begin{aligned} \alpha &\triangleq K_1 = \frac{1}{8}\left(-\lambda^2 - 8\lambda + (\lambda+4)\sqrt{\lambda^2 + 8\lambda}\right) \\ \beta &\triangleq TK_2 = \frac{1}{4}\left(\lambda^2 + 4\lambda - \lambda\sqrt{\lambda^2 + 8\lambda}\right), \end{aligned}$$

where

$$\lambda = \frac{T^2\, \sigma_w}{\sigma_v}$$

is the "maneuvering index". Essentially, it is the ratio of the state noise to the measurement noise. The behavior of the optimal gain parameters as a function of λ is illustrated in the next figure.

It can also be shown that

$$(P^+)_{11} = \alpha\, \sigma_v^2$$

so that α also represents the improvement in position estimation variance as compared with that of a single measurement.

8.3 Higher-Order Models

Our basic model so far was $a(t) = w(t)$, which essentially corresponds to a constant velocity motion (with white noise perturbation). In some cases a constant acceleration model may be more appropriate. In this case we can increase the model order and consider

$$\dot{a}(t) = w(t)\,.$$

This leads to a 3rd order system. The resulting steady-state filer is called the "α-β-γ filter", and has the form:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \begin{bmatrix} \alpha \\ \beta/T \\ \gamma/T^2 \end{bmatrix} \tilde{z}_k\,.$$

The coefficients α, β, γ again depend only on the maneuvering index $\lambda = \frac{T^2\sigma_w}{\sigma_v}$.

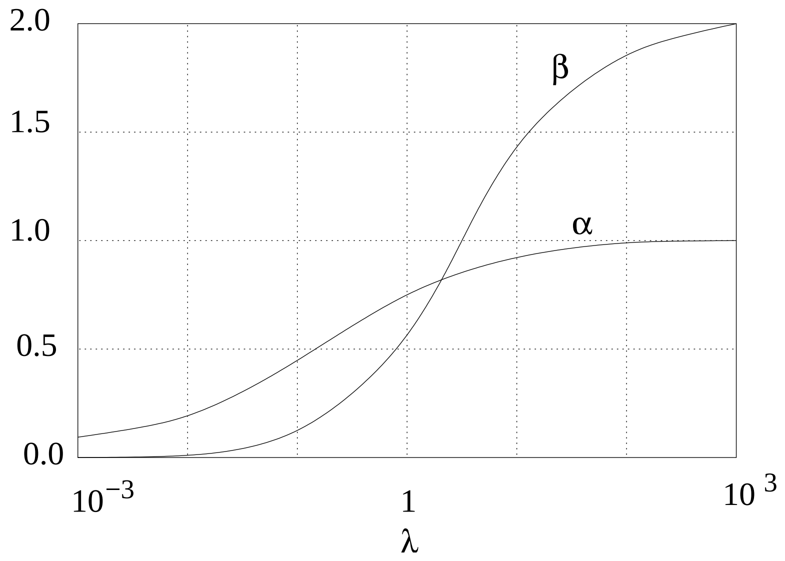

Figure 1: The gain coefficients as function of λ (α-β filter)

Another option to regularize the velocity change in the 2nd-order filter is to use filtered noise in place of white noise. This simplest such model is:

$$\dot{a}(t) = -b\,a(t) + w(t)\,.$$

Note that this may be viewed as the (second-order) noise acceleration model $a(t) = \tilde{w}(t)$ with low-pass filtered noise $\tilde{w} = w/(s+b)$. This model is useful for tracking maneuvering targets, where velocity changes cannot be too abrupt.

8.4 Target Tracking in General

Target tracking in practice involves many additional issues with varying degrees of difficulty. Among those we mention:

- Correlated motion in several dimensions.
- Polar measurements (leading to nonlinear filters).
- Partial measurements (such as bearings-only measurements in sonar).
- Spurious measurements (”clutter”).
- Different target maneuvers (which requires adaptive or multiple models).
- Multi-target tracking.