Technion – Israel Institute of Technology, Department of Electrical Engineering
Estimation and Identification in Dynamical Systems (048825)
Lecture Notes, Fall 2009, Prof. N. Shimkin

6 The Steady-State Filter

6.1 Problem and Main Results

We focus here on the stationary (time-invariant) model:

$$\begin{aligned} x_{k+1} &= F x_k + w_k \\ z_k &= H x_k + v_k \end{aligned} \tag{1}$$

where

$$E(w_k w_l^T) = Q\delta_{kl}, \quad E(v_k v_l) = R\delta_{kl}, \quad E(w_k v_l^T) = 0$$

$$x_0 \sim (\overline{x}_0, P_0), \quad x_0 \perp \{v_k, w_k\}$$

$R > 0$ ("non-singular problem").

The KF equations, with $\hat{x}_k \equiv \hat{x}_{k|k-1}$ and $P_k \equiv P_{k|k-1}$:

$$\begin{aligned} \hat{x}_{k+1} &= F\hat{x}_k + K_k(z_k - H\hat{x}_k) \\ &= (F - K_k H)\hat{x}_k + K_k z_k \ , \quad k \geq 0 \\ K_k &= F P_k H^T (H P_k H^T + R)^{-1} \\ P_{k+1} &= F[P_k - P_k^T H^T (H P_k H^T + R)^{-1} H P_k] F^T + Q \ . \end{aligned}$$

Note: this filter is still time-varying in general !

Question: Does the filter become stationary *asymptotically* ?

The main results of this section are summarized as follows.

Theorem 1. Assume that:

(i) The pair $[F, H]$ is detectable.

(ii) The pair $[F, G_1]$ is stabilizable, where $G_1 = \sqrt{Q}$ (i.e., $G_1 G_1^T = Q$).

Then

(a) $P_k \to \overline{P} \geq 0$ as $k \to \infty$, for any $P_0 \geq 0$.

The limit $\overline{P}$ is the *unique* non-negative-definite solution of the Algebraic Riccati Equation (ARE):

$$P = F[P - PH^T(HPH^T + R)^{-1}\, HP]F^T + Q\ .$$

(b) $(F - \overline{K}H)$ is stable, where

$$\overline{K} = \lim_{k\to\infty} K_k = F\overline{P}\, H^T(H\overline{P}H^T + R)^{-1}\ .$$

We shall explain and prove these results below.

<u>Remarks</u>

1. Conditions (i) + (ii) hold trivially when the system is asymptotically stable (i.e., F is stable).

2. If the optimal filter is started with $P_0 = \overline{P}$, it is immediately stationary.

3. In many applications the (sub-optimal) stationary filter is employed, i.e., K_k is replaced by $\overline{K}$.

4. The stationary filter $K_k := \overline{K}$ is optimal w.r.t. the asymptotic criterion:

$$\lim_{t\to\infty} E\{(x_t - \hat{x}_t)(x_t - \hat{x}_t)^T\}\ .$$

6.2 Basic Properties of LTIV State Systems

Consider the linear time-invariant (LTIV) system:

$$\begin{aligned} x_{k+1} &= Fx_k + Gu_k \; ; \quad x_0 = x^0 \\ y_k &= Hx_k \end{aligned}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^m$.

Stability:

The system is *asymptotically stable* ($x_k \to 0$ for $u_k \equiv 0$) iff $|\lambda_i(F)| < 1$, $i = 1 \ldots n$. We call such a matrix F *stable*.

Controllability:

The pair $[F, G]$ is controllable iff any one of the following equivalent conditions is satisfied:

1. $\forall\, x^0\,,\, x^f \in \mathbb{R}^n \quad \exists \{u_0\,, \ldots,\, u_{n-1}\}$ s.t. $x_n = x^f$.
2. rank $[G,\; FG\,, \ldots,\; F^{n-1}G] = n$.
3. $\sum_{i=0}^{n-1} F^i G G^T (F^T)^i > 0$.
4. rank $[\lambda I - F \,\vdots\, G] = n \quad \forall\, \lambda$
   (equivalently, $\forall\ \lambda = \lambda_i(F)$).
5. The modes ($\equiv$ eigenvalues) of $(F{+}GK)$ can be assigned arbitrarily by choosing K.

If $[F, G]$ is not controllable, there exists a similarity transform T s.t.

$$T^{-1}FT = \begin{bmatrix} F_1 & F_3 \\ 0 & F_2 \end{bmatrix}, \quad T^{-1}G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

$$\text{and} \quad [F_1,\, G_1] \quad \text{is controllable .}$$

The modes of F_1 are the "controllable modes", and those of F_2 are the "uncontrollable modes".

Stabilizability:

$[F, G]$ is stabilizable iff any one of the following equivalent statements holds:

(i) $\exists\, K$ s.t. $(F + GK)$ is stable.

(ii) All uncontrollable modes of F are "stable": $|\lambda_i(F_2)| < 1$.

(iii) rank $[\lambda I - F \,\vdots\, G] = n \quad \forall\, |\lambda| \geq 1$.

Observability and Detectability:

Observability means that x_0 can be determined from
$\{u_k,\, y_k\,;\ k = 0, 1\,, \ldots\,, n-1\}$.

Detectability means that the non-observable modes of F are stable.
In particular, $\ni K$ s.t. the "observer matrix" $[F - KH]$ is stable.

Algebraically:

- $[F, H]$ is *observable* iff $[F^T, H^T]$ is controllable.
- $[F, H]$ is *detectable* iff $[F^T, H^T]$ is stabilizable.

Obviously,

- controllability $\Longrightarrow$ stabilizability
- observability $\Longrightarrow$ detectability
- asymptotic stability $\Longrightarrow$ {stabilizability+detectability}.

<u>Stationary behavior of LTIV systems</u>

Consider the system

$$x_{k+1} = Fx_k + Gw_k\,,\ k \geq k_0\,,$$

with the usual assumptions on $\{w_k\}$ and x_{k_0}.
Let $E(x_{k_0}) = m_0,\ \mathrm{cov}(x_{k_0}) = \Pi^0$.

Recall that $\Pi_{k+1} = F\,\Pi_k\,F^T + GQG^T$, where $\Pi_k = \mathrm{cov}(x_k)$.

Theorem 2. Suppose F is stable. Then

(i) As $k_0 \to -\infty$, the processes $\{x_k\}$ converge to a (wide-sense) stationary process $\{\overline{x}_k\}$, with $E(\overline{x}_k) = 0,\ E(\overline{x}_k\overline{x}_k^T) = \overline{\Pi}$.
Moreover, $E(\overline{x}_k\overline{x}_l^T) = F^{k-l}\,\overline{\Pi}$ for $k \geq l$.

(ii) $\overline{\Pi}$ is the unique non-neg.-definite solution of

$$\Pi - F\,\Pi\,F^T = GQG^T\ . \tag{2}$$

Note: It follows that

1. For fixed k_0, $E(x_k) \to 0$ and $\mathrm{cov}(x_k) \to \overline{\Pi}$ as $k \to \infty$.

2. If we start with $m_0 = 0$ and $\Pi^0 = \overline{\Pi}$, then $\{x_k\}_{k\geq k_0}$ is stationary.

Lyapunov's equation and stability

Equation (2) is the discrete-time *Lyapunov equation* (for Π). Let us write it in the following form:

$$P - FPF^T = Q \ . \tag{3}$$

This (linear) equation has a unique solution P iff $\lambda_i(F)\lambda_j(F) \neq 1$ for all $1 \leq i, j \leq n$. This solution is symmetric if Q is. We henceforth assume that Q is symmetric.

The following basic relations exist between stability of F and positive-definite solutions to the Lyapunov equation.

1. F is stable *iff* there exist $P > 0$, $Q > 0$ that satisfy (3).
2. If F is stable, then the solution P is unique, symmetric, and $P = \sum_{i=0}^{\infty} F^i Q (F^T)^i$.
3. If F is stable and $Q > 0$ ($Q \geq 0$), then $P > 0$ ($P \geq 0$).
4. If F is stable, $Q \geq 0$, and $[F, \sqrt{Q}]$ is controllable, then $P > 0$.

6.3 Proof of Theorem 1

The proof proceeds along the following steps:

1. $\forall\, P_0 \geq 0\,,\ \{P_k\}$ is bounded [provided (F, H) is detectable].
2. The map f from P_k to P_{k+1} is monotone, i.e.,
   $P \geq \tilde{P} \geq 0 \Rightarrow f(P) \geq f(\tilde{P})$.
3. For $P_0 = 0$, $P_k \nearrow \overline{P}$ for some $\overline{P} \geq 0$.
4. $(F - \overline{K}H)$ is stable [provided $(F, \sqrt{Q})$ is stabilizable].
5. $P_k \to \overline{P}$ for any $P_0 \geq 0$ (with $\overline{P}$ as in 3.).

Step 1.

Since (F, H) is detectable, $\exists\, K_1$ s.t. $(F - K_1 H)$ is stable. Consider the sub-optimal filter

$$\hat{\hat{x}}_{k+1} = F\hat{\hat{x}}_k + K_1(z_k - H\hat{\hat{x}}_k)\ .$$

It is easily verified that

$$\tilde{x}_{k+1} := x_{k+1} - \hat{\hat{x}}_{k+1} = (F - K_1 H)\,\tilde{x}_k + (w_k - K_1 v_k)\ .$$

By stability of $(F - K_1 H)$ and the above-quoted results, it follows that $\Pi_k :=$ cov $(\tilde{x}_k)$ is bounded; however $\hat{\hat{x}}$ is sub-optimal, so that $P_k \leq \Pi_k$.

Step 2.

Recall that $P_{k+1} = \min\limits_K g(P_k, K)$, where

$$g(P, K) = (F - KH)P\,(F - KH)^T + KRK^T + Q\ .$$

Thus, if $P_k \geq \tilde{P}_k$,

$$\begin{aligned} P_{k+1} &= \min_K g(P_k, K) = g(P_k, K^*) \geq g(\tilde{P}_k, K^*) \\ &\geq \min_K g(\tilde{P}_k, K) = \tilde{P}_{k+1} \ . \end{aligned}$$

Step 3.

Suppose $P_0 = 0$. Then $P_1 \geq P_0 = 0$. But from Step 2 it follows that $P_2 \geq P_1$ etc., namely $P_{k+1} \geq P_k,\ k \geq 0$. But since $\{P_k\}$ is bounded by Step 1, then $P_k \to \overline{P}$ for some $\overline{P} \geq 0$. Obviously, $\overline{P}$ must be a stationary point of the covariance update equation, hence solves the ARE.
(Uniqueness of the solution will follow from Step 5).

Step 4: Stability of $(F - \overline{K}H)$.

With $\overline{K}$ the (stationary) gain corresponding to $\overline{P}$, the ARE is

$$\overline{P} = (F - \overline{K}H)\,\overline{P}(F - \overline{K}H)^T + \overline{K}R\overline{K}^T + \overbrace{G_1 G_1^T}^{Q} \ . \tag{4}$$

Let v be a left-eigenvector of $(F - \overline{K}H)$ with eigenvalue λ. Then

$$(v\overline{P}v^*) = |\lambda|^2\,(v\overline{P}v^*) + \underbrace{v(\overline{K}R\overline{K}^T + G_1 G_1^T)v^*}_{\geq 0} \ . \tag{5}$$

Obviously this implies that $|\lambda| \leq 1$. It only remains the show that $|\lambda| = 1$ is impossible. If $|\lambda| = 1$, we have from (5) and the definition of v:

(1) $\quad v(F - \overline{K}H) = \lambda v$

(2) $\quad v\overline{K} = 0 \quad$ (recall the $R > 0$)

(3) $\quad vG_1 = 0$.

But (1) + (2) imply $vF = \lambda v$ or $v(\lambda I - F) = 0$. Together with (3) this gives $v[\lambda I - F,\, G_1] = 0$. This contradicts the assumption that $(F,\, G_1)$ is stabilizable.

Step 5: $P_k \to \overline{P} \quad \forall\, P_0 \geq 0.$

Suppose we use the stationary suboptimal filter $K_k \equiv \overline{K}$ to obtain the estimator $\hat{\hat{x}}_k$. We show that its error covariance converges to $\overline{P}$. Defining $\tilde{x}_k \stackrel{\triangle}{=} x_k - \hat{\hat{x}}_k$ we obtain

$$\tilde{x}_{k+1} = (F - \overline{K}H)\,\tilde{x}_k - \overline{K}v_k + w_k\ .$$

Since $(F - \overline{K}H)$ is stable, it follows from above-quoted results on stationary behavior that $\Sigma_k := \mathrm{cov}\ (\tilde{x}_k) \to \tilde{P} \geq 0$, where $\tilde{P}$ is the unique non-negative solution of the (Lyapunov) equation:

$$\tilde{P} = (F - \overline{K}H)\,\tilde{P}(F - \overline{K}H)^T + \overline{K}R\overline{K}^T + Q\ .$$

However, substituting $\overline{K}$ this is just the ARE which is satisfied by $\overline{P}$, hence $\tilde{P} = \overline{P}$.

Now, $\hat{\hat{x}}_k$ is sub-optimal so that $P_k \leq \Sigma_k \to \overline{P}$. On the other hand, by monotonicity of $f\ :\ P_k \to P_{k+1}$, it follows that $P_k \geq P_k^0 \to \overline{P}$, where P_k^0 is the covariance for $P_0 = 0$.

Hence $P_k \to \overline{P}$. $\square$

6.4 Spectral Factorization in State Space

a. Definitions (reminder)

- Let $\{X_k\}$ be a (wide-sense) stationary process, with covariance $R_k = E(X_l\, X_{l+k}^T)$.
- The power spectrum of X_k is

$$S_x(z) = \mathcal{Z}\{R_k\} \;:= \sum_{k=-\infty}^{\infty} z^{-k}\, R_k \;.$$

  By symmetries in $\{R_k\}$ it follows that $S_x(z) = S_x^T(z^{-1})$. Thus, poles and zeros appear in inverse pairs, $(z_i,\; , z_i^{-1})$.
  E.g., $S_x(z) = \dfrac{1}{(z-2)(z^{-1}-2)}$.

- A (Wiener-Hopf, or canonical) spectral factorization of S_x is

$$S_x(z) = W(z)\, W(z^{-1})^T \;,$$

  where $W(z)$ has no poles nor zeros outside the unit circle; thus, $W(z)$ is stable and minimum-phase.

The problem of spectral factorization is non-trivial in the MIMO case. The use of state-space methods (and Kalman Filter formulas) provides an explicit solution.

b. The Factorization Formula

Suppose that $\{z_k\}$ is a stationary process, which may be modeled as the output of a LTIV system driven by white noise:

$$\begin{cases} x_{k+1} &= F x_k + w_k \\ z_k &= H x_k + v_k \end{cases}$$

with the usual noise assumptions.

We assume that F is *stable*. Further, the initial conditions ($x_0 = 0$ and P_0) are such that x_k (hence z_k) is stationary. It follows that

$$S_z(z) = T_w(z)\, Q T_w(z^{-1})^T + R$$

where $T_w(z) \stackrel{\triangle}{=} H \cdot (zI - F)^{-1}$.

Theorem 3. Let $\overline{P}$ and $\overline{K}$ be the steady-state Kalman covariance and gain for the model system, and $T_0(z) \stackrel{\triangle}{=} I + H(zI - F)^{-1}\,\overline{K}$. Then

$$W(z) \stackrel{\triangle}{=} T_0(z) \cdot [R + H\,\overline{P}\,H^T]^{1/2}$$

provides a spectral factorization of $S_z(z)$. That is: $W(z)$ is stable, with stable inverse, and $S_z(z) = W(z) W(z^{-1})^T$.

c. <u>Proof</u>

The theorem above follows from the following two lemmas, which can be proved using some algebraic manipulations. It is however much easier (and more illuminating) to use previous results on the *innovations process*.

Lemma: Consider the model system above. Then (even with F unstable, but assuming that $\overline{P}$ and $\overline{K}$ are well defined)

$$S_z(z) = T_0(z)\,[R + H\overline{P}\,H^T]\,T_0(z^{-1})$$

where $T_0(z) \stackrel{\triangle}{=} I + H(zI - F)^{-1}\,\overline{K}$.

Proof: Recall the renewal representation of z_k (with $P_0 = \overline{P}$):

$$\begin{cases} \hat{x}_{k+1} &= F\hat{x}_k + \overline{K}\tilde{z}_k \\ \\ z_k &= H\hat{x}_k + \tilde{z}_k \end{cases}$$

where

$$\tilde{z}_k = z_k - E(z_k | Z_{k-1}) = H(x_k - \hat{x}_k) + v_k\,.$$

We know that $\tilde{z}_k$ is white, with covariance $E(\tilde{z}_k\,\tilde{z}_k^T) = H\overline{P}H^T + R$. Hence $S_{\tilde{z}}(z) = H\overline{P}H^T + R$. Noting that $T_0(z)$ is the transfer function from $\tilde{z}$ to z, the required expression for S_z follows. $\square$

Lemma: If F is stable, then $T_0(z)$ is stable and with stable inverse.

Proof: Stability is trivial by definition of T_0. To compute $T_0(z)^{-1}$, note that we have an explicit "inverse" state model that creates $\tilde{z}$ from z:

$$\begin{cases} \hat{x}_{k+1} &= (F - \overline{K}H)\hat{x}_k + \overline{K}z_k \\ \tilde{z}_k &= -H\hat{x}_k + z_k \end{cases}$$

It follows that

$$T_0(z)^{-1} = -H\Big(zI - (F - \overline{K}H)\Big)^{-1}\overline{K} + I. \tag{6}$$

But we know that $(F - \overline{K}H)$ is stable if F is stable. $\square$

We note that the last expression for $T_0(z)^{-1}$ can also be derived using the "matrix inversion lemma":

$$[A + BC^{-1}D^T]^{-1} = A^{-1} - A^{-1}B[C + D^T A^{-1} B]^{-1} D^T A^{-1}\,.$$