On Union Bounds for Random Serially Concatenated Turbo Codes with Maximum Likelihood Decoding

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Abstract

The input-output weight enumeration (distribution) function of the ensemble of serially concatenated turbo codes is derived, where the ensemble is generated by a uniform choice over the interleaver and a uniform choice of the component codes from the set of time varying recursive systematic convolutional codes. The conventional union bound is employed to obtain an upper bound on the bit error probability of the ensemble of serially concatenated turbo codes for the binary-input additive white Gaussian noise (AWGN) channel with coherent detection of antipodal signals and maximum likelihood decoding. The influence of the interleaver length N and the memory length of the component codes m on the ensemble performance is investigated and compared with relevant results for parallel concatenated multiple-turbo codes.

I. Introduction

The discovery of turbo codes in 1993 [3] was an exciting development in coding theory. These codes demonstrated near Shannon-limit performance on a Gaussian channel with relatively simple component codes and large interleavers. An extensive literature on turbo codes has already appeared, as evidenced by [1]–[16] and references therein.

In addition to performance obtained from simulations, upper bounds on the bit-error rates of turbo codes have been developed. Transfer function bounding techniques have been applied to obtain union bounds on the bit-error rate for maximum-likelihood decoding of turbo codes constructed with random interleaver permutations [6]. Because it has not been tractable to obtain analytic results on bit error rate for a particular interleaver, the bounds on bit-error rate have been developed as averages over certain ensembles with *random coding* properties. Moreover, since most of these bounds are based on a union bounding technique [14],[16], they give useless results for energy-per-bit to spectral-noise-density $\left(\frac{E_b}{N_0}\right)$ ratios for which the code rate is above the resulting cutoff rate (R_0) of the channel, which is the region of particular interest for turbo code operation.

For component codes with a known weight distribution, the weight distribution of the parallel concatenated turbo code with a hypothetical 'optimal' interleaving was calculated in [13] and used to derive union upper bounds on the bit-error rate. Turbo codes with random component codes have also been investigated and lower bounds on the $\frac{E_b}{N_0}$ needed to achieve essentially error free decoding have been given. For component codes that satisfy the Gilbert-Varshamov bound, an upper bound on the ratio of the minimum distance of the turbo code to that of the component codes has been derived [13].

Two distinct modes of concatenation have been used in conjunction with turbo coding, namely parallel concatenation and serial concatenation. Fig. 1 illustrates these two forms of concatenation.

An upper bound on the bit error probability of a parallel concatenated coding scheme averaged over all interleavers of a given length was proposed in [1]. A probabilistic interleaver, called the 'uniform interleaver', was introduced that permits an easy derivation of the weight distribution of the parallel concatenated code from those of its component codes. However, the analysis in [1] used an exponential extrapolation of coefficients and was thus only approximate. Although the validity of the approximation was verified numerically, the results with truncation might be misleading and might fail to give a valid upper bound on bit-error rate for rates above the cutoff rate (R_0), as was pointed out in [14],[16].

An interesting result, which was proved in [2] and mentioned also in [16], about the so called 'interleaver gain' of serially concatenated codes with recursive convolutional inner codes is that every term in the union bound on bit error probability decreases asymptotically at least as fast as $N^{-\left\lfloor \left(d_f^{(0)}+1\right)/2 \right\rfloor}$, where N is the interleaver size, $d_f^{(0)}$ is the free distance of the outer code and $\lfloor x \rfloor$ is the integer part of x. Therefore when a serially concatenated code is considered and the component codes are recursive systematic convolutional (RSC) codes, it is advisable to use the RSC code with the greater free distance as the outer code in order to enhance the interleaver gain of the resulting serially concatenated code. A recent analysis [10] shows that the minimum distance of a serially concatenated coding scheme converges to $N^{\frac{d_f^{(0)}-2}{d_f^{(0)}}}$. This result holds regardless of the structure of the outer codes, which need not be recursive. For parallel concatenation with K component codes, the minimum distance behaves as $N^{\frac{K-2}{K}}$, and again the result does not depend on the structure of the K component recursive codes nor on their memory.

Union bounds for serially concatenated codes with maximum likelihood decoding for the binary-input additive white Gaussian noise (AWGN) channel were studied in [16]. The component codes were explicitly chosen while the interleaver was random and uniform. A set of recursions was developed, based on the chosen component codes, to facilitate numerical computation of the union bound. These union bounds indicate that the interleaver gain was actually achieved above the value of $\frac{E_b}{N_0}$ that corresponds to the cutoff rate. The ensemble considered in [16] was that of all interleaver permutations rather than the ensemble of all codes as is usually considered in random coding applications [15]. The union bound is still the weak link, as was demonstrated in [16] by computer simulations for iterative maximum a posteriori (MAP) soft-output decoding. These simulations also demonstrated an interleaver gain for $\frac{E_b}{N_0}$ values corresponding to code rates above the cutoff rate but below the channel capacity.

Analytical bounds on the performance of concatenated codes on a tree structure were obtained in [8] based on the conventional union bound. Analytical results for the binary-input AWGN and Rayleigh fading channels were applied to examples of parallel concatenation of two codes (turbo codes), serial concatenation of two codes, hybrid concatenation of three codes and self concatenated codes. This yielded design criteria for the selection of the component codes.

An efficient algorithm for obtaining the distance spectrum of turbo codes was presented in [5]. This algorithm is an important tool for calculating upper bounds on the bit error rate that depends on the distance spectrum of the code, such as the conventional union bound used in this paper and in [1],[2],[16],[14],[8],[5]. The focus of [5] was directed to parallel concatenated convolutional codes (PCCC), but the results of [5] can be extended to serial concatenation and to general coding networks. For simplification of the analysis, it was assumed in [5] that the component codes in the PCCC are identical, but the results extend easily to the general case. Improved bounding techniques have recently been presented in [9],[12] and their advantages when applied to serial and parallel concatenated turbo coding was demonstrated.

In this paper, we derive the random ensemble weight enumerator for serially concatenated turbo codes. The ensemble is generated by a uniform random choice of the interleaver and a uniform random choice of the component codes from the set of time varying recursive convolutional codes with a given memory length. As in the result on parallel concatenation reported in [14], this reduces the performance behavior of random turbo codes to two parameters: the interleaving length N, and the memory length m of the inner and outer codes which we assume to be the same.

Following the derivation of the ensemble weight enumerator, we use a union bound argument to obtain an upper bound on the bit error probability of the ensemble of serially concatenated turbo codes for maximum likelihood decoding on the AWGN channel. We focus on rate 1/2 inner and outer codes, which results in an overall code of rate 1/4. As a direct consequence of using the standard union bound, we obtain a meaningful upper bound only for code rates below the cutoff rate, as in [6],[8],[13],[14],[16]. This excludes the portion of the rate region where the performance of turbo codes is most appealing. Nevertheless, as demonstrated in [6]–[8], [14],[16], union bounds which are based on the distance spectra of long enough codes are useful analytical tools for investigating and gaining insight into the performance of turbo codes at rates below the channel cutoff rate. We compare the ensemble performance of random parallel concatenated turbo codes, as presented in [14], to the case of random serially concatenated turbo codes and conclude that serial concatenation is superior for moderate values of $\frac{E_b}{N_0}$. We then show that increasing the memory length m of the component codes above $\lfloor \log_2 N \rfloor$ is not advisable since it improves the union ensemble performance of the serially concatenated turbo codes only slightly while increasing the decoding complexity of the codes considerably. A similar result has also been demonstrated for parallel concatenated turbo codes [14]. The asymptotic behavior in terms of $\frac{E_b}{N_0}$ is discussed in light of recent observations in [10] about the minimum distances of parallel and serial turbo coding constructions.

In the next section we present our underlying assumptions and explain our notation. The analysis is given in Section III and the results are described in Section IV. A summary and some concluding remarks are presented in Section V.

II. Preliminaries

In this section, we state the underlying assumptions on which our analysis is based, introduce notation and basic relations from [1]–[14], which apply to parallel and serially concatenated turbo codes, and state some further useful relations and results.

A. Assumptions:

In our analysis, the channel is assumed to be a binary-input AWGN channel with binary antipodal signaling. The detection is coherent and the decoding is *maximum likelihood* (ML). The coding structure considered is serial concatenated turbo codes: the component codes are assumed to be time-varying RSC codes with the same memory length (see Fig. 2). A uniform random interleaver is incorporated at the output of the outer RSC encoder (see Fig. 1b). The random (uniform) interleaving takes into consideration all possible permutations, including the option of no interleaving (for the identity permutation) as a particular case, with all interleaving permutations equally likely.

B. Notation and relations:

A serially concatenated turbo code c_s with components c_0 and c_i as the outer and inner codes, respectively, is considered. The rate in units of $\frac{\text{bits}}{\text{symbol}}$ of the code c_s is R, which is also the product of the rates of the component codes. The uniform interleaver situated between the component codes operates on *bits* and has a length of N, which is also the length of a codeword of c_0. The number of information bits is the product of N and the rate of the outer code $R^{(0)}$. The common memory length of the component codes is m.

The number of codewords of Hamming weight h of the code c_s that result from information sequences of Hamming weight w is denoted by $A^{c_s}_{w,h}$. As the component codes are assumed to be systematic, the serial concatenated code c_s is also systematic. It follows that $A^{c_s}_{w,h} = 0$ if $w > h$. The quantities $A^{c_0}_{w,\ell}$ and $A^{c_i}_{\ell,h}$ are similarly defined for the component codes. Since the component codes are systematic, it follows that $A^{c_0}_{w,\ell} = 0$ for $w > \ell$ and that for $\ell > h$. $A^{c_i}_{\ell,h} = 0$ for $w > \ell$ and that for $\ell > h$. Termination of the codes has been assumed here, but we omit the details of how to terminate both shift registers. This notation is consistent with that used in [2],[16], which deal with serially concatenated turbo codes. However, unlike in [1],[14] which address parallel concatenated turbo codes, the parameter h of $A^{c_s}_{w,h}$ is here the Hamming weight of the entire codeword of c_s, not just that of its parity bits.

For a serially concatenated code c_s with a uniform interleaver of length N, it has been shown [2],[16] that

$$A^{c_s}_{w,h} = \sum_{\ell=0}^{N} \frac{A^{c_o}_{w,\ell}\, A^{c_i}_{\ell,h}}{\binom{N}{\ell}} \,. \tag{1}$$

The following notation is adopted from [14] with some modifications for dealing with serial rather than parallel concatenated codes. Let the random variable T be the number of steps that a time-varying recursive shift register of length m initially in a non-zero state remains *active*, i.e., remains in a non-zero state, when a non-zero input bit is applied. Let p_k be the probability that the random variable T equals k, $p_k = \text{Prob}\,\{T = k\}$. Moreover, as long as the shift register is in a non-zero state, the "state sequence" $\{s_1(n),\, n \geq 1\}$ (see Fig. 2) at the output of the first modulo-two adder will be a sequence of i.i.d. random variables, uniformly distributed on $GF(2)$, and independent of the input process x [14].

We first consider the expected number of steps, $E[T]$, for which the shift register remains active after an initial non-zero input. This problem was already solved in a more general context by Nielsen [11] who determined the expectation, $E[x]$, of the position x of the first digit in the first occurrence of an arbitrary pattern within an i.i.d. sequence of equiprobable letters over an arbitrary alphabet. For a binary pattern of m consecutive 0's, the Corollary in [11] gives $E[x] = 2^{m+1} - m - 1$. But T is just the position in which this pattern ends, i.e.,

$T = x + m - 1$, so that

$$E[T] = 2^{m+1} - 2\ . \tag{2}$$

Note that this result implies that if $m = \lfloor \log_2 N \rfloor$, then the average activity time of a time-varinging recursive shift register with memory length m is at least N.

Let p_k be the probability that the random variable T is equal to k, The probabilities $p_k = Prob\{T = k\}$ are easily computed recursively as we now show. Trivially

$$p_k = 0 \text{ for } k < m \text{ and } p_m = 2^{-m}\,, \tag{3}$$

since the shift register remains active until m consecutive 0's appear in the state sequence. For $k > m$, the event that $T = k$ coincides with the event that the first k digits of the state sequence are

$$b_1, b_2, \ldots, b_{k-m-1}, 1, 0, 0, \ldots, 0\,, \tag{4}$$

where $b_1, b_2, \ldots, b_{k-m-1}$ is a binary sequence containing no run of m consecutive 0's. But the event that the first $k - m - 1$ digits of the state sequence are equal to such a sequence $b_1, b_2, \ldots, b_{k-m-1}$ is just the event that $T \geq k - m$. By the independence of the digits in the state sequence, it follows that

$$p_k = \text{Prob}\,\{T = k\} = \text{Prob}\,\{T \geq k - m\}\frac{1}{2}2^{-m},\ \text{all } k > m\,. \tag{5}$$

But $\text{Prob}\,\{T \geq k - m\} = 1$ for $k \leq 2m$ and hence

$$p_k = 2^{-m-1} \text{ for } m < k \leq 2m. \tag{6}$$

Upon setting $k = 2m + i$ and noting that

$$\text{Prob}\,\{T \geq m + i\} = 1 - \sum_{j=0}^{i-1} \text{Prob}\,\{T = m + j\},\ \text{all } i > 0\,, \tag{7}$$

we can rewrite the above equation (based in Eq. (5)) as

$$p_{2m+i} = [1 - \sum_{j=0}^{i-1} p_{m+j}]2^{-m-1},\ \text{all } i > 0\,, \tag{8}$$

which is our desired recursion. We see that the calculation of each probability p_{2m+i} by this recursion requires one subtraction and one multiplication by 2^{-m-1}, which is just a shift in binary arithmetic.

Since we are interested in the output of the shift register over a finite interval, we define $p_{k,\ell}$ as in [14] to be

$$p_{k,\ell} = \begin{cases} p_k & \text{if} \quad k < \ell \\ 1 - \sum_{j=0}^{\ell-1} p_j & \text{if} \quad k = \ell \\ 0 & \text{if} \quad k > \ell \end{cases} . \tag{9}$$

Given a linear systematic block code c of dimension N, let $A^c(W,Z) = \sum_{w=0}^{N} A^c_w(Z)\, W^w$ be its *input-output weight enumeration (distribution) function*, where $A^c_w(Z)$, the *conditional* weight enumeration function, is the polynomial $A^c_w(Z) = \sum_{\ell} A^c_{w,\ell}\, Z^\ell$.

A serially concatenated turbo code can be regarded as a linear systematic block code of dimension equal to the product of the interleaver length (N) and the rate of the outer code ($R^{(0)}$). As the outer code is assumed to be RSC code of rate $\frac{1}{2}$, the serially concatenated code has dimension $\frac{N}{2}$, which is the total number of information bits encoded. For a linear systematic block code C of dimension N, S_w is the set of binary N-tuples of Hamming weight w.

The dimensions of the component codes of a parallel concatenation are the same and equal to (N). However, related to the outer code of a serially concatenated code, $S^{(0)}_w$ is defined as the set of binary $NR^{(0)}$-tuples of Hamming weight w ($R^{(0)} = \frac{1}{2}$ in our analysis). Related to the inner code of the serial concatenation, $S^{(i)}_w$ is defined similar to S_w for parallel concatenation (these sets are the same, since the input to the inner code is of length N).

As in paper [14], for $x \in S_w$, we let $i_1, i_2 \ldots i_w$ $(0 \leq i_1 < i_2 < \cdots < i_w \leq N-1)$ be the positions of the non-zero inputs. After the first non-zero input enters the shift register at time i_1, the register stays active for a time T. If there exists an index j, $1 < j \leq w$, such that $i_1 + T < i_j$, then the non-zero input at time i_j will activate the shift register again. As before, the shift register will stay active for a time T which is independent of the first activity time. The time span that the output is active is relevant (being the time span in which either the input is non-zero or the shift register is active) over the observation span of N (the length of binary tuples in S_w). Let E_w be the random variable describing this time span, averaged over a uniform choice over S_w, and let Prob $\{E_w = k\} = q^w_k$. In a similar manner, based on the definitions of $S^{(i)}_w$ and $S^{(0)}_w$ for the component codes of a serially concatenated turbo code, the random variables $E_w^{(i)}$ and $E_w^{(0)}$ and the corresponding probabilities $q_k^w(c_i)$ and $q_k^w(c_0)$ are defined (related to the inner and outer codes of the serially concatenated code c_s, respectively). The probabilities $\{q_k^w(c_0)\}$ and $\{q_k^w(c_i)\}$ related to the outer and inner codes respectively, were calculated with the aid of the statistical algorithm described in [14].

C. Comments:

For a serially concatenated code of rate $R \, \frac{\text{bits}}{\text{symbol}}$, the value of energy per bit to spectral noise density $\frac{E_b}{N_0}$ that corresponds to the cutoff rate is

$$\frac{E_b}{N_0} = -\frac{\ln\left(2^{1-R}-1\right)}{R} = \begin{cases} 1.85\,\text{dB} \ @\, R = \frac{1}{4} \, \frac{\text{bit}}{\text{symbol}} \\ 2.03\,\text{dB} \ @\, R = \frac{1}{3} \, \frac{\text{bit}}{\text{symbol}} \end{cases} . \tag{10}$$

The conventional union bound is expected to be useless for values of $\frac{E_b}{N_0}$ smaller than this cutoff-rate value. The value of $\frac{E_b}{N_0}$ that corresponds to the channel capacity of the binary-input AWGN channel is

$$\frac{E_b}{N_0} = \begin{cases} -0.79\,\text{dB} \quad @\, R = \frac{1}{4} \ \text{bit/symbol} \\ -0.50\,\text{dB} \quad @\, R = \frac{1}{3} \ \text{bit/symbol} . \end{cases} \tag{11}$$

The union bound derived next is a tightened version of the conventional union bound that was used in [2],[9],[14],[15],[16].

By definition, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} \, dt$ is the probability that a random Gaussian variable with zero mean and unit variance exceeds the value x. For the exponential form of the union bound, the inequality

$$Q(x) \leq \frac{1}{2} \, e^{-\frac{x^2}{2}} \quad \text{for} \quad x \geq 0 \, , \quad \text{is invoked} . \tag{12}$$

A tighter upper bound on the bit error rate of codes can be derived from the identity

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \quad \text{for } x \geq 0 \tag{13}$$

given in [4]. Equation (13) implies the following upper bound on bit-error probability

$$P_b \leq \frac{1}{\pi N} \int_0^{\frac{\pi}{2}} W \left. \frac{\partial A^c(W,Z)}{\partial W} \right|_{W=Z=e^{-\frac{RE_b}{N_0 \sin^2 \theta}}} d\theta \tag{14}$$

in place of the upper bound

$$P_b \leq \frac{W}{2N} \left. \frac{\partial A^c(W,Z)}{\partial W} \right|_{W=Z=e^{-\frac{RE_b}{N_0}}} ,$$

which results from the conventional union bound and which is obviously looser. Finally, the integration over θ in the upper bound (14) is performed numerically.

III. Analysis

In this section, we first derive the input-output weight enumeration function of the ensemble of random serially concatenated turbo codes as described in Section II. As for the results on parallel concatenation reported in [14], this reduces the performance analysis of random turbo codes to a two parameter family where the parameters are the interleaver length N and the memory length m of the component codes of the serial concatenation.

Following the derivation of the ensemble weight enumeration function, we employ the union bound to provide an upper bound on the bit-error probability of the ensemble of random serially concatenated turbo codes for the binary-input AWGN channel with maximum likelihood decoding of coherently detected antipodal signals.

By linearity, a zero input sequence results in a zero output and therefore, the corresponding conditional weight enumeration function of the serial concatenated code satisfies

$$A_0^{c_s}(Z) = 1 \, . \tag{15}$$

This implies also that $A_{0,0}^{c_s} = 1$.

Following the analysis in [14] for the case when the Hamming weight of the encoded information sequence is 1 and taking into account that the power of Z here is the Hamming weight of the whole systematic codeword (and not only of its parity bits), the corresponding conditional weight enumeration function $A_1(Z)$ of the considered set of convolutional codes is obtained by multiplying the expression of $B_1(Z)$ in [14] by Z.

Because the outer code in our analysis is assumed to be of rate $\frac{1}{2}$, the length of the input sequence at the input of the outer encoder is $\frac{N}{2}$. The interleaver length N is also the length of the input sequence to the inner encoder of the serial concatenation. Therefore, the corresponding weight enumeration functions $A_1^{c_0}(Z)$ and $A_1^{c_i}(Z)$ of the rate $\frac{1}{2}$ outer and inner codes, respectively, are

$$A_1^{c_0}(Z) = \sum_{i=0}^{\frac{N}{2}-1} \sum_{k=0}^{\frac{N}{2}-1-i} p_{k,\,\frac{N}{2}-1-i}\, 2^{-(k+1)} \sum_{j=0}^{k+1} \binom{k+1}{j} Z^{j+1} \tag{16}$$

$$A_1^{c_i}(Z) = \sum_{i=0}^{N-1} \sum_{k=0}^{N-1-i} p_{k,\,N-1-i}\, 2^{-(k+1)} \sum_{j=0}^{k+1} \binom{k+1}{j} Z^{j+1} \,. \tag{17}$$

When the uncoded information supplied to the outer encoder has Hamming weight $w \geq 2$, then, following the derivation of equation (14) in paper [14], we obtain, after multiplying $B_w(Z)$ of paper [14] by Z^w and changing N to $\frac{N}{2}$, the corresponding conditional weight enumeration function of the outer code as

$$A_w^{c_0}(Z) = \binom{\frac{N}{2}}{w} \sum_{k=0}^{\frac{N}{2}} q_k^w(c_0)\, 2^{-k} \sum_{j=0}^{k} \binom{k}{j} Z^{j+w} \quad \text{for} \quad 2 \leq w \leq \frac{N}{2} \,. \tag{18}$$

Similarly, because the uncoded input to the inner encoder is of length N, when its Hamming weight is $\ell \geq 2$, then the conditional weight enumeration function of the inner code is

$$A_\ell^{c_i}(Z) = \binom{N}{\ell} \sum_{k=0}^{N} q_k^\ell(c_i)\, 2^{-k} \sum_{j=0}^{k} \binom{k}{j} Z^{j+\ell} \quad \text{for} \quad 2 \leq \ell \leq N \,. \tag{19}$$

To apply equation (1), which relates the weight enumeration function of the serially concatenated codes to the weight enumeration functions of its component codes, we need first to find the coefficients $A_{\ell,h}^{c_i}$ and $A_{w,\ell}^{c_o}$ for any combination of values of w, ℓ, h, with the aid of (15)–(19). However, since the component codes are systematic, the Hamming weight of a codeword is not less than the Hamming weight of its information bits. This implies that $A_{w,\ell}^{c_o} = 0$ and $A_{\ell,h}^{c_i} = 0$ for $w > \ell$ and $\ell > h$, respectively. Also, since the length of a codeword received at the output of the outer encoder is N and the length of the uncoded information sequence supplied to the outer encoder is $\frac{N}{2}$, then the systematic feature of the outer code implies that the Hamming weight of a codeword received at the output of the outer encoder is greater than the Hamming weight of its information bits by not more than

$\frac{N}{2}$ (the number of parity bits). This implies that $A^{c_o}_{w,\ell} = 0$, if $\ell > \frac{N}{2} + w$. Because $w \leq \frac{N}{2}$ (since the Hamming weight of the input sequence of the outer encoder is not greater than its length), the inequality $\frac{N}{2} + w \leq N$ follows. Because we are considering rate $\frac{1}{2}$ inner and outer codes, the Hamming weight of the serially concatenated code cannot exceed $\frac{3N}{2} + w$ (as $\frac{3N}{2}$ is the number of parity bits of the serially concatenated code). Therefore, equation (1) can be simplified, under our assumptions, to yield

$$A^{c_s}_{w,h} = \sum_{\ell=w}^{\min\left(\frac{N}{2}+w,h\right)} \frac{A^{c_0}_{w,\ell}\, A^{c_i}_{\ell,h}}{\binom{N}{\ell}} \tag{20}$$

for values of w and h that satisfy $1 \leq w \leq \frac{N}{2}$ and $w \leq h \leq \frac{3N}{2} + w$. On the other hand, if $w > h$ or $h > \frac{3N}{2} + w$, then $A^{c_s}_{w,h} = 0$ as explained above.

We proceed now to derive some expressions required in the analysis. From (16) and (17), we obtain

$$\begin{aligned}
A^{c_0}_{1,\ell} &= \sum_{i=0}^{\frac{N}{2}-1} \sum_{k=0}^{\frac{N}{2}-1-i} p_{k,\,\frac{N}{2}-1-i}\, 2^{-(k+1)} \binom{k+1}{\ell-1} \quad \text{for} \quad 1 \leq \ell \leq N/2+1 \\
A^{c_i}_{1,h} &= \sum_{i=0}^{N-1} \sum_{k=0}^{N-1-i} p_{k,\,N-1-i}\, 2^{-(k+1)} \binom{k+1}{h-1} \quad \text{for} \quad 1 \leq h \leq 3N/2+1\,.
\end{aligned}$$

But the binomial coefficient $\binom{n}{m}$ is zero for $n < m$, so that

$$\left\{\begin{aligned}
A^{c_0}_{1,1} &= \sum_{i=0}^{\frac{N}{2}-1} \sum_{k=0}^{\frac{N}{2}-1-i} p_{k,\,\frac{N}{2}-1-i}\, 2^{-(k+1)} \\
A^{c_i}_{1,1} &= \sum_{i=0}^{N-1} \sum_{k=0}^{N-1-i} p_{k,\,N-1-i}\, 2^{-(k+1)} \\
A^{c_o}_{1,\ell} &= \sum_{i=0}^{\frac{N}{2}-1} \sum_{k=\ell-2}^{\frac{N}{2}-1-i} p_{k,\,\frac{N}{2}-1-i}\, 2^{-(k+1)} \binom{k+1}{\ell-1} \quad \text{for} \quad 2 \leq \ell \leq \tfrac{N}{2}+1 \\
A^{c_i}_{1,h} &= \sum_{i=0}^{N-1} \sum_{k=h-2}^{N-1-i} p_{k,\,N-1-i}\, 2^{-(k+1)} \binom{k+1}{h-1} \quad \text{for} \quad 2 \leq h \leq \tfrac{3N}{2}+1
\end{aligned}\right. . \tag{21}$$

Obviously, according or our assumptions, if $\ell > \frac{N}{2} + 1$ or $h > \frac{3N}{2} + 1$, then $A^{c_0}_{1,\ell} = 0$ or

$A^{c_i}_{1,h} = 0$, respectively. Similarly, (18) and (19) yield

$$A^{c_0}_{w,\ell} = \binom{\frac{N}{2}}{w} \sum_{k=0}^{\frac{N}{2}} q^w_k(c_0)\, 2^{-k} \binom{k}{\ell - w} \quad \text{for} \quad 2 \leq w \leq \frac{N}{2} \quad \text{and} \quad w \leq \ell \leq w + \frac{N}{2}\,. \tag{22}$$

$$A^{c_i}_{\ell,h} = \binom{N}{\ell} \sum_{k=0}^{N} q^\ell_k(c_i)\, 2^{-k} \binom{k}{h - \ell} \quad \text{for} \quad 2 \leq \ell \leq N \quad \text{and} \quad \ell \leq h \leq \ell + N\,. \tag{23}$$

Because $q^w_k(c_0)$ and $q^\ell_k(c_i)$ are by definition zero for $k < w$ or $k < \ell$, respectively, and because the binomial coefficients $\binom{k}{\ell - w}$ and $\binom{k}{h - \ell}$ are zero for $k < \ell - w$ or $k < h - \ell$, respectively, it follows that

$$A^{c_0}_{w,\ell} = \binom{\frac{N}{2}}{w} \sum_{k=\max(w,\ell-w)}^{\frac{N}{2}} q^w_k(c_0)\, 2^{-k} \binom{k}{\ell - w} \quad \text{for } 2 \leq w \leq \frac{N}{2} \text{ and } w \leq \ell \leq w + \frac{N}{2} \tag{24}$$

$$A^{c_i}_{\ell,h} = \binom{N}{\ell} \sum_{k=\max(\ell,h-\ell)}^{N} q^\ell_k(c_i)\, 2^{-k} \binom{k}{h - \ell} \quad \text{for } 2 \leq \ell \leq N \text{ and } \ell \leq h \leq \ell + N\,. \tag{25}$$

Equations (20),(24) and (25), for integer values of w, h such that $2 \leq w \leq \frac{N}{2}$ and $w \leq h \leq \frac{3N}{2} + w$, yield

$$A^{c_s}_{w,h} = \binom{\frac{N}{2}}{w} \sum_{\ell=\max(w,h-N)}^{\min\left(\frac{N}{2}+w,h\right)} \left\{ \sum_{k_1=\max(w,\ell-w)}^{\frac{N}{2}} q^w_{k_1}(c_0)\, 2^{-k_1} \binom{k_1}{\ell - w} \cdot \sum_{k_2=\max(\ell,h-\ell)}^{N} q^\ell_{k_2}(c_i)\, 2^{-k_2} \binom{k_2}{h - \ell} \right\}. \tag{26}$$

Combining (20) and (21) gives

$$A^{c_s}_{1,1} = \frac{A^{c_0}_{1,1}\, A^{c_i}_{1,1}}{N} = \frac{1}{4N} \left(\sum_{i_1=0}^{\frac{N}{2}-1} \sum_{k_1=0}^{\frac{N}{2}-1-i_1} 2^{-k_1}\, p_{k_1,\, \frac{N}{2}-1-i_1} \right) \left(\sum_{i_2=0}^{N-1} \sum_{k_2=0}^{N-1-i_2} 2^{-k_2}\, p_{k_2,\, N-1-i_2} \right). \tag{27}$$

Finally, from (20),(21) and (25), we conclude that, for values of h such that $2 \leq h \leq \frac{3N}{2}$,

$$A^{c_s}_{1,h} = \sum_{\ell=1}^{\min\left(\frac{N}{2}+1,h\right)} \frac{A^{c_0}_{1,\ell}\, A^{c_i}_{\ell,h}}{\binom{N}{\ell}} = \frac{A^{c_0}_{1,1}\, A^{c_i}_{1,h}}{N} + \sum_{\ell=2}^{\min\left(\frac{N}{2}+1,h\right)} \frac{A^{c_0}_{1,\ell}\, A^{c_i}_{\ell,h}}{\binom{N}{\ell}}$$

$$A^{c_s}_{1,h} = \frac{1}{N}\left(\sum_{i_1=0}^{\frac{N}{2}-1}\sum_{k_1=0}^{\frac{N}{2}-1-i_1} p_{k_1,\,\frac{N}{2}-1-i_1}\,2^{-(k_1+1)}\right)\left(\sum_{i_2=0}^{N-1}\sum_{k_2=h-2}^{N-1-i_2} p_{k_2,\,N-1-i_2}\,2^{-(k_2+1)}\,\binom{k_2+1}{h-1}\right)$$
$$+\sum_{\ell=2}^{\min\left(\frac{N}{2}+1,h\right)}\left\{\left(\sum_{i_1=0}^{\frac{N}{2}-1}\sum_{k_1=\ell-2}^{\frac{N}{2}-1-i_1} p_{k_1,\,\frac{N}{2}-1-i_1}\,2^{-(k_1+1)}\,\binom{k_1+1}{\ell-1}\right)\sum_{k_2=\max(\ell,h-\ell)}^{N} 2^{-k_2}\,q^{\ell}_{k_2}(c_i)\,\binom{k_2}{h-\ell}\right\} \tag{28}$$

for integer values of h, such that $2 \leq h \leq \frac{3N}{2}+1$.

Equations (26),(27) and (28) provide the foundation for the evaluation of the input-output weight enumeration function of serially concatenated codes, when the ensemble of rate $\frac{1}{2}$ inner and outer time-varying RSC codes is considered with a random interleaver of length N. From the definition of the weight enumeration function, we obtain

$$A^{c_s}(W,Z) = \sum_{w,h} A^{c_s}_{w,h}\,W^w\,Z^h = 1 + A^{c_s}_{1,1}\,WZ + W\sum_{h=2}^{\frac{3N}{2}+1} A^{c_s}_{1,h}\,Z^h + \sum_{w=2}^{\frac{N}{2}}\sum_{h=w}^{\frac{3N}{2}+w} A^{c_s}_{w,h}\,W^w\,Z^h \tag{29}$$

where the first, second, third and fourth terms above are based on equations (15),(27),(28) and (26) respectively.

The union upper bound on the bit error probability of the ensemble of serially concatenated turbo codes for the AWGN channel based on maximum likelihood decoding and coherent detection of antipodal signals may now be formulated as

$$P_b \leq \frac{2}{N}\sum_{w=1}^{\frac{N}{2}}\sum_{h=w}^{\frac{3N}{2}+w} w\,A^{c_s}_{w,h}\,Q\left(\sqrt{\frac{2hRE_b}{N_0}}\right) \tag{30}$$

where we have normalized by the input sequence length $\frac{N}{2}$ at the input of the outer encoder.

IV. Results

Fig. 3 compares the normalized average distance spectrum of serially concatenated codes in Fig. 2b (with component codes that are random time-varying RSC codes of rate $\frac{1}{2}$, memory length $m = 5$ and a uniform interleaver of length $N = 200, 400$ bits) with the normalized binomial distribution of a fully random binary block code of the same length $n = 2N$ and rate $R = \frac{1}{4}$. The match of the two curves is quite good for Hamming distances larger than twice the Gilbert-Varshamov (GV) distance, i.e. for normalized distances that are above

$2h^{-1}(1-R) = 0.429$, where $R = \frac{1}{4}$ bit/symbol. In Fig. 4a the block length is $n = 2N = 400$ and, for Hamming distances larger than $0.429 \cdot 400 = 171.6$, there is a good match between the two curves.

Fig. 4 shows that there is a good match between the normalized average distance spectrum of serially concatenated codes and the normalized distribution of a fully random block code of the same rate for Hamming distances that are adequately large. Yet, the normalized average distance spectrum of the random parallel or serial concatenated codes becomes significantly larger than the corresponding normalized binomial distribution, especially for relatively low values of Hamming weights (see Figs. 3,4), and this relative increase may explain an inherent degradation in performance as compared to optimal even fully random block codes.

As shown in Section IIc, the value of $\frac{E_b}{N_0}$ that corresponds to the cutoff rate is 2.03 dB or 1.85 dB for an overall code rate of $R = \frac{1}{3}$ or $R = \frac{1}{4}$, respectively. From these results and from Figures 3,4,7,8,9 and 10, we see again that the union bound is useless at rates exceeding the cutoff rate. Truncating the summation of terms associated with the union bound may lead to somewhat misleading results at rates above the cutoff rate [2].

Fig. 5 shows the behavior of the probabilities p_k versus k for various values of m. The exponential decay of the probabilities p_k with k is obvious from this figure. The smaller the memory length m, the faster the probabilities p_k decay with k. On the other hand, for large values of memory length, the curves of p_k versus k decay very slowly. It can also be seen from the curves of Fig. 5 that $p_m = 2^{-m}$ and $p_{m+1} = \frac{p_m}{2}$. Fig. 6 shows the minimal number of steps required such that a time varying recursive shift register with memory length of m will return to the zero state with a probability that is not less than a given threshold p for two different values of p.

Another interesting fact, which is also derived from the properties of the probabilities $\{p_k\}$, is that increasing the memory length of the component codes above $\log_2 N$ is not very effective since it affects negligibly the union ensemble performance of concatenated turbo codes while increasing the decoding complexity. This result has been demonstrated for parallel concatenated turbo codes [14] without explicit analytic justification. This result was proved analytically here by showing first that the average activity time of a time-varying recursive convolutional code with a memory length of m is $2^{m+1} - 2$, thus exponentially increasing with m. Choosing $m = \lfloor \log_2 N \rfloor$ yields an average activity time of at least N. Following the explanation in [14], this result explains why, for values of m exceeding $\lfloor \log_2 N \rfloor$, the weight distribution is dominated by the distribution of the first non-zero input (since with high probability, the encoder once activated will again become inactive within the observation interval). Therefore, it follows analytically (see appendix A) that a reasonably good choice of m is $\lfloor \log_2 N \rfloor$. The illustration of this result for serially concatenated turbo codes is given in Fig. 7 for a uniform interleaver of length $N = 200$ bits. For this case, the recommended values of m is 7. Increasing the memory length by just 1 (above the corresponding recommended value of m) gives a slight coding gain of only 0.02 dB. It is also clear from the curve in Fig. 7 that the coding gain achieved by increasing the memory length from an arbitrary value of m to $m+1$ decreases while m is increased.

The influence of the value of the interleaver length N on the random-ensemble performance of serially concatenated turbo codes is illustrated by Figs. 8 and 9. In Fig. 8 the ensemble performance (union bound) of serially concatenated codes of rate 1/4 and $m = 5$ is shown for $N = 10, 20, 50, 100$ and 200. Fig. 9 demonstrates the asymptotic performance for these parameters. We notice, as predicted by the conventional union bound, for rates below the cutoff rate, that the random-ensemble performance of serial concatenated turbo codes improves with the increase of the interleaver length N. Increasing N is penalized by increased memory demands. This improvement was predicted theoretically in [2] where it was shown that, for serially concatenated codes with recursive convolutional inner codes, every term of the bit error probability appearing in the union bound decreases asymptotically at least as $N^{-\left\lfloor \frac{d_f^{(0)}+1}{2} \right\rfloor}$ where $d_f^{(0)}$ is the free distance of the outer code and $\lfloor x \rfloor$ is the integer part of x.

Finally, the asymptotic behavior of the bit error rate is shown for random interleaver lengths $N = 10, 20, 50, 100$ and 200 in Fig. 10, where $m = 5$, $R = \frac{1}{4}$. Note that the slope of the curve steepens with the increase of $\frac{E_s}{N_0}$. The fact that increasing N does not reflect the expected minimum distance behavior of $N^{\frac{d_f^{(0)}-2}{d_f^{(0)}}}$ established in [10] can be attributed to the ensemble averaging which is dominated by relatively 'bad' codes with a small minimum distance, though their total probability of appearance is apparently low. Most codes (with probability approaching 1) follow the minimum distance behavior of [10]. For the particular case of the original turbo codes with $K = 2$ (see Fig. 2a), one sees from [10] that the minimum distance depends neither on the interleaver length nor on the component codes. On the other hand, for serially concatenated codes, we see that we can achieve growth rate of the minimum distance close to linear ($d_{\min} \approx N$) if we pick an outer code with large free distance, which is also consistent with the result derived in [2]. Therefore, for high signal-to-noise ratios, the relatively modest minimum distance of most turbo codes with $K = 2$ components implies that most serially concatenated codes with RSC codes as component codes will perform better than for parallel concatenation.

Fig. 10 depicts a comparison between the union bounds on the bit error probability of the random serial and parallel concatenated codes. The component codes of the two random concatenated codes are time-varying RSC codes with memory length of $m = 3, 4, 5$ and 7 and of rate $\frac{1}{2}$ (the component codes in each of the four cases have the same memory length). The serially concatenated codes are generated by a uniform choice over all possible interleavers of length $N_1 = 200$ (as in Fig. 2b). The parallel concatenated codes are generated by a uniform choice over all possible interleavers of length $N_2 = 100$ and three ($K = 3$) component codes. The two random interleavers of length N_1 of the parallel concatenated turbo codes are chosen uniformly and independently (see Fig. 2c). This comparison between serially and parallel concatenated turbo codes is done under equal rate and interleaving delay. The union bounds on the bit-error probability of the serially concatenated turbo codes are slightly smaller than those for the parallel concatenation at low and moderate values of $\frac{E_b}{N_0}$ (at rates above the cut-off rate), except for $m = 3$. Moreover, the difference between the curves of the parallel and serially concatenated codes for low and moderate values of $\frac{E_b}{N_0}$ increases with the memory length m of the component codes. However, since the union bound is not tight for rates above the cutoff rate, we were motivated to investigate these concatenated codes by improved upper bounds [12]. For rates slightly below the cutoff rate, the union bounds indicate that the parallel concatenation is advantageous over the serial concatenation (see Fig. 10).

V. Summary and Conclusions

The input-output weight enumeration function of the ensemble of serially concatenated RSC codes was derived. This ensemble is generated by a uniform choice over all possible interleavers of length N and a uniform choice over all the component codes taken from the set of time-varying RSC codes of rate $\frac{1}{2}$ and memory length m.

Similarly to the result for parallel concatenation in [14], the performance behavior of random serially concatenated RSC codes was reduced to a two parameter family: the interleaver length N and the memory length m of the inner and outer codes of the serial concatenation, assumed here to be equal.

The union bound was applied to provide an upper bound on the bit-error probability of the ensemble of serially concatenated RSC codes for the binary-input AWGN channel assuming *maximum likelihood* decoding and coherent detection of antipodal signals.

We compared the ensemble performance of random parallel concatenated multiple-turbo codes with three RSC component codes to the case of random serially concatenated RSC codes (under the base of the same interleaving delay and code rate). The union bounds demonstrate analytically that serial concatenation is preferable over parallel concatenation for moderate values of $\frac{E_b}{N_0}$. Further work based on improved tangential sphere upper bounds shows that this result remains valid [12].

It was demonstrated that increasing the memory length m of the component codes above $\lfloor \log_2 N \rfloor$ is not effective, since it improves negligibly the ensemble performance of the concatenated turbo codes while increasing considerably the decoding complexity of the codes.

The influence of the interleaver length N on the ensemble performance of concatenated turbo codes was also investigated. It was shown that at rates below the cutoff rate the ensemble performance of the concatenated turbo codes is improved by increasing the value of N. This conforms with the results of [2],[10] and [16].

It was demonstrated that there is a good match between the normalized average distance spectrum of random serially concatenated codes and the normalized binomial distribution of a fully random block code of the same length and rate for Hamming distances that are adequately large. The two distance spectra nearly coincides for Hamming distances larger than twice the Gilbert-Varshamov distance, i.e., for normalized Hamming distances above $2h^{-1}(1-R)$ (where R is the overall code rate in bit/symbol and h^{-1} is the inverse of the binary entropy function). However, the normalized average distance spectrum of random serially concatenated codes becomes significantly larger than the corresponding normalized binomial distribution especially for relatively low values of Hamming weights. This observation may explain an inherent degradation in performance compared to optimal even fully random binary block codes.

Acknowledgement. We are indebted to James Massey for his very constructive comments, for providing reference [11] and an elegant alternative way to evaluate the mean activity time, which is incorporated in the text.

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Figure Captions

Figure 1: A time-varying recursive shift register of length m.
The state equations of the shift register at time n:

$$\begin{cases} s_1(n+1) = \sum_{k=1}^{m} s_k(n)\, c_k(n) + x(n) \\ s_k(n+1) = s_{k-1}(n) \quad k = 2, 3, \ldots, m \\ y(n) = s_m(n) \;, \end{cases}$$

when $c_k(n)$ $(k = 1, 2, \ldots, m)$ are i.i.d random variables that get the values '0' or '1' with the same probability $\left(\frac{1}{2}\right)$, for every moment of time n.

Figure 2: Serial and parallel concatenated turbo codes.
a. Parallel concatenation of rate $\frac{1}{2}$ time-varying RSC codes ([7]) of overall rate $\frac{1}{3}$.
b. Serial concatenation of rate $\frac{1}{2}$ time-varying RSC codes of overall rate $\frac{1}{4}$ (discussed here).
c. Parallel concatenation of these components of rate $\frac{1}{2}$ time-varying RSC codes of overall rate $\frac{1}{4}$.

Figure 3: a. A comparison between the normalized average distance spectrum of random parallel concatenated code (memory length of its components: $m = 5$, overall rate: $R = \frac{1}{3}$ and a random interleaver of length $N = 200$) and the normalized binomial distribution of a fully random block code of the same code length $n = 600$ bits and of the same rate.
b. A comparison between the normalized average distance spectrum of random parallel concatenated code (memory length of its components: $m = 5$, overall rate: $R = \frac{1}{3}$ and a random interleaver of length $N = 400$) and the normalized binomial distribution of a fully random block code of the same code length $n = 1200$ bits and of the same rate.

Figure 4: The normalized average distance spectrum of random serially concatenated codes with inner and outer codes that are time varying recursive systematic convolutional codes of rate $\frac{1}{2}$ and of memory length $m = 5$. The overall rate is $R = \frac{1}{4}$ and a random interleaver between the component codes is of length $N = 50, 100, 200, 400$ bits.

Figure 5: The minimal number of cycles required such that a time varying recursive shift-register with memory length of m will return to the zero state (after being activated), with a probability that is not less than a given threshold.

Figure 6: The ensemble performance of serially concatenated turbo codes with an overall rate of $\frac{1}{4}$, as a function of $m = 3, 4, 5, 6, 7, 8$ - the memory length of its components. The length of the random uniform interleaver is $N = 200$.

Figure 7: Comparison of the ensemble performance of serially concatenated turbo codes with overall rate of $\frac{1}{4}$, fixed memory length of component codes ($m = 5$) and different values of an interleaver length ($N = 10, 20, 50, 100, 200$).

Figure 8: The asymptotic (union bound) behavior of the bit error probability for the ensemble of serially concatenated turbo codes with overall rate of $\frac{1}{4}$, fixed memory length of component codes $m = 5$ and different interleaver lengths, $N = 10, 20, 50, 100, 200$.

Figure 9: A comparison between the union bounds on the bit error probabilities of serially concatenated random codes with inner and outer codes of rate $\frac{1}{2}$ (an overall rate of $\frac{1}{4}$) and a random uniform interleaver of length $N = 200$ and of parallel concatenated random (turbo) codes with three component codes of rate $\frac{1}{2}$ (the same overall rate $R = \frac{1}{4}$) and two random uniform interleavers of length $N = 100$ (the same interleaving delay), as a function of the memory length of its components $m = 3, 4, 5, 7$.

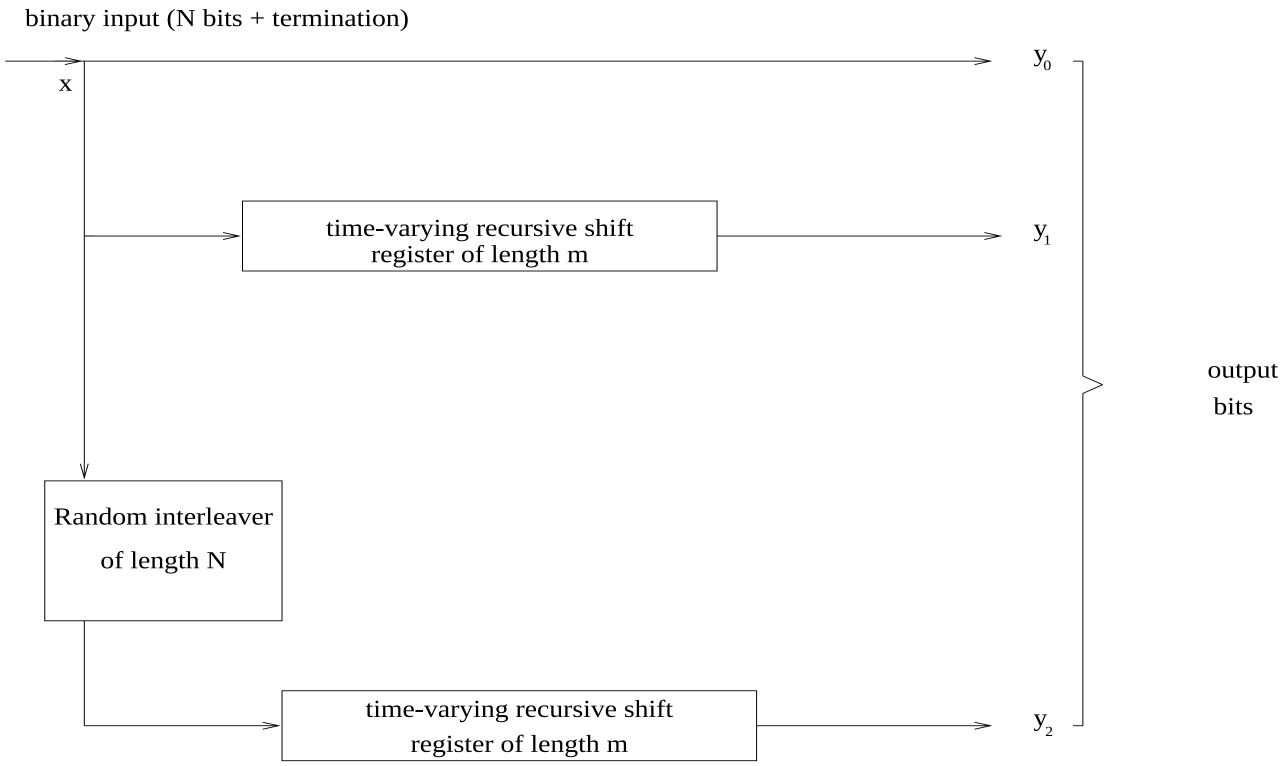
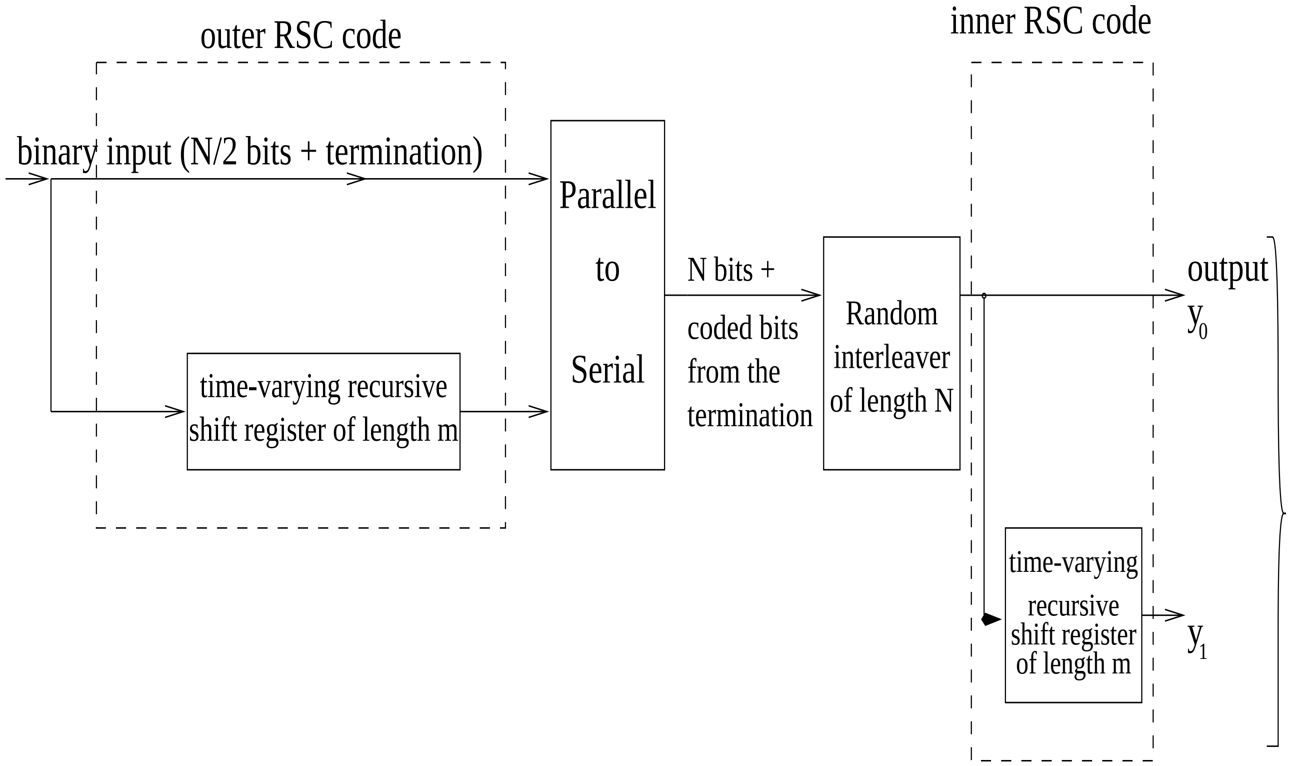
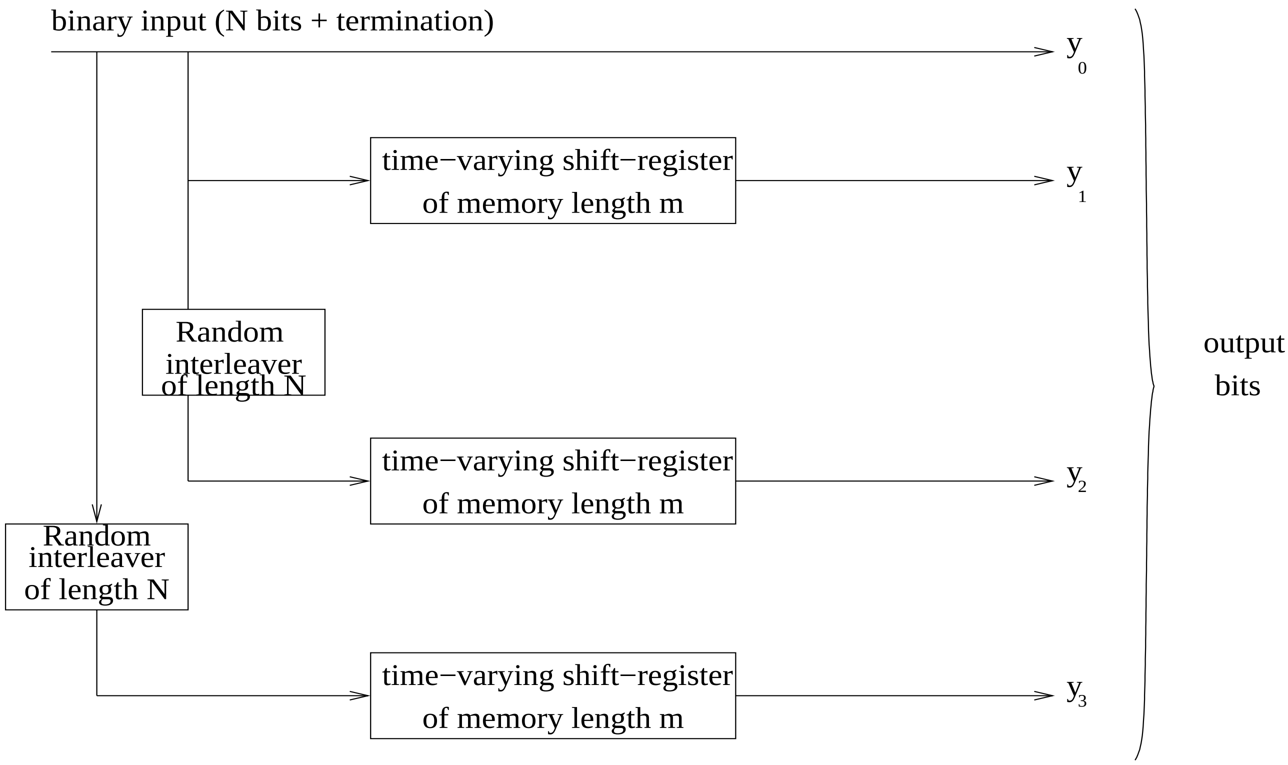


Figure 1: Serial and parallel concatenated turbo codes.

a. Parallel concatenation of unpunctured rate $\frac{1}{2}$ time-varying RSC codes ([7]), with an overall code rate of $\frac{1}{3}$.

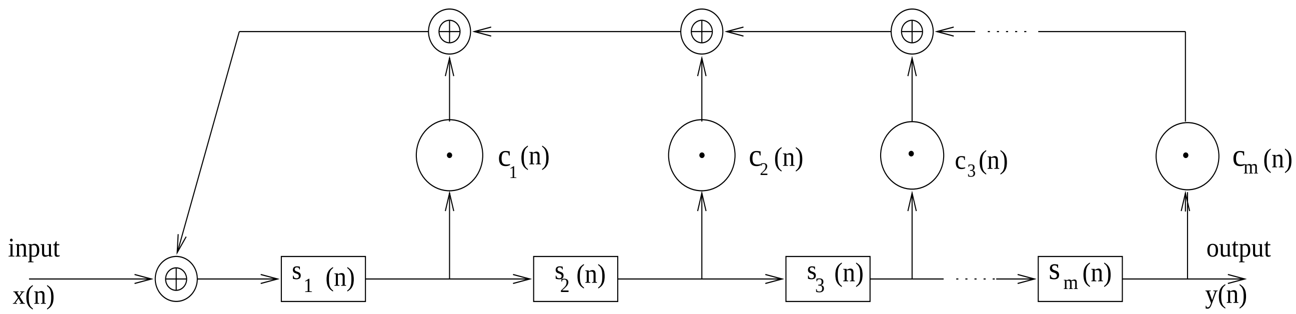


(b)



(c)

Serial and parallel concatenated turbo codes.
b. Serial concatenation of overall rate $\frac{1}{4}$ of rate $\frac{1}{2}$ time-varying RSC codes (discussed here).
c. Parallel concatenation of overall rate $\frac{1}{4}$ of these components of rate $\frac{1}{2}$ time-varying RSC codes.



(a)

Figure 2: a time-varying recursive shift register of length m.

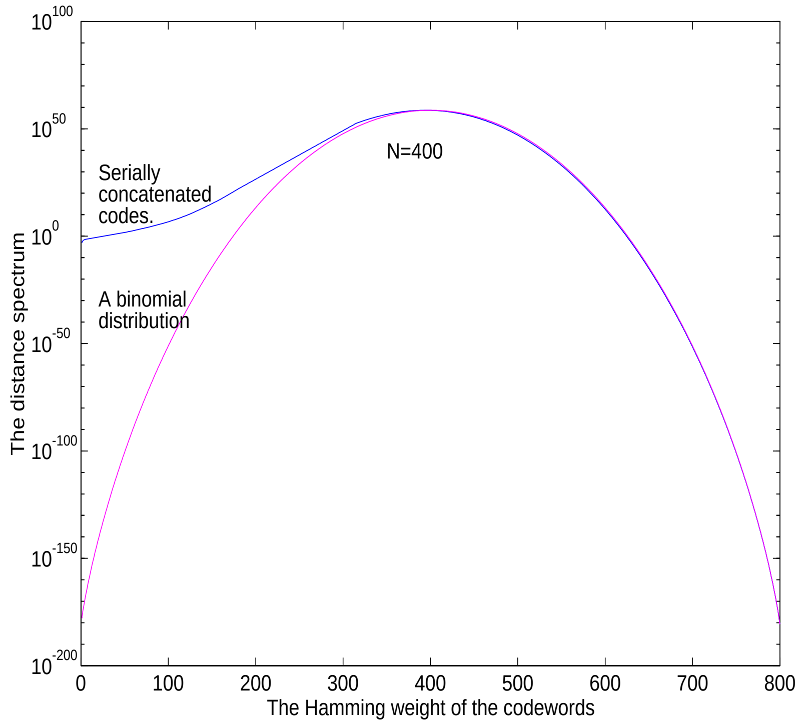
The state equations of the shift register at time n:

$$s_1(n+1) = \sum_{k=1}^{m} s_k(n)\, c_k(n) + x(n)$$

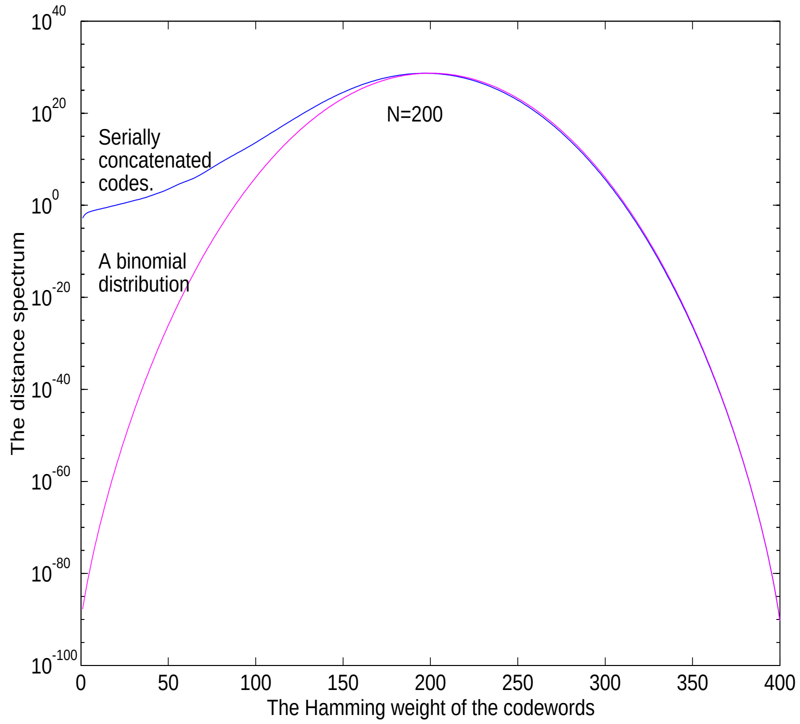
$$s_k(n+1) = s_{k-1}(n) \quad k = 2, 3, \ldots, m$$

$$y(n) = s_m(n)$$

where $c_k(n)$ $(k = 1, 2, \ldots, m)$ are i.i.d random variables that take on the values '0' or '1' with the same probability $\left(\frac{1}{2}\right)$ at every time instant n.



(a)



(b)

Figure 3: a. A comparison between the normalized average distance spectrum of random parallel concatenated code (memory length of its components: $m = 5$, overall rate: $R = \frac{1}{3}$ and a random interleaver of length $N = 200$) and the normalized binomial distribution of a fully random block code of the same code length $n = 600$ bits and of the same rate.
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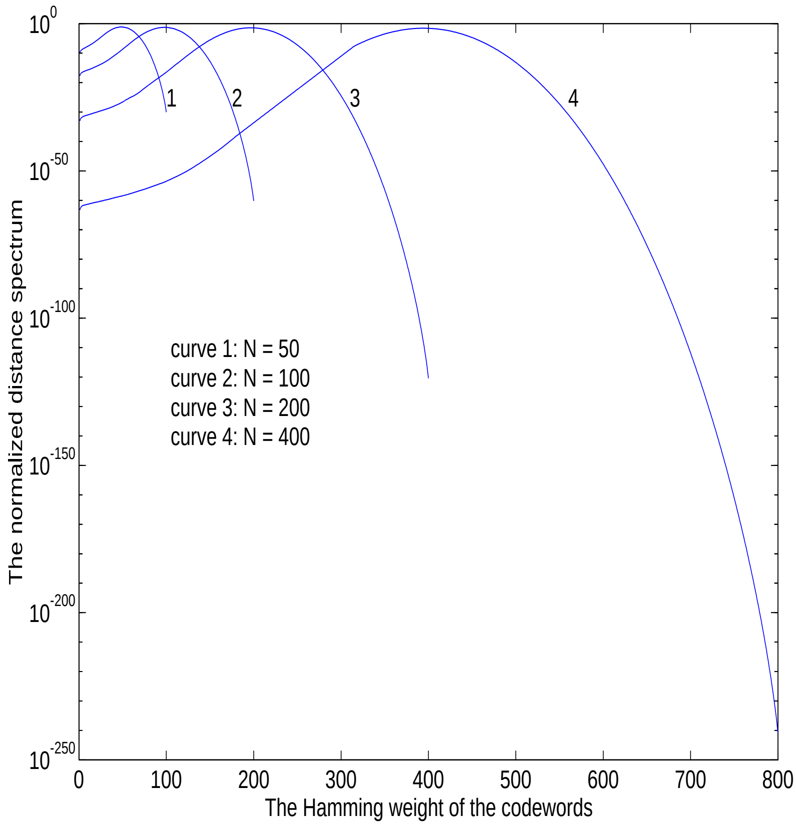

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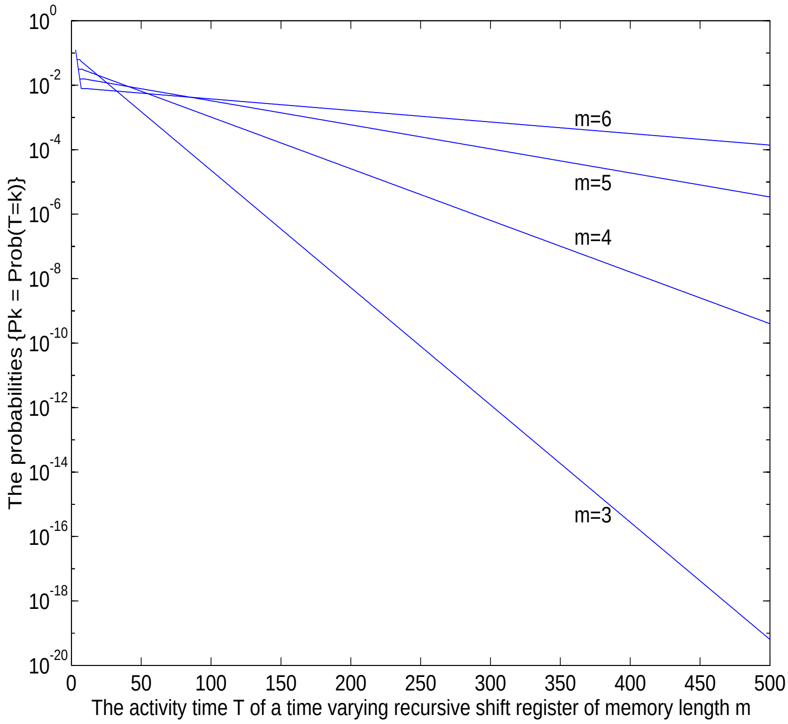
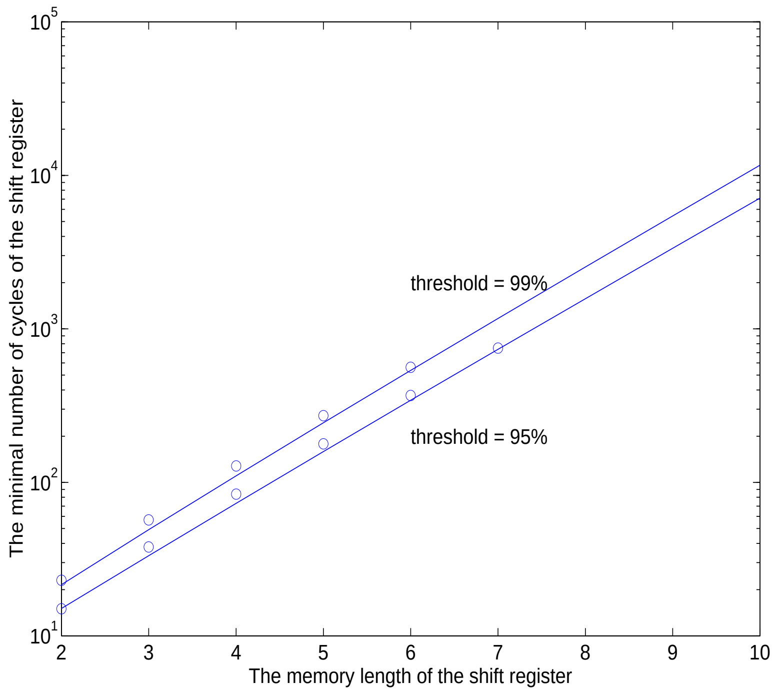


Figure 5: The probabilities p_k versus k, for different values of memory length $m = 3, 4, 5, 6$.



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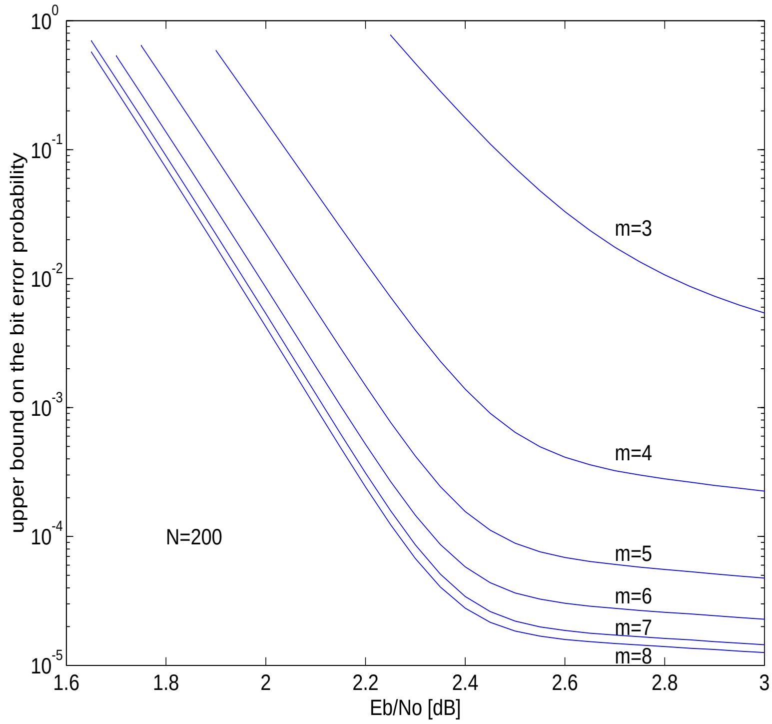

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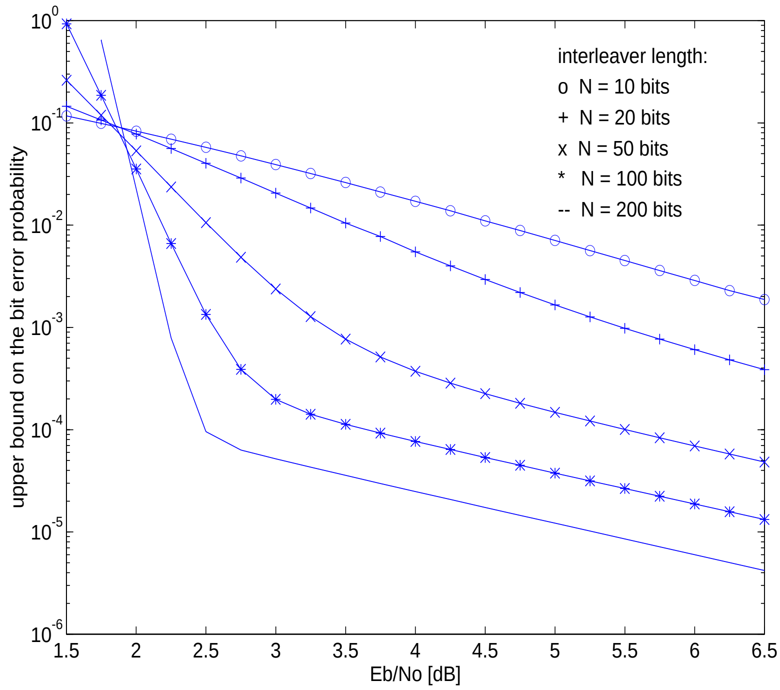

Figure 8: Comparison of the ensemble performance of serially concatenated turbo codes with overall rate of $\frac{1}{4}$, fixed memory length of component codes ($m = 5$) and different values of an interleaver length ($N = 10, 20, 50, 100, 200$).

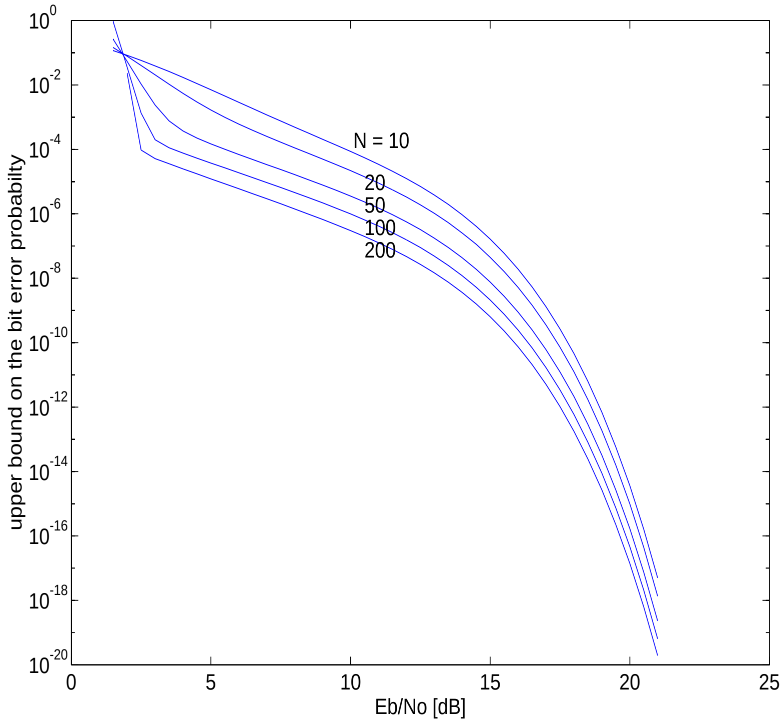

Figure 9: The asymptotic (union bound) behavior of the bit error probability for the ensemble of serially concatenated turbo codes with overall rate of $\frac{1}{4}$, fixed memory length of component codes $m = 5$ and different interleaver lengths, $N = 10, 20, 50, 100, 200$.

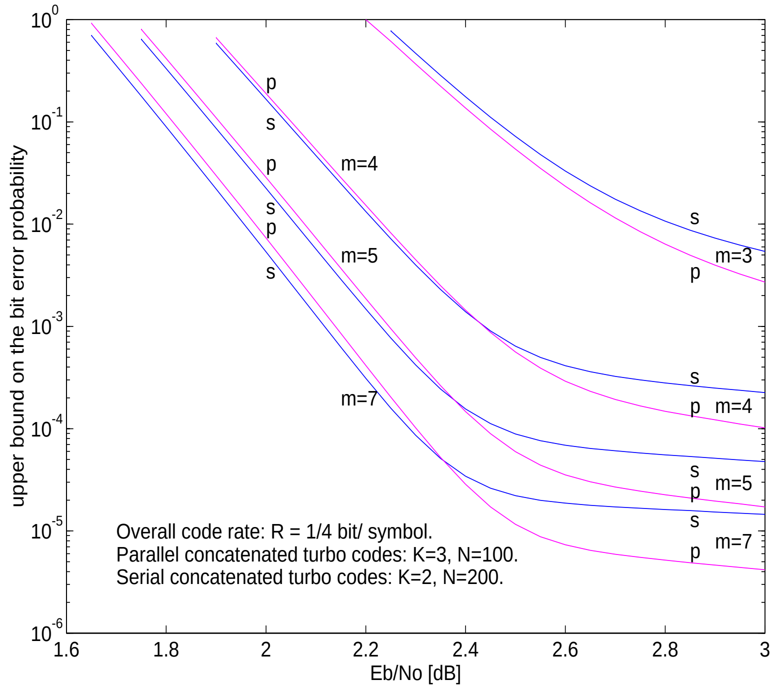

Figure 10: A comparison between the union bounds on the bit error probabilities of serially concatenated random codes with inner and outer codes of rate $\frac{1}{2}$ (an overall rate of $\frac{1}{4}$) and a random uniform interleaver of length $N = 200$ and of parallel concatenated random (turbo) codes with three component codes of rate $\frac{1}{2}$ (the same overall rate $R = \frac{1}{4}$) and two random uniform interleavers of length $N = 100$ (the same interleaving delay), as a function of the memory length of its components $m = 3, 4, 5, 7$.