Exponential Bounds for Discrete Time, Conditionally Symmetric Martingales with Bounded Increments

Igal Sason

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel

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A Simple Example to Motivate this Work

Let

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- $d > 0$ and $\gamma \in (0, 1]$ be fixed numbers.
- $\{U_k\}_{k \in \mathbb{N}}$ be i.i.d. random variables where

$$\mathbb{P}(U_k = +d) = \mathbb{P}(U_k = -d) = \frac{\gamma}{2}, \quad \mathbb{P}(U_k = 0) = 1 - \gamma, \quad \forall\, k \in \mathbb{N}.$$

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Consider the Markov process $\{X_k\}_{k \geq 0}$ where $X_0 = 0$ and

$$X_k = X_{k-1} + U_k, \quad \forall k \in \mathbb{N}.$$

Large Deviation Analysis

From Cramér's theorem in $\mathbb{R}$, for every $\alpha \geq \mathbb{E}[U_1] = 0$,

$$\lim_{n \to \infty} -\frac{1}{n} \ln \mathbb{P}(X_n \geq \alpha n) = I(\alpha)$$

where the rate function is given by

$$I(\alpha) = \sup_{t \geq 0} \left\{ t\alpha - \ln \mathbb{E}[\exp(tU_1)] \right\}.$$

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Let $\delta \triangleq \frac{\alpha}{d}$. Calculation shows that

$$I(\alpha) = \delta x - \ln\Big(1 + \gamma\big[\cosh(x) - 1\big]\Big)$$

$$x \triangleq \ln\left(\frac{\delta(1-\gamma) + \sqrt{\delta^2(1-\gamma)^2 + \gamma^2(1-\delta^2)}}{\gamma(1-\delta)}\right).$$

Large Deviation Analysis (Cont.)

Lets examine the martingale approach in this simple case, where

$$X_k = \sum_{i=1}^{k} U_i, \quad \forall\, k \in \mathbb{N}, \quad X_0 = 0$$

with the natural filtration

$$\mathcal{F}_k = \sigma(U_1, \ldots, U_k), \quad \forall\, k \in \mathbb{N}, \quad \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

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In this case, the martingale has bounded increments, and for every $k \in \mathbb{N}$

$$\begin{aligned} &|X_k - X_{k-1}| \leq d \\ &\mathbb{E}[(X_k - X_{k-1})^2 \,|\, \mathcal{F}_{k-1}] = \mathsf{Var}(U_k) = \gamma d^2 \end{aligned}$$

almost surely.

Large Deviation Analysis (Cont.)

Via the martingale approach, the following exponential inequality follows:

$$\mathbb{P}\left(X_n - X_0 \geq \alpha n\right) \leq \exp\left(-n\, D\left(\frac{\delta+\gamma}{1+\gamma} \Big|\Big| \frac{\gamma}{1+\gamma}\right)\right)$$

where $\delta \triangleq \frac{\alpha}{d}$, and

$$D(p||q) \triangleq p \ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right), \quad \forall\, p, q \in [0,1]$$

is the divergence (relative entropy) between the probability distributions $(p, 1-p)$ and $(q, 1-q)$. If $\delta > 1$, then the above probability is zero.

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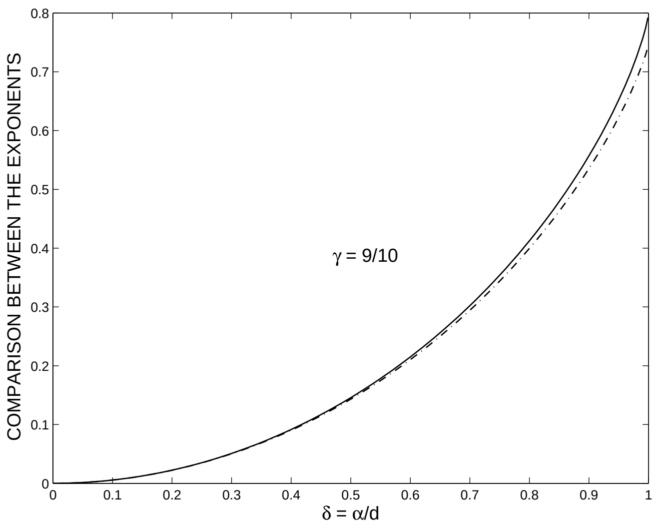
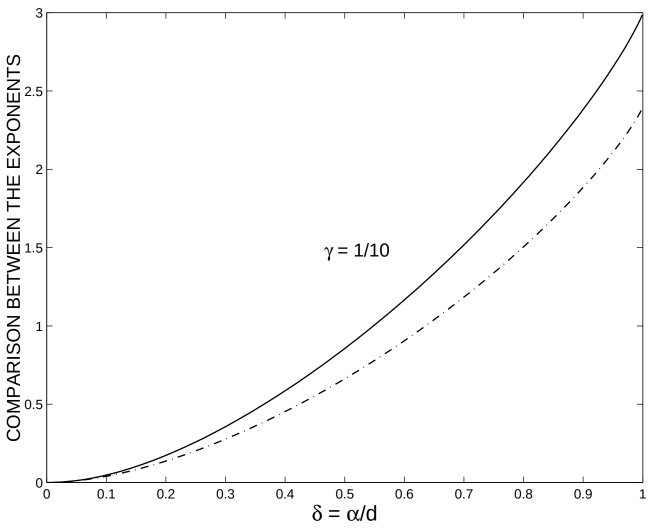
is the divergence (relative entropy) between the probability distributions $(p, 1-p)$ and $(q, 1-q)$. If $\delta > 1$, then the above probability is zero.

This exponential bound is <u>not tight (unless $\gamma = 1$)</u>, and the gap to the exact asymptotic exponent increases as the value of $\gamma \in (0, 1]$ is decreased.

Comparison of the Exact Exponent and Its Lower Bound



Comparison of the Exact Exponent and Its Lower Bound (Cont.)



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- Why the lower bound on the exponent is not asymptotically tight ?
- Why this gap increases as the value of γ is decreased ?
- How this situation can be improved via the martingale approach ?

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This exponential bound relies on Bennett's inequality: Since $U_k \leq d$ and $\text{Var}(U_k) = \gamma d^2$, then

$$\mathbb{E}\left[\exp(tU_k)\right] \leq \frac{\gamma \exp(td) + \exp(-\gamma td)}{1+\gamma}, \quad \forall\, t \geq 0.$$

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The probability distribution that achieves Bennett's inequality with equality is asymmetric (unless $\gamma = 1$), and is equal to

$$\mathbb{P}(\tilde{U}_k = d) = \frac{\gamma}{1+\gamma}, \quad \mathbb{P}(\tilde{U}_k = -\gamma d) = \frac{1}{1+\gamma}.$$

By reducing the value of $\gamma \in (0,1]$, the above asymmetry grows. Indeed, this enlarges the gap to the exact asymptotic exponent.

Interim Conclusion

An improvement is likely to be obtained by a refinement of Bennett's inequality when X_k is conditionally symmetric given $\mathcal{F}_{k-1}$.

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Definition: Conditionally Symmetric Martingales

Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$, where $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$, be a discrete-time and real-valued martingale, and let $\xi_k \triangleq X_k - X_{k-1}$ for every $k \in \mathbb{N}$. Then $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ is a *conditionally symmetric martingale* if, conditioned on $\mathcal{F}_{k-1}$, the RV ξ_k is symmetrically distributed around zero.

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An improvement is likely to be obtained by a refinement of Bennett's inequality when X_k is conditionally symmetric given $\mathcal{F}_{k-1}$.

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Definition: Conditionally Symmetric Sub/ Supermartingales

Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a discrete-time real-valued sub or supermartingale, and let $\eta_k \triangleq X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}]$ for every $k \in \mathbb{N}$. Then it is conditionally symmetric if, conditioned on $\mathcal{F}_{k-1}$, the RV η_k is symmetrically distributed around zero.

Construction of Conditionally Symmetric Martingales

Example 1: Let

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space
- $\{U_k\}_{k \in \mathbb{N}} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ be independent random variables that are symmetrically distributed around zero
- $\{\mathcal{F}_k\}_{k \geq 0}$ be the natural filtration where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(U_1, \ldots, U_k), \ \forall\, k \in \mathbb{N}$.
- For $k \in \mathbb{N}$, let $A_k \in L^\infty(\Omega, \mathcal{F}_{k-1}, \mathbb{P})$ be an $\mathcal{F}_{k-1}$-measurable random variable with a finite essential supremum.

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- For $k \in \mathbb{N}$, let $A_k \in L^\infty(\Omega, \mathcal{F}_{k-1}, \mathbb{P})$ be an $\mathcal{F}_{k-1}$-measurable random variable with a finite essential supremum.

Define a new sequence of random variables in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ where

$$X_n = \sum_{k=1}^{n} A_k U_k, \quad \forall\, n \in \mathbb{N}$$

and $X_0 = 0$. Then, $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a conditionally symmetric martingale.

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Example 2: Let

- $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ be a conditionally symmetric martingale
- $A_k \in L^\infty(\Omega, \mathcal{F}_{k-1}, \mathbb{P})$ be an $\mathcal{F}_{k-1}$-measurable random variable with a finite essential supremum.

Define $Y_0 = 0$ and $Y_n = \sum_{k=1}^n A_k(X_k - X_{k-1}), \quad \forall\, n \in \mathbb{N}$.
Then, $\{Y_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a conditionally symmetric martingale.

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Then, $\{Y_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a conditionally symmetric martingale.

Example 3: A sampled Brownian motion is a discrete time, conditionally symmetric martingale with un-bounded increments.

Goal and Related Literature

- Our goal is to demonstrate how the assumption of the conditional symmetry improves existing exponential inequalities for discrete-time real-valued martingales with bounded increments.
- Earlier results, serving as motivation, appear in the works by
  1. V. H. de la Pena (Annals of Probability 1999, 2004)
  2. K. Dzhaparide and J. H. van Zanten (SPA, 2001).
- The new exponential bounds in this work are also extended to conditionally symmetric sub or supermartingales.
- Additional results addressing weak-type inequalities, maximal inequalities and ratio inequalities for conditionally symmetric martingales were derived by
  1. A. Oskowski (2010)
  2. G. Wang (Proceedings of the AMS, 1991).

Exponential Inequalities for Conditionally Symmetric Martingales

Lemma

Let X be a real-valued RV with a symmetric distribution around zero, a support $[-d, d]$, and assume Var$(X) \leq \gamma d^2$ for some $d > 0$ and $\gamma \in [0, 1]$. Let h be a real-valued convex function, and assume that $h(d^2) \geq h(0)$. Then,

$$\mathbb{E}[h(X^2)] \leq (1 - \gamma)h(0) + \gamma h(d^2)$$

with equality if $\mathbb{P}(X = \pm d) = \frac{\gamma}{2}, \quad \mathbb{P}(X = 0) = 1 - \gamma.$

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Proof

- h convex, $X \in [-d, d]$ a.s. $\Rightarrow h(X^2) \leq h(0) + \left(\frac{X}{d}\right)^2 \big(h(d^2) - h(0)\big)$.
- Taking expectations on both sides gives the inequality, with an equality for the above symmetric distribution.

Exp. Inequalities for Conditionally Symmetric Martingales (Cont.)

Corollary

Let X be a real-valued RV with a symmetric distribution around zero, a support $[-d, d]$, and assume Var$(X) \leq \gamma d^2$ for some $d > 0$ and $\gamma \in [0, 1]$. Then,

$$\mathbb{E}\big[\exp(\lambda X)\big] \leq 1 + \gamma\big[\cosh(\lambda d) - 1\big], \quad \forall\, \lambda \in \mathbb{R}$$

with equality if $\mathbb{P}(X = \pm d) = \frac{\gamma}{2}, \quad \mathbb{P}(X = 0) = 1 - \gamma.$

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Proof

- Symmetric distribution of $X \quad \Rightarrow \quad \mathbb{E}\big[\exp(\lambda X)\big] = \mathbb{E}\big[\cosh(\lambda X)\big].$
- The corollary follows from the lemma since, for every $x \in \mathbb{R}$, $\cosh(\lambda x) = h(x^2)$ where $h(x) \triangleq \sum_{n=0}^{\infty} \frac{\lambda^{2n} |x|^n}{(2n)!}$ is a convex function, and $h(d^2) = \cosh(\lambda d) \geq 1 = h(0)$.

Theorem 1

Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a discrete-time real-valued and conditionally symmetric martingale. Assume that, for some fixed numbers $d, \sigma > 0$, the following two requirements are satisfied a.s.

$$|X_k - X_{k-1}| \leq d, \qquad \text{Var}(X_k | \mathcal{F}_{k-1}) = \mathbb{E}\big[(X_k - X_{k-1})^2 \,|\, \mathcal{F}_{k-1}\big] \leq \sigma^2$$

for every $k \in \mathbb{N}$.

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for every $k \in \mathbb{N}$. Then, for every $\alpha \ge 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{1\le k\le n} |X_k - X_0| \ge \alpha n\right) \le 2\exp\big(-nE(\gamma,\delta)\big)$$

where

$$\gamma \triangleq \frac{\sigma^2}{d^2}, \quad \delta \triangleq \frac{\alpha}{d}$$

and for $\gamma \in (0,1]$ and $\delta \in [0,1)$, the exponent $E(\gamma,\delta)$ is given as follows:

Theorem 1 (Cont.)

$$E(\gamma, \delta) \triangleq \delta x - \ln\Big(1 + \gamma\big[\cosh(x) - 1\big]\Big)$$

$$x \triangleq \ln\left(\frac{\delta(1-\gamma) + \sqrt{\delta^2(1-\gamma)^2 + \gamma^2(1-\delta^2)}}{\gamma(1-\delta)}\right).$$

If $\delta > 1$, then the probability is zero (so $E(\gamma, \delta) \triangleq +\infty$), and $E(\gamma, 1) = \ln\big(\frac{2}{\gamma}\big)$.

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Asymptotic Optimality of the Exponent

The example we studied earlier shows that the exponent of the bound in Theorem 1 is asymptotically optimal. That is, there exists a conditionally symmetric martingale, satisfying the conditions in this theorem, that attains the exponent $E(\gamma, \delta)$ in the limit where $n \to \infty$.

Theorem 1 should be compared to the following existing bound which does not require the conditional symmetry property.

Theorem 2

[McDiarmid 1989, book of Dembo & Zeitouni (Corollary 2.4.7)]

Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a discrete-time real-valued martingale with bounded jumps. Assume that the two conditions on the bounded increments and conditional variance from Theorem 1 are satisfied a.s. for every $k \in \mathbb{N}$.

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$$\mathbb{P}\left(\max_{1\leq k\leq n} |X_k - X_0| \geq \alpha n\right) \leq 2\exp\left(-n\, D\left(\frac{\delta+\gamma}{1+\gamma}\Big|\Big|\frac{\gamma}{1+\gamma}\right)\right)$$

where γ and δ are introduced in Theorem 1, and

$$D(p||q) \triangleq p\ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right), \quad \forall\, p, q \in [0,1].$$

If $\delta > 1$, then the probability is zero.

Proof Technique for Theorems 1 and 2

- Define $X_n - X_0 = \sum_{k=1}^{n} \xi_k$ where $\xi_k \triangleq X_k - X_{k-1}$.

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- $|\xi_k| \le d$, $\mathbb{E}\big[\xi_k \,|\, \mathcal{F}_{k-1}\big] = 0$, $\text{Var}(\xi_k \,|\, \mathcal{F}_{k-1}) \le \gamma d^2$. The RV ξ_k is $\mathcal{F}_k$-measurable.

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- Exponentiation and the maximal inequality for submartingales give that, for every $\alpha \geq 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} (X_k - X_0) \geq n\alpha\right) \leq e^{-n\alpha t}\, \mathbb{E}\left[\exp\left(t \sum_{k=1}^{n} \xi_k\right)\right], \quad \forall\, t \geq 0.$$

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- Since $\{\mathcal{F}_i\}$ forms a filtration, then for every $t \geq 0$

$$\mathbb{E}\left[\exp\left(t \sum_{k=1}^{n} \xi_k\right)\right] = \mathbb{E}\left[\exp\left(t \sum_{k=1}^{n-1} \xi_k\right) \mathbb{E}\big[\exp(t\xi_n) \,|\, \mathcal{F}_{n-1}\big]\right].$$

Proof Technique for Theorems 1 and 2 (Cont.)

- For proving Theorem 2, the use of Bennett's inequality gives

$$\mathbb{E}\left[\exp(t\xi_k) \,|\, \mathcal{F}_{k-1}\right] \leq \frac{\gamma \exp(td) + \exp(-\gamma td)}{1+\gamma}.$$

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- For the refinement in Theorem 1 for conditionally symmetric martingales with bounded increments, use of Lemma 1 gives

$$\mathbb{E}\left[\exp(t\xi_k) \,|\, \mathcal{F}_{k-1}\right] \leq 1 + \gamma\big[\cosh(td) - 1\big].$$

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- In both cases, one gets exponential bounds with the free parameter t.
- Optimization over $t \geq 0$ & union bound to get two-sided inequalities.
- The asymptotic optimality of the exponent in Theorem 1 follows from the simple example that was shown earlier.

Theorem 3

Let $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ be a discrete-time real-valued and conditionally symmetric martingale. Assume that there exists a fixed number $d > 0$ such that $\xi_k \triangleq X_k - X_{k-1} \leq d$ a.s. for every $k \in \mathbb{N}$. Let

$$Q_n \triangleq \sum_{k=1}^{n} \mathbb{E}[\xi_k^2 \,|\, \mathcal{F}_{k-1}]$$

with $Q_0 \triangleq 0$, be the predictable quadratic variation of the martingale up to time n.

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$$Q_n \triangleq \sum_{k=1}^{n} \mathbb{E}[\xi_k^2 \,|\, \mathcal{F}_{k-1}]$$

with $Q_0 \triangleq 0$, be the predictable quadratic variation of the martingale up to time n. Then, for every $z, r > 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} (X_k - X_0) \geq z,\, Q_n \leq r \text{ for some } n \in \mathbb{N}\right) \leq \exp\left(-\frac{z^2}{2r} \cdot C\left(\frac{zd}{r}\right)\right)$$

where

$$C(u) \triangleq \frac{2[u \sinh^{-1}(u) - \sqrt{1+u^2} + 1]}{u^2}, \quad \forall\, u > 0.$$

Theorem 3 refines the following result that was stated without the requirement for the conditional symmetry of the martingale.

Theorem 4 (Freedman 1975, Dembo & Zeitouni's book (Ex. 2.4.21))

Let $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ be a discrete-time real-valued martingale. Assume that there exists a fixed number $d > 0$ such that $\xi_k \triangleq X_k - X_{k-1} \leq d$ a.s. for every $k \in \mathbb{N}$. Then, for every $z, r > 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} (X_k - X_0) \geq z, \, Q_n \leq r \text{ for some } n \in \mathbb{N}\right) \leq \exp\left(-\frac{z^2}{2r} \cdot B\left(\frac{zd}{r}\right)\right)$$

where

$$B(u) \triangleq \frac{2[(1+u)\ln(1+u) - u]}{u^2}, \quad \forall \, u > 0.$$

Related Refs. to this Work & Conditionally Symmetric Martingales

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