SEQUENTIAL HYPOTHESIS TESTING AND CHANNEL CODING WITH FEEDBACK

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"No more fiction for us: we calculate; but that we may calculate, we had to make fiction first".

$\big($*Friedrich Wilhelm Nietzsche, 1844–1900*$\big)$

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Abstract

In this thesis, we review concepts and methods of both sequential hypothesis testing and communication systems with feedback, where variable length coding is used. We begin with a concise review of many aspects of sequential hypothesis testing problems, starting with the problem of sequentially inferring between two simple hypotheses, through problems of composite and multiple hypotheses, and ending with problems in which there exists some feedback mechanism that enables the receiver to control future observations, using the knowledge gathered thus far. The discussion regarding hypothesis testing is concentrated around tests with the property that, as the expected number of observation grows, the average error probability tends to zero, and most of the results stated willhold under account this asymptotic regime. Nevertheless, some non-asymptotic results will also be presented, along with very basic and simple sequential tests that have been proposed in the literature. One of the main goals is to illuminate the interplay between sequential hypothesis testing and variable length coding communication problem. This connection is established in the second part of the thesis, along with a review of some important and relevant results regarding communication systems with feedback. In addition to a review of existing work, we provide some novel results. Specifically, a new communication scheme for variable length coding is proposed and is proven optimal in the error-exponent sense. The novelty here is that it is purely sequential. In addition, for the "stop-feedback" constraint, in which only one feedback bit per message is allowed, bounds on the error exponent function are obtained in the random coding regime.

Summary of Abbreviations

SPRT	Sequential Probability Ratio Test
LR	Likelihood Ratio
LRT	Likelihood Ratio Test
LLR	Log Likelihood Ratio
i.i.d.	Independent and Identically Distributed
MGF	Moment Generating Function
EB	Exponentially Bounded
NP	Neyman-Pearson
UMP	Uniformly Most Powerful
GLR	Generalized Likelihood Ratio
GLRT	Generalized Likelihood Ratio Test
ML	Maximum Likelihood
WSPRT	Weighted Sequential Probability Ratio Test
MSPRT	Multiple Hypotheses Sequential Probability Ratio Test
KL	Kullback-Leibler
DP	Dynamic Programming
MDP	Markov Decision Process
POMDP	Partially Observed Markov Decision Process
DMC	Discrete Memoryless Channel
AWGN	Additive White Gaussian Noise
AWGNC	Additive White Gaussian Noise Channel
VL	Variable Length
ACK	Acknowledgement
NACK	unacknowledge
BSC	Binary Symmetric Channel
GV	Gilbert-Varshamov
ARQ	Automatic Repeat reQuest
SP	Sphere Packing
MMI	Maximum Mutual Information
LDPC	Low Density Parity Check
VNC	Very Noisy Channel
IR-ARQ	Incremental Redundancy Automatic Repeat reQuest
MAP	Maximum A Posteriori

Chapter 1

Introduction

This work is an overview on the interplay between two fields that attracted a lot of attention in the last three decades: sequential hypothesis testing and variable length channel coding with feedback. Although the two problems have been widely studied separately over the years, the number of works in which the two are combined is relatively small. This is somewhat surprising since there is an obvious similarity between the two problems, as will be discussed in Part II of this work. Nevertheless, over the last few years, some researchers have harnessed results from sequential hypothesis testing to obtain bounds on performance of communication systems with feedback and variable block length. These results, along with the strong analogy between the two problems, is the main motivation behind this work. We aimed at the following goals: (a) to give a concise and comprehensive summary of results on statistical sequential hypothesis testing with special focus on those that help in the analysis of communication systems, and (b) to show how some of these results were utilized in both classic and state-of-the-art work. However, the scope of this work does not enable one to cover the entire volume of literature on sequential hypothesis testing and coding with feedback. Therefore we have limited the discussion only to the interplay between sequential discrimination between simple hypotheses, and its connection to the error exponent function of some variable-length schemes with feedback. In this sense, Chapter 3 is an exception, since it deals with composite hypotheses. The reason this topic is covered is twofold: first, these types of hypotheses play a pivotal role in sequential hypothesis testing theory, and, second, results on composite hypothesis testing may be useful both in gaining intuition on general hypothesis testing problems and in communication with feedback where the decoder chooses a set of messages (also known as list-decoding) rather than just one message.

The operational meaning of the aforementioned focus on simple hypotheses and the exponential behaviour of the average error probability, is that we consider only hypotheses of the form:

$$H_i \;:\; \Pr(\mathbf{Y}) = \mathbf{P_i}(\mathbf{Y}), \tag{1.0.1}$$

where the different hypotheses are denoted by $\{H_i\}_{i=0}^{M-1}$, and $\{\mathbf{P_i}(\mathbf{Y})\}_{i=0}^{M-1}$ are known probability distributions. Here $\mathbf{Y} = Y_1, Y_2, \ldots$ is assumed an infinite sequence of random variables, distributed according to one of the $\mathbf{P_i}$'s. The objective is to find both a stopping rule and a decision rule that guarantees that the average stopping time would be as small as possible and the average probability of error would be as small as possible. The rigorous mathematical formulation of these requirements will be given in the first part of this work.

In the first part, four families of sequential hypothesis testing problems will be covered:

Binary hypothesis testing: For the case of two hypothesis only, Wald's test is optimal in the sense of the trade-off between the expected stopping time and the two kinds of error probabilities.

Multiple hypothesis testing: M hypotheses to be tested, where $M > 2$.

Multiple hypothesis testing with control: Testing among $M > 2$ hypotheses with some control over future observations. Given the observations $Y_1, \ldots, Y_{n-1}$, the experimenter can decide on one out of K actions to take, each one corresponding to a different statistical behavior of the next observation Y_n.

Composite hypothesis testing: Here, the hypotheses are of the form:

$$H_i \;:\; \Pr(\mathbf{Y}) = \mathbf{P}_\theta(\mathbf{Y}), \; \theta \in \Theta_i \tag{1.0.2}$$

where the Θ_i's are are unknown.

Since an optimal test has not been found only for the binary case, the part of the work in which multiple simple hypotheses are discussed will focus on practical examples that were analysed over the years, as well as some tests with "good behavior" at least asymptotically. Although less attention was given to the problem of composite hypothesis testing, the interested reader can still find results in that field in the sequel, along with a list of references to some of the leading work on the subject.

In the second part of this work, we deal with communication problems in which the decoder takes as many observations as needed to obtain a reliable decision. The fidelity criteria under which different decoding schemes are compared to each other is the error exponent function, defined as $\lim \frac{\log(P_{\text{error}})}{\mathbb{E}[N]}$, where P_{error} and $\mathbb{E}[N]$ denote, respectively, the average error probability and the average time it takes the decoder to make its decision; the limit is taken as $\mathbb{E}[N] \to \infty$ and $P_{\text{error}} \to 0$, in a way that ensures that the rate is constant. The available feedback is assumed to be utilized in one of the following ways:

Perfect instantaneous feedback, in which the feedback channel has zero delay, unlimited capacity, and it is available at any time instant. For this type of feedback, the error exponent function is known exactly for all rates. A few different proofs of this claim are covered in the sequel, where each of them illuminates this result in a different light, using various mathematical techniques and giving rise to a different intuitive explanation. A novel scheme that uses sequential analysis tools is also given.

Single bit feedback, in which the feedback is limited to one bit per message. This restricts the use of feedback to the case where the decoder cannot control the symbols sent through the channel, but only signal back as soon as it has gathered enough samples to make a decision. For this feedback channel, two types of coding and decoding schemes are addressed. The first is where coding is done in blocks, i.e., where a codebook is chosen in advance and some codeword is sent consecutively until the decoder makes a decision. The second is more general and it is named "stop-feedback scheme". Here, each message is mapped to a codeword that is infinitely long, and the decoder can stop its transmission at *any* time and declare its estimate. Some novel results are obtained regarding the best achievable error exponent.

The remainder of the thesis is organized as follows: In Part I, we deal mostly with sequential hypothesis testing. In Section 2 the sequential binary hypothesis testing problem is addressed. Some of the most fundamental relations of sequential analysis are given, along with the definition and optimality property of Wald's test. Section 3 describes composite hypothesis testing, along with a unified theory of asymptotic optimality. In Section 4, we return to simple hypotheses. We begin the discussion with three simple sequential tests,are easy to implement, and discuss their performance. In addition, two asymptotically optimal tests are given. We conclude with a review of results on multiple hypotheses with observation control. Part II is devoted to variable-length coding. In Section 5 the case of perfect feedback is discussed. Specifically, an optimality claim on the error exponent function in such a setup is made, along with outlines of some of its known proofs. A novel sequential coding scheme is also shown to achieve optimal performance. In Section 6, the problem of limited feedback is reviewed. The main focus is on feedback channels over which only a single bit per symbol is allowed to be transmitted. Some classical and state of the art results are discussed, and novel results regarding a certain feedback scheme are given. Chapter 7 concludes the thesis, and it suggests topics for further research.

Part I

Sequential Hypothesis Testing

Chapter 2

Sequential Binary Hypothesis Testing

2.1 Introduction

In traditional binary hypothesis testing, after a random sample is observed, one of the two possible actions is taken: accept the null hypothesis H_0, or accept the alternative hypothesis H_1. We will assume a Bayesian framework where prior probabilities are assigned to each hypothesis. In the case of simple hypotheses, the strength of the evidence for H_1 is given by likelihood ratio function, which is defined as the ratio of the probability of the data under H_1 and the probability of the data under H_0. We will denote this function by λ. For the non-Bayesian setting, the Neyman-Pearson lemma implies that the likelihood ratio test (LRT) is the most powerful test. The LRT accepts H_1 if λ is large enough, and H_0 otherwise. However, in some cases the evidence regarding H_0 and H_1 may not be convincing, nevertheless, a decision must be made. In sequential tests, there is a third possible course of action when the evidence is ambiguous: take more observations. Such a test is typically continued until the evidence strongly favors one of the two hypotheses.

There are various sequential procedures for deciding when to continue sampling. Each such procedure consists of a stopping rule and a terminal decision rule. The nature of the latter depends on the nature of the problem. On the other hand, the mathematical structure stopping rule is mostly common to all statistical procedures and is a prescription adopted by the "experimenter" when to stop the sampling. Any stopping rule gives rise to a random sample size N, also called *stopping time*. It has to satisfy the condition that the event $\{N = n\}$ depends only on $Y_1, Y_2, \ldots, Y_n$. In general, the event $\{N = \infty\}$ is allowed, and we then say that the sequential procedure does not terminate. If $P(N = \infty) = 0$, we say that the procedure *terminates with probability 1* under P.

In 1943, Wald [149] proposed the *sequential probability ratio test* (SPRT). In its simplest form, the SPRT is described as follows. Suppose that Y is a random variable (or vector) distributed according to f (that is, f is a probability mass function if Y is discrete or a density if Y is continuous). The problem is to test the hypothesis H_0: $f = P$ against H_1: $f = Q$, where P and Q are specified. The instructions provided by the SPRT are: observe values of Y successively where each sample is independent of the rest. We denote the observation sequence by $y_1, y_2, \ldots$. Then the random variables Y_i, corresponding to y_i, are independent and identically distributed (i.i.d.) with a common distribution f. Let $\lambda_n = \prod_{i=1}^{n} \frac{Q(y_i)}{P(y_i)}$ be the observed likelihood ratio at stage n. Choose two constants or *boundaries*, A and B, satisfying $0 < B < A < \infty$. Accept H_0 if $\lambda_n \leq B$, accept H_1 if $\lambda_n \geq A$, and continue to stage $n+1$ if $B < \lambda_n < A$. In practice, the constants A and B in the SPRT are determined by the desired error probabilities and do not depend on the distributions of the likelihood ratios $\lambda_1, \lambda_2, \ldots$. When A and B are replaced by A_n and B_n $(n \geq 1)$, one gets a *generalized* SPRT. A more formal and general definition of the SPRT is given in the next section, but many details of SPRTs have been omitted in the sequel. These can be found in numerous articles and books, for example, [59], [131], [146] , [155].

2.2 Sequential Likelihood-Ratio Tests - General Results

Although we will mostly focus on the i.i.d. case, there are important examples where this is not the case. In this section, we describe certain properties of SPRTs which hold under very general conditions on the sequence of observations. The results are direct consequences of the definition of likelihood ratio.Consider sequence of random variables (or vectors), not necessarily i.i.d. $Y_1, Y_2, \ldots$ on a measurable space $(\Omega, \mathcal{F})$. The hypotheses H_0 and H_1 are represented by probability measures P and Q satisfying $P \not\equiv Q$. We assume that hypothesis H_1 occurs with prior probability π, and H_0 with prior probability $1 - \pi$. We denote the joint distribution of $(Y_1, Y_2, \ldots, Y_n)$ under P and Q by P_n and Q_n, respectively, and denote the filtration generated by it by

$\{\mathcal{F}_n\ ,\ n=1,2,\ldots\}$, that is

$$\mathcal{F}_n=\sigma\left(Y_1,\ldots,Y_n\right),\ n=1,2,\ldots\ ,\mathcal{F}_0=\left\{\Omega,\emptyset\right\}. \tag{2.2.1}$$

For simplicity, we assume that for each n, P_n and Q_n are mutually absolutely continuous with a *likelihood ratio* $\lambda_n=\frac{dQ_n}{dP_n}$ (that is, we assume that the Radon-Nikodym derivative exists and is denoted by λ_n). In particular, if P_n has density f_n and Q_n has density g_n, then $\lambda_n=\frac{g_n(y_1\ldots,y_n)}{f_n(y_1\ldots,y_n)}$, which is the likelihood ratio defined earlier. We denote the random variables $L_n=\lambda_n\left(Y_1,\ldots,Y_n\right)$, defined on Ω, and for every $\mathscr{A}_n\in\mathcal{F}_n$

$$\int_{\mathscr{A}_n}L_ndP_n=Q_n\left(\mathscr{A}_n\right). \tag{2.2.2}$$

A *sequential test* is a pair (N,d) consisting of a *stopping time* $N\in\mathcal{N}$, where $\mathcal{N}$ denotes the set of all stopping times with respect to the filtration $\{\mathcal{F}_n\}$ (that is, for each n, the events $\{N=n\}$ and $\{N>n\}$ each belong to $\mathcal{F}_n$), and a *terminal decision rule* d, which is an $\mathcal{F}_N$-measurable random variable taking values in $\{0,1\}$ (such that the events $\{d=H_0\}\cap\{N=n\}$ and $\{d=H_1\}\cap\{N=n\}$ belong to $\mathcal{F}_n$ for each n). Let $\mathcal{D}$ denote the set of all such d. The random variable N designates the time to stop sampling, and once the value of N is given, d takes the value 0 or 1, depending on the two accepted hypotheses.

It then follows that

$$\begin{aligned} Q_n\left(N>n\right) &= \int_{\{N>n\}}L_ndP_n, && (2.2.3)\\ Q_n\left(\{d=H_1\}\cap\{N=n\}\right) &= \int_{\{d=H_1\}\cap\{N=n\}}L_ndP_n. && (2.2.4)\end{aligned}$$

Furthermore, equation (2.2.2) can also be modified in the following way

$$Q\left(\mathscr{A}\cap\{N<\infty\}\right)=\mathbb{E}_P\left[L_N\mathbb{I}\left\{\mathscr{A}\cap\{N<\infty\}\right\}\right], \tag{2.2.5}$$

where L_N denotes the likelihood ratio at the stopping time N (that is, $L_N=L_n$ on the set $\{N=n\}$). The relation (2.2.5) is known as *Wald's LR identity*, whose proof is in [155, Theorem 1.1] and in [131, Proposition 2.24].

Assuming $P\left(N<\infty\right)=1$, let $\mathscr{A}$ be such that $\mathscr{A}\cap\{N=n\}\in\mathcal{F}_n$. Then, a useful generalization of (2.2.2) is

$$\int_{\mathscr{A}}L_NdP=Q\left(\mathscr{A}\right). \tag{2.2.6}$$

In other words, the relation above holds for A determined by the observations up to the stopping time. Suppose next that H_0 is true. Then, the event that a decision rule d would make an error is called an *error of type 1* or *type 1 error*, and its probability is denoted by α, i.e,

$$\alpha = P(d = H_1). \tag{2.2.7}$$

Similarly, if H_1 is true, then the event that a decision rule d would make an error is called an *error of type 2* or *type 2 error*, and its probability is denoted by β, where

$$\beta = Q(d = H_0). \tag{2.2.8}$$

An important inequality follows from (2.2.6), Jensen's inequality and (2.2.7). For any sequential test with finite N (a.s.) and any convex function g, we have

$$\begin{aligned} \int_\Omega g(L_N)\, dP \;&\geq\; \alpha g\left(\frac{Q(d = H_1)}{\alpha}\right) + (1-\alpha)\, g\left(\frac{Q(d = H_0)}{1-\alpha}\right) & (2.2.9)\\ &=\; \alpha g\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\, g\left(\frac{\beta}{1-\alpha}\right). & (2.2.10)\end{aligned}$$

If g is strictly convex, equality holds only if L_N is constant on $\{d = H_0\}$ and on $\{d = H_1\}$. In particular, for $g(x) = -\log x$ one gets

$$\int_\Omega (\log L_N)\, dP \leq \alpha \log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right). \tag{2.2.11}$$

Using (2.2.11), it is possible to prove a weak, but general optimality property of the SPRT (e.g., [131]) and to obtain a lower bound on the stopping time of the SPRT. This will be discussed in the next subsection.

As the SPRT stops (that is $\{N = n\}$ is determined) at the first time instant satisfying $L_n \leq B$ with $d = H_0$, or $L_n \geq A$, with $d = H_1$, it holds that $B < L_n < A$ for $n < N$. From these simple relations, one can derive some important properties regarding the SPRT. For example, it follows from (2.2.3) that $BP(N > n) \leq Q(N > n) \leq AP(N > n)$ for each n. For $0 < B < A < \infty$, this implies that $P(N = \infty) = \lim_{n\to\infty} P(N > n) = 0$ if and only if $Q(N = \infty) = 0$. It also implies that for the average sample size $\mathbb{E}_P[N] < \infty$ if and only if $\mathbb{E}_Q[N] < \infty$, and that the stopping times are exponentially bounded[1] under P if and only if they are exponentially bounded under Q.

[1]A nonnegative random variable N is said to be *exponentially bounded* (EB) under probability P is there exists a constant $c > 0$ and $0 < \rho < 1$ such that $P(N > n) < c\rho^n$, $n = 1, 2, \ldots$. This property implies that the moment generating function (MGF) of N is finite on any subset of $\mathbb{R}$. As a consequence, all the moments of N are finite. A fortiori, $P(N < \infty) = 1$

Since $L_n \geq A$ on $\{d = H_1\} \cap \{N = n\}$, it follows from (2.2.4) that

$$Q\{d = H_1 \cap N < \infty\} \geq AP(d = H_1 \cap N < \infty). \tag{2.2.12}$$

Thus, if $P(N < \infty) = 1$, then $1 - \beta = Q(d = H_1) \geq AP(d = H_1) = A\alpha$. By a similar argument, one also gets $1 - \alpha = P(d = H_0) \geq B^{-1}Q(d = H_0) = \frac{\beta}{B}$. Altogether, we have, for the SPRT with finite stopping time,

$$1 - \beta \geq A\alpha, \quad 1 - \alpha \geq \frac{\beta}{B}. \tag{2.2.13}$$

These relations are fundamental in the construction and analysis of SPRTs. The inequalities show that, for given values of A and B, the possible values of α and β are limited to a convex set as is shown in Figure 2.2.1.

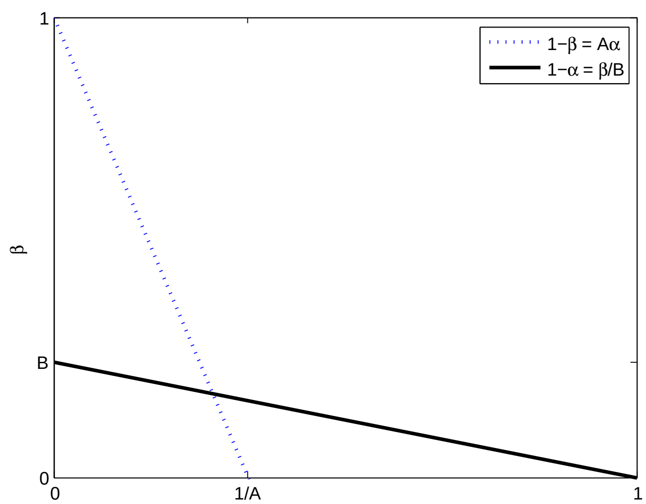

Figure 2.2.1: For given boundary values A and B, the region of the achievable values of (α, β) is given by the convex hull of the set $\left\{(0,0), (0, 1/A), \left(\frac{1-B}{A-B}, \frac{B(A-1)}{A-B}\right), (B, 0)\right\}$

An important question is how to choose A and B to achieve some given error probabilities (α_0, β_0). Although an exact answer to this question is not known in general, a "conservative" choice $A = \frac{1}{\alpha_0}$ and $B = \beta_0$ (and then $\alpha \leq \frac{1}{A} = \alpha_0$ and $\beta \leq B = \beta_0$).

Another choice of the boundaries is a consequence of a well known approximation, often called the *Wald approximation*. Under this approximation, the excess of the test statistic over the boundaries when the test ends is neglected, i.e., $L_N = A$ when

$d = H_1$ and $L_N = B$ when $d = H_0$ are assumed. These approximate relations yield the *Wald boundaries*

$$A = \frac{1-\beta_0}{\alpha_0}, \quad B = \frac{\beta_0}{1-\alpha_0}. \tag{2.2.14}$$

In this case, $\alpha = \alpha_0$ and $\beta = \beta_0$. This crude and somewhat heuristic approach took a more analytical form after results from renewal theory were harnessed to the field of sequential testing (see [155] for a comprehensive survey on the subject).

Even from the simple analysis leading to Figure 2.2.1, one can learn something about the nature of α and β when changing the A and B. It is clear that if one increases A and decreases B, the set of possible values of the error probabilities shrinks. It is possible, however, that the actual error probabilities will increase. In [48] it is shown that, if an alternative SPRT with boundaries A' and B' has $\alpha' \leq \alpha$ and $\beta' \leq \beta$ then $P(N \leq N') = 1$. It also follows that if $\alpha' = \alpha$ and $\beta' = \beta$ then $P(N = N', d = d') = 1$ so that, although the boundaries of the two tests may differ, the tests are equivalent.

2.3 Fundamental Results for Stopping Times of Random Walks

In many cases, it is both reasonable and simple to model the observations as a sequence of *i.i.d.* random variables (or vectors) distributed according to some probability measure P. The cumulative sum of such a sequence is a *random walk*. More precisely, let $Y_1, Y_2 \ldots$ be i.i.d. random variables distributed according to P with $\mathbb{E}_P[Y_1] = \mu_P$. We will denote $\hat{S}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. In general, we would like to evaluate decision rules and stopping times based on $\hat{S}_n$. The literature on random walks and stopped random walks is far too vast to cover in this work, so we cover here only very few important statements.

One property of interest is whether or not a sequential test terminates with probability 1. The following theorem by Chernoff [23] helps to answer this question, provided the moment generating function (MGF) $\varphi(t) = \mathbb{E}_P[\exp\{tY\}]$ exists and is finite for all $t \in \mathbb{R}$.

Theorem 1 *Let $Y_1, Y_2, \ldots$ be i.i.d. real-valued random variables with a finite MGF and distributed according to P. Let $\mathbb{E}_P[Y_1] = \mu_P$, and let $\hat{S}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Then, given any $x < \mu_P$, there exists $0 < \rho < 1$ such that*

- $P\left(\hat{S}_n < x\right) \leq \rho^n$ *for* $n = 1, 2, \ldots$

- $\frac{1}{n}\log\left[P\left(\hat{S}_n < x\right)\right] \to \log\rho$ *as* $n \to \infty$

Actually, Chernoff's theorem is stronger in that it also gives the value of ρ in terms of the MGF of Y. Many generalizations of Theorem 1 have been established over the years [38].

In [145], Wald also proved the following property, relating $S_n = \sum_{i=1}^{n} Y_i$ to a stopping time defined on it in terms of the MGF of Y:

Theorem 2 (Wald's identity) *If N is a stopping time for $S_1, S_2, \ldots$ such that $|S_n| < \gamma$ for any $n \in \mathbb{N}$ satisfying $n \leq N$ and $\mathbb{E}_P[N] < \infty$, then for any real $t \neq 0$ such that $1 \leq \varphi(t) < \infty$*

$$\mathbb{E}_P\left[e^{tS_N}\varphi(t)^{-N}\right] = 1. \tag{2.3.1}$$

Theorem 2 follows from the fact that for any (deterministic) time n, the process $\left(e^{tS_n}\varphi(t)^{-n}\right)_{n\geq 0}$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ (which is the σ-algebra generated by $Y_1, \ldots, Y_n$), and therefore (2.3.1) holds under any condition which implies the satisfiability of Doob's optional sampling theorem[2] (see, e.g., [152], Sec. 10.10) for this martingale. The proof of this theorem appears in [15].

Theorem 2 implies the following:

Corollary 3 *Suppose $\varphi(t) < \infty$ in a neighborhood of $t = 0$, then*

1. *If $P(Y > 0) > 0$, $P(Y < 0) > 0$, and $\mu_P \neq 0$ there exists two non-zero numbers t_0 and t_1 such that $\varphi(t_0) = \varphi(t_1) = 1$ and Wald's identity implies that*

$$\mathbb{E}_P\left[e^{t_0 S_N}\right] = \mathbb{E}_P\left[e^{t_1 S_N}\right] = 1. \tag{2.3.2}$$

2.

$$\mathbb{E}_P[S_N] = \mu_P \mathbb{E}_P[N]. \tag{2.3.3}$$

3. *Define $\sigma^2 \triangleq Var(Y_1)$. Then*

$$\mathbb{E}_P\left[(S_N - N\mu_P)^2\right] = \sigma^2 \mathbb{E}_P[N]. \tag{2.3.4}$$

[2]Generally, Doob's optional sampling theorem says that, under certain conditions, the expected value of a martingale at a stopping time is equal to the expected value of its initial value.

The relations (2.3.3) and (2.3.4) are often referred to as "Wald's first equation" and "Wald's second equation", respectively. The first part of this corollary is proved in [51], and parts 2 and 3 are simply applications of the monotone convergence theorem combined with Theorem 2 [120][3].

Recall that Wald's SPRT consists of choosing real numbers $0 < B < 1 < A < \infty$, and defining the stopping time

$$N = \inf_{n \geq 1} \{L_n \geq A \text{ or } L_n \leq B\}. \tag{2.3.5}$$

The decision rule is then to accept H_1 if $L_N \geq A$ and accept H_0 if $L_N \leq B$.

In order to connect the results in this section to Wald's SPRT, it is convenient to express the test in terms of the log-likelihood ratio $\log L_n$. Let

$$Z = \log \left[\frac{Q(Y)}{P(Y)}\right] \quad , \quad Z_k = \log \left[\frac{Q(Y_k)}{P(Y_k)}\right] , k = 1, 2, \ldots \tag{2.3.6}$$

where Q is another probability measure under which $Y_1, Y_2, \ldots$ are i.i.d., and define $S_n \triangleq \log(L_n) = \sum_{k=1}^{n} Z_k$. For any fixed n, the random variable S_n is then a random walk with $\mathbb{E}_P[Z] < 0 < \mathbb{E}_Q[Z]$. Furthermore, by setting $a = \log A$, $b = \log B$ $(b < 0 < a)$, N can be rewritten as

$$N = \inf_{n \geq 1} \{S_n \geq a \text{ or } S_n \leq b\}. \tag{2.3.7}$$

This representation allows us to obtain some of the important properties of N, using the results above. Using Theorem 1, we conclude that N is EB under P and under Q (and, as a matter of fact, N is EB under any measure $\mathbb{P}$ excluding measures under which $\mathbb{P}(Z = 0) = 1$). Consequently, if $\Pr(Z = 0) < 1$, then $\Pr(N < \infty) = 1$ and N has moments of all orders. Thus, Wald's SPRT terminates with probability 1. Also, applying Wald's first equation to S_n and using (2.2.11), a lower bound for the expected stopping time in terms of the error probabilities can be found, that is,

$$\mathbb{E}_P[N] \geq \mu_P^{-1} \left[\alpha \log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)\right], \tag{2.3.8}$$

$$\mathbb{E}_Q[N] \geq \mu_Q^{-1} \left[(1-\beta) \log\left(\frac{1-\beta}{\alpha}\right) + \beta\log\left(\frac{\beta}{1-\alpha}\right)\right]. \tag{2.3.9}$$

[3] More accurately, it follows from the theorem that both $U_n \triangleq S_n - n\mu$ and $V_n \triangleq (S_n - \mu n)^2 - n\sigma^2$ are martingales with respect to their natural filtration. The result follows using Doob's Sampling Theorem for Martingales.

Notice that in order to obtain these lower bounds all that was used is the fact that the observations are i.i.d., combined with (2.2.6) and Jensen's inequality, and hence they are valid for any sequential test with error probabilities α and β and a stopping time N.

In order to gain more insight on equations (2.3.8) and (2.3.9), we rewrite, say, equation (2.3.9) in the following form, using the definition of the Kullback-Leibler divergence [29] $\mu_Q = \mathbb{E}_Q[Z] = D(Q \parallel P)$:

$$\mathbb{E}_Q[N]\, D(Q \parallel P) \geq \left[(1-\beta)\log\left(\frac{1-\beta}{\alpha}\right) + \beta\log\left(\frac{\beta}{1-\alpha}\right)\right]. \tag{2.3.10}$$

Now, denoting by Q^N and P^N the distributions of the observations under Q and P restricted to $\mathcal{F}_N$, i.e., $Q^N(\mathcal{A}) = Q(\{Y_1, \ldots, Y_N\} \in \mathcal{A})$ and $P^N(\mathcal{B}) = P(\{Y_1, \ldots, Y_N\} \in \mathcal{B})$, the left hand side of (2.3.10) equals $D\left(Q^N \parallel P^N\right)$ (this can be checked using Wald's first equation). In order to give an information-theoretic interpretation of the right hand side of (2.3.10), notice that the final decision of the decision function can be described using the following random variable: under the measure Q,

$$d(Y_1, \ldots, Y_N) = \begin{cases} 0 & \text{w.p. } 1-\beta \\ 1 & \text{w.p. } \beta \end{cases} \tag{2.3.11}$$

and under the measure P,

$$d(Y_1, \ldots, Y_N) = \begin{cases} 0 & \text{w.p. } \alpha \\ 1 & \text{w.p. } 1-\alpha \end{cases} \tag{2.3.12}$$

Next, define the measures $\tilde{Q}^N$ and $\tilde{P}^N$ such that $\tilde{Q}^N(i) = Q(d(Y_1, \ldots, Y_N) = i)$ and $\tilde{P}^N(i) = P(d(Y_1, \ldots, Y_N) = i)$ $(i = 0, 1)$. Using these notation, (2.3.10) can be written as:

$$D\left(Q^N \parallel P^N\right) \geq D\left(\tilde{Q}^N \parallel \tilde{P}^N\right) \tag{2.3.13}$$

and equality holds if the random variable $d(Y_1, \ldots, Y_N)$ provides a full description of the random vector $Y_1, \ldots, Y_n$ on the event $\{N = n\}$, which is the case where there is no excess over the boundary values of the test statistic of the SPRT (with probability 1).

By applying Wald's approximation to the SPRT, it is easy to verify that the inequality signs are replaced by equalities, and so, under this crude approximation, Wald's SPRT is the optimal test in the sense that for given error probabilities, it minimizes the expected value of the stopping time under both hypotheses. This optimality property holds also in the general case, as is implied from the Wald-Wolfowitz Theorem, to be discussed in the next section.

2.4 Optimality Property of Wald's SPRT

The SPRT exhibits minimal expected stopping time, both under P and Q, among all sequential tests between a simple hypothesis against a simple alternative with i.i.d. observations, given error probabilities and finite $\mathbb{E}_P[N]$ and $\mathbb{E}_Q[N]$. In particular, we have the following well-known result by Wald and Wolfowitz [147],[148]:

Theorem 4 (Wald and Wolfowitz) *Suppose that (N, d) is the SPRT with boundary values A and B (with $0 < A < 1 < B < \infty$) and error probabilities α and β. Let (d', N') be any other sequential decision rule with finite $\mathbb{E}_P[N']$ and $\mathbb{E}_Q[N']$ and error probabilities α' and β' satisfying*

$$\alpha' \leq \alpha, \quad \beta' \leq \beta. \tag{2.4.1}$$

Then

$$\mathbb{E}_P[N'] \geq \mathbb{E}_P[N], \quad \mathbb{E}_Q[N'] \geq \mathbb{E}_Q[N]. \tag{2.4.2}$$

Another feature of Wald's SPRT is its similarity to the classical Neyman-Pearson fixed-sample-size test of the simple null hypothesis H_0 versus the simple alternative H_1 subject to the type I error constraint. Both tests involve the likelihood ratios and are, if fact, solutions to natural optimization problems. While the Neyman-Pearson optimization criterion is aimed to minimize the type II error probability for the given sample size and type I error bound, the Wald and Wolfowitz criterion is aimed to minimize both $\mathbb{E}_P[N]$ and $\mathbb{E}_Q[N]$ under the type I and II error constraints.

There are essentially two parts in the proof: first, under a given loss (for making the wrong decision) and cost (for taking a single observation), show that a Bayes test[4] is an SPRT; second, given a SPRT, show that there is a loss and cost structure that makes it Bayes. A walk through of the main steps of the proof is given in [26]. Note that in the classical proofs of this theorem (as appear in [147] and [6]), there are some flaws. More rigorous and clear proofs of this theorem can be found in [91], [60], [102], [27], [17] (with relaxation of the finiteness of expected stopping time), [51] and [97] for a more modern analysis using Markov chain properties.

[4]A Bayes test is a sequential procedure which minimizes the Bayes risk. The notions "Bayes risk", "cost" and "loss" will be formally defined in the next section of the text.

2.5 A Note on the Derivation of Wald's SPRT

Let us consider a probability space $(\mathbb{R}^\infty, \mathcal{B}^\infty, P_\pi)$ where

$$P_\pi = (1-\pi)\, P^\infty + \pi Q^\infty, \quad 0 < \pi < 1 \tag{2.5.1}$$

and where P^∞ and Q^∞ are probability measures on $(\mathbb{R}^\infty, \mathcal{B}^\infty)$ with marginal distribution P and Q on $(\mathbb{R}, \mathcal{B})$ such that, under P^∞ (Q^∞) $Y_1, Y_2, \ldots$ are i.i.d. with marginal distribution P (Q). Note that as $n \to \infty$

$$\lambda_n = \prod_{k=1}^{n} \frac{Q(Y_k)}{P(Y_k)} \to \begin{cases} 0 & \text{a.s. under} \quad P^\infty \\ \infty & \text{a.s. under} \quad Q^\infty \end{cases} \tag{2.5.2}$$

and so, if we are observing $Y_1, Y_2, \ldots$, one can decide perfectly between the two hypotheses of interest.

In his paper, Wald [145] used basic mathematical and probabilistic tools in order to develop his sequential probability ratio test. In general, Wald's idea was to update the posteriori probabilities of the two hypotheses at each time step, using the information given by the observations available up to that time, and to stop taking observations as soon as one of the two computed a posteriori probabilities passes a certain, predetermined constant. This constant may differ from one hypothesis to the other and is set according to the prior probabilities and the desired error probabilities α and β. This simple decision protocol (and stopping procedure) comprises the very essence of sequential hypothesis testing: as soon as we can declare that one of the two hypotheses is true with a reasonable certainty, we stop taking observations and decide on this reasonable hypothesis. Here "reasonable certainty" is in the sense that the a-posteriori probability of one hypothesis is "large enough" so that the chance of making an error is small. This heuristic approach led Wald to come up with what is now known to be the optimal sequential hypothesis test (in the sense of Theorem 4). In addition, the classical sequential detection problem can be viewed as an *optimal stopping problem* (or, more accurately, a *Markov optimal stopping problem*) [5] in a Bayesian framework. It is possible to prove that Wald's SPRT is the solution of the optimal stopping problem. This was done, for example, in [27] and [119]. Since the

[5] An optimal stopping problem is a problem in which we try to choose a stopping time N to minimize $\mathbb{E}[Z_N]$ over some interesting class of stopping times where Z_k represents a loss or a cost to be paid at time k. A Markov optimal stopping problem is an optimal stopping problem whose loss sequence can be represented as $Z_k = g_k(X_k)$ where $g_k(\cdot)$ are measurable functions and X_k is a Markov process.

latter approach gives more insight into the fundamental structure of the SPRT, we will concentrate on it rather than on Wald's original techniques.

The starting point of the derivation of an optimal sequential decision rule is to introduce costs $c_j > 0$ for falsely rejecting hypothesis H_j $(j = 0, 1)$ and a cost $C > 0$ per sample, so that the cost of taking n samples is nC. To understand the necessity of theses costs, notice that (2.5.2) implies that the quality of a decision will keep on improving as the number of samples grows, and thus, a mathematical tool is needed in order to temper the net benefit of that improvement and to implement the tradeoff between the error probabilities and the time until the test terminates. To that end, we define, for any sequential test (N, d), two performance indices of interest: the *average cost of errors*,

$$c_e\left(N, d\right) = \left(1 - \pi\right) c_0 P\left(d = 1\right) + \pi c_1 Q\left(d = 0\right) \tag{2.5.3}$$

and the *average cost of sampling*,

$$C\mathbb{E}_\pi\left[N\right], \tag{2.5.4}$$

where $\mathbb{E}_\pi[\cdot]$ denotes expectation with respect to P_π. The sum of the two costs is often referred to as the *Bayes risk* or simply as the *risk*, denoted by $R(\pi, N, d)$, and is given by

$$R\left(\pi, N, d\right) = c_e\left(N, d\right) + C\mathbb{E}_\pi\left[N\right]. \tag{2.5.5}$$

To see the structure of the optimum decision rule, it is useful to consider the function $V(\pi, \mathcal{N}, \mathcal{D})$ defined as the solution to the following optimization problem:

$$V\left(\pi, \mathcal{N}, \mathcal{D}\right) \triangleq \inf_{N\in\mathcal{N},\, d\in\mathcal{D}} \left[c_e\left(N, d\right) + C\mathbb{E}_\pi\left[N\right]\right]. \tag{2.5.6}$$

The first step in solving (2.5.6) is to notice that, given a stopping time $N \in \mathcal{N}$, it is easy to determine a terminal decision rule $d \in \mathcal{D}$ which minimizes the function $V(\pi, \mathcal{N}, \mathcal{D})$ for fixed values of c_0, c_1 and π. To that end, note that the summand $V(\pi, \mathcal{N}, \mathcal{D})$ that depends on d, $c_e(N, d)$, satisfies ([119], Proposition 4.1)

$$\inf_{d\in\mathcal{D}} c_e\left(N, d\right) = \mathbb{E}_\pi\left[\min\left\{c_1\pi_N^\pi, c_0\left(1 - \pi_N^\pi\right)\right\}\right] \tag{2.5.7}$$

and the sequence $\{\pi_k^\pi\}$ is defined by the recursion

$$\pi_k^\pi = \frac{\pi_{k-1}^\pi Q\left(Y_k\right)}{\pi_{k-1}^\pi Q\left(Y_k\right) + \pi_{k-1}^\pi P\left(Y_k\right)}\ ,\ k = 1, 2, \ldots\ \ ,\ \ \pi_0^\pi = \pi \tag{2.5.8}$$

where the random variable π_k^π is the posterior probability that H_1 is true, given $Y_1, \ldots, Y_k$.

Moreover, the infimum in (2.5.7) is achieved by the terminal decision rule

$$d'_N = \begin{cases} 1 & \text{if} \quad \pi_N^\pi \geq c_0/\left(c_0 + c_1\right) \\ 0 & \text{if} \quad \pi_N^\pi < c_0/\left(c_0 + c_1\right) \end{cases} \tag{2.5.9}$$

and thus the problem of (2.5.6) reduces to the alternative problem defined by the function $V(\pi, \mathcal{N})$:

$$V(\pi, \mathcal{N}) = \inf_{N \in \mathcal{N}} \left[\mathbb{E}_\pi \left[\min\left\{c_1 \pi_N^\pi, c_0\left(1 - \pi_N^\pi\right)\right\}\right] + CN\right]. \tag{2.5.10}$$

Hence finding a pair (N, d) that achieves (2.5.6) amounts to solving the following problem: given $0 < \pi < 1$, let $Y_1, Y_2, \ldots$ have a joint density function P_π. For given $c_0, c_1 > 0$, let

$$\begin{aligned} h(\lambda) &= \min\left\{c_0 \lambda, c_1 (1 - \lambda)\right\} \quad 0 \leq \lambda \leq 1 && (2.5.11) \\ v_n &= h\left(\pi_n^\pi\right) + Cn. && (2.5.12) \end{aligned}$$

The goal is then to find a stopping time $N \in \mathcal{N}$ such that $\mathbb{E}_\pi[v_N]$ is minimized. We will denote the random variable v_n as *the loss at time n*. Note that in the present context we are allowed to take no observations and decide in favor of H_0 or H_1 with $v_0 = h(\pi)$; also, a stopping time which takes the value 0 (that is, stops the test before starting to take observations) must do so with probability 0 or 1, since $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The problem is trivial if $c_0 \leq C$ or $c_1 \leq C$ since then $h(\lambda) < 1$ and $v_0 < v_n$ for all n, so that the optimal rule is $N = 0$. We thus assume that the costs are both larger than C.

The following non-rigorous argument has a considerable intuitive appeal. We ask initially whether we should take a first observation. We compute

$$V^\star(\pi) \triangleq V\left(\pi, \mathcal{N}^{(1)}, \mathcal{D}\right) = \inf_{N \in \mathcal{N}^{(1)},\, d \in \mathcal{D}} \left[c_e(N, d) + C\mathbb{E}_\pi[N]\right], \tag{2.5.13}$$

where $\mathcal{N}^{(1)}$ is the class of all stopping times that take at least one observation. $V^\star(\pi)$ then represents our minimum expected cost if we take at least one observation. If we take no observations, our loss is $h(\pi)$ and hence we should take a first observation if and only if $h(\pi) > V^\star(\pi)$.

Next, notice that: (a) v_n depends on $Y_1, \ldots, Y_n$ only through the value of π_n^π and (b) the sequence π_k^π forms a stationary Markov process [119]. Suppose that we take an initial observation. If we stop, our loss is $h\left(\pi_1^\pi\right) + C$. If we continue, our prospects of the future are exactly the same as those with no samples in the sense that we still have infinitely many i.i.d. samples at our disposal and the costs are the same, but we have already "paid" a cost C and the priori is now π_1^π. Hence the minimum expected loss among rules taking at least one more observation is $V^\star(\pi_1^\pi)+C$, and we should take a second observation if and only if $h(\pi_1^\pi) + C > V^\star(\pi_1^\pi) + C$. The argument is repeated by induction, so that the natural candidate for an optimal stopping time is $N = \inf\{n > 0 : \ h(\pi_n^\pi) \le V^\star(\pi_n^\pi)\}$. Let $\mathscr{A} = \{\pi : \ h(\pi) \le V^\star(\pi)\}$. Since $V^\star(\cdot)$ is concave with $V^\star(0) = V^\star(1) = C$ ([51], 7.6 Lemmas 1 and 2), assuming $\mathscr{A} \neq [0,1]$, then it is not hard to show that there are unique numbers π_U and π_L such that $\mathscr{A} = \{\pi : \pi \le \pi_L \text{ or } \pi \ge \pi_U\}$ and that (N, d'_N) is Wald's SPRT. This fact is thoroughly dealt with in [120] which, follows the footsteps of [51] and is illustrated graphically in both.

Chapter 3

Composite Hypotheses Testing

3.1 Introduction

So far, we have only considered the problem of sequentially testing two hypotheses where the observation sequence under each one of the hypotheses is i.i.d. with a given distribution. Such hypotheses, that specify exactly one distribution, are called *simple* hypotheses. In contrast, in the problem of testing two *composite* hypotheses, there is an additional uncertainty which will be represented by specifying a collection of possible models for each hypothesis. In the problems discussed in the sequel, these collections will be indexed by a parameter. The binary composite hypothesis testing problem is thus given by

$$H_0: \quad \Pr\left(Y_1^n\right) = \textstyle\prod_{i=1}^n P_{\theta_0}\left(Y_i\right), \quad \theta_0 \in \Theta_0 \tag{3.1.1}$$

$$H_1: \quad \Pr\left(Y_1^n\right) = \textstyle\prod_{i=1}^n P_{\theta_1}\left(Y_i\right), \quad \theta_1 \in \Theta_1, \tag{3.1.2}$$

where P_{θ_0} and P_{θ_1} represent families of probability measures indexed by the parameters θ_0 and θ_1 respectively, which can be thought of as elements of some vector space $\Theta(\Theta_0 \cap \Theta_1 = \emptyset$ and $\Theta_0 \cup \Theta_1 \subset \Theta)$. As discussed in the previous section, the problem of sequentially testing two simple hypotheses was solved by Wald and Wolfowitz [148]. More precisely, the Wald and Wolfowitz theorem implies that among all tests with given upper bounds on the error probabilities of type I and type II, Wald's SPRT minimizes simultaneously the expected sample size under the two hypotheses (provided that these expected values exist and are finite). The theory of optimal sequential tests of composite hypotheses, however, is much less complete, and many basic problems are still open. In addition, the "naive" extension of the optimal finite-sample-size test will not yield such a good performance as in the case of the binary sequential hypothesis test with simple hypotheses.

In the case of fixed sample size tests, the error probabilities cannot be controlled simultaneously. It is therefore customary to assign a bound on the probability of rejecting H_0 when it is true, and to attempt to minimize the other probability of error subject to this condition. Thus, one selects a number $\alpha \in [0, 1]$, called *the level of significance* or simply *the level*, and imposes the condition

$$P_\theta\,(\text{choose } H_1) \leq \alpha \quad \text{for all} \quad \theta \in \Theta_0. \tag{3.1.3}$$

Subject to this condition, it is desired to maximize

$$P_\theta\,(\text{choose } H_1) \quad \text{for all} \quad \theta \in \Theta_1. \tag{3.1.4}$$

The probability (3.1.4), evaluated at a given $\theta \in \Theta_1$, is called the *power* of the test against the alternative θ. Considered as a function of θ, the probability (3.1.4) is called the *power function* of the test. Typically, the test that maximizes the power against a particular alternative in Θ_1 depends on the alternative. In special cases, it may turn out that the same test maximizes the power for all alternatives in Θ_1 - in this case, we say that the test is *uniformly most powerful* (UMP).

In the case of testing two simple hypotheses, it is well known that the optimal test, in the sense of maximizing the power function for a given level, is the Neyman-Pearson (N-P) test. This test is constructed by comparing the likelihood ratio of the fixed-length observation sequence to a given threshold, the value of which is determined according to the required level of the test, and deciding on the most likely hypothesis. The first step to extend the N-P theory from simple to composite hypotheses is to consider one-sided composite hypotheses of the form $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta \geq \theta_1\,(> \theta_0)$ in the case of parametric families with a monotonic likelihood ratio indexed by a real parameter θ. In this case, it can be shown that the level of the N-P test of $H\colon \theta = \theta_0$ versus $K\colon \theta = \theta_1\,(> \theta_0)$ does not depend on the alternative θ_1. Due to this fact, and the fact that the power function is strictly increasing, it is possible to reduce the composite problem to the problem of simple hypotheses and, in turn, develop an optimal test, in the sense of a uniformly most powerful test (the full derivation can be found, for example, in [128] and [91]).

In the sequential problem, however, this is not possible. As an illustration of this, let $Y_1, Y_2, \ldots$ be i.i.d. normal random variables with mean θ and variance 1.

To test the composite null hypothesis $H_0\colon \theta \leq \theta_0$ versus the composite alternative $H_1 : \theta \geq \theta_1 \, (> \theta_0)$ subject to an error constraint $P_\theta \, (\text{Reject } H_0) \leq \alpha$ for $\theta \leq \theta_0$ and $P_\theta \, (\text{Reject } H_1) \leq \beta$ for $\theta \geq \theta_1$, one can use the SPRT of $H\colon \theta = \theta_0$ versus $: \theta = \theta_1$ with type I and II error probabilities α and β, respectively. The monotonic likelihood ratio structure implies that the SPRT also satisfies the error constraints for the composite hypothesis problem. Although the SPRT that satisfies the error probability constraint has a minimal expected stopping time at $\theta = \theta_0$ and $\theta = \theta_1$, its expected stopping time can be far from optimal at other values of θ. For example, for $\alpha = \beta$, the expected sample size is maximized when the true value of the parameter is equal to $\frac{\theta_0+\theta_1}{2}$. In this case, the expected sample size needed in order to satisfy the error probability constraints can be larger than the sample size in an optimal fixed size test, under the same constraints. This was also pointed out by Kiefer and Weiss [81] who suggested some approaches to deal with problems like the one described. These include a minimax approach of finding a sequential test which minimizes $\sup\limits_{\theta} \mathbb{E}_\theta \, [N]$ over the set of feasible values of θ, where N denotes the stopping time (this problem is better known as the "Kiefer-Weiss problem"). In the following sections, a few different variants of this problem are reviewed, and suggestions for further work on the subject are given.

3.2 Composite Hypothesis Testing - First Steps

In this section, we focus on some of the early ideas of sequentially testing composite hypotheses. Although, as seen in the previous section, it is not wise to use the natural extension of the N-P test constructed for the case of fixed (and finite) sample size, one can still use certain concepts from the finite size problem and apply them on the unlimited one. Two such concepts are the following:

Generalized Likelihood Ratio Test (GLRT): In the construction of the generalized likelihood ratio, we model $\theta \in \Theta$ as a fixed, but unknown, parameter. The GLRT, adopted from the classical fixed-sample size theory, is defined for some $0 < A < 1 < B < \infty$ as

$$N_{\text{GLRT}} = \min_{n \geq 0} \left\{ GLR_n \triangleq \frac{\sup_{\theta \in \Theta_0} \prod_{i=1}^{n} P_{\theta_0} \left(Y_i \right)}{\sup_{\theta \in \Theta_1} \prod_{i=1}^{n} P_{\theta_1} \left(Y_i \right)} \notin (A, B) \right\}. \qquad (3.2.1)$$

The test statistics GLR_n can be constructed by estimating θ using the Maximum Likelihood (ML) method under H_0 and H_1 from the data available up to time n, and plugging the estimate into the GLR. The decision function d_{WSPRT} rejects H_0 if $GLR_n \leq A$ and rejects H_1 if $GLR_n \geq B$. Note that, although intuitively appealing, the GLRT is not necessarily the optimal test. This claim holds for the non-sequential case as well [91].

Weighted SPRT: The next sequential test for composite hypotheses was suggested by Wald and it is described in Section 6 of [145]. When either one or both hypotheses H_0 or H_1 are composite, Wald proposed the use of *weight functions*: the idea is to construct "new" simple hypotheses out of the composite ones, and to use Wald's SPRT for these simple hypotheses. This test is often referred to as the weighted SPRT (WSPRT) or weighted mixtures. The simple hypotheses are constructed using the weight functions $w_0(\theta), w_1(\theta)$, which satisfy $w_i(\theta) \geq 0$ and $\int w_i(\theta)\, d\theta = 1$ $(i = 0, 1)$, in the following way:

$$H_0: \quad \Pr(Y_1^n) = \int_{\Theta_0} w_0(\theta_0) \prod_{i=1}^n P_{\theta_0}(Y_i)\, d\theta_0 \tag{3.2.2}$$

$$H_1: \quad \Pr(Y_1^n) = \int_{\Theta_1} w_1(\theta_1) \prod_{i=1}^n P_{\theta_1}(Y_i)\, d\theta_1. \tag{3.2.3}$$

The stopping time takes the form

$$N_{WSPRT} = \min_{n \geq 0} \left\{ WLR_n \triangleq \frac{\int_{\Theta_0} w_0(\theta) \prod_{i=1}^n P_{\theta_0}(Y_i)\, d\theta}{\int_{\Theta_1} w_1(\theta) \prod_{i=1}^n P_{\theta_1}(Y_i)\, d\theta} \notin (A, B) \right\} \tag{3.2.4}$$

and the decision function d_{WSPRT} rejects H_0 if $WLR_n \leq A$, and it rejects H_1 if $WLR_n \geq B$.

The expressions for $\Pr(Y_1^n)$ can be interpreted as the probability measure on the sample space of n observations if we use the model that assumes θ_0 and θ_1 be the independent random variables with prior probability distribution $w_0(\theta)$ and $w_1(\theta)$. In this sense, the WSPRT can be viewed as a method of comparing the *average* model (with respect to the priors $w_i(\theta)$, $i = 0, 1$). It is also clear from these expressions why the WSPRT is sometimes referred to as "Bayes factor". Using the same methods described in Section 2.2 for bounding the error probabilities, it is possible to obtain the following bounds:

$$\int_{\Theta_0} w_0(\theta_0) P_{\theta_0}(d_{\mathrm{WSPRT}} = H_1)\, d\theta_0 \leq \frac{1}{B}, \tag{3.2.5}$$

$$\int_{\Theta_1} w_1(\theta_1) P_{\theta_1}(d_{\mathrm{WSPRT}} = H_0)\, d\theta_1 \leq A. \tag{3.2.6}$$

The WSPRT's greatest advantage is the fact that it is amenable to simple mathematical analysis as the one used in order to derive (3.2.5) and (3.2.5) , and the fact that it allows the observer to define a weight corresponding to each $\theta \in \Theta$. This, in turn, can be used in order to mitigate large error probability on θ's that are "more important" in some sense. However, this method of dealing with composite hypotheses has many downsides. For the non-Bayesian problem, where the weight functions have no specific meaning and are to be chosen somehow arbitrarily, it is not clear whether the probabilities (3.2.5) and (3.2.6) are of any real significance. Clearly, for practical purposes, one would strongly prefer to upper-bound rather then the maximal error probabilities of Types I and II, the average error probabilities, which depend on a particular choice of weights. However, in general, it is not clear how to obtain the upper bounds on maximal error probabilities of the WSPRT and the GSLRT. In addition, even when considering a Bayesian setup, in which the weight function are assumed given and represent the prior on the parameter space Θ, it has not been possible to obtain analytical results regarding the optimal sequential test for this composite hypotheses problem (In contrast to the case of testing simple hypotheses). The fact that exact result seem to be out of reach triggered the search for asymptotic optimal solutions in a sense soon to be defined.

3.3 The Modified Kiefer-Weiss Problem and the 2-SPRT

In this section, a different type of test is presented, taking into account only three possible "states of Nature". In other words, a sequential hypothesis testing procedure, for the case where three simple hypotheses are to be tested about a certain parameter, is discussed. Although this type of problems will be reviewed more thoroughly in the next chapter, the motivation and connection to our current discussion, regarding composite hypotheses, can be understood through the following example, which was also discussed in the introduction to this section: say a normally distributed i.i.d. sequence is observed, and is to be tested about its mean, $\theta \in \mathbb{R}$, where the error probability constraints of types I and II are given by

$$P_\theta\,(\text{choose } H_0) \;\leq\; \alpha \quad \text{for all} \quad \theta \leq \theta_0 \tag{3.3.1}$$

$$P_\theta\,(\text{choose } H_1) \;\leq\; \beta \quad \text{for all} \quad \theta \geq \theta_1\,(> \theta_0) \tag{3.3.2}$$

for some $\alpha, \beta \in [0,1]$ and $\theta_0 < \theta_1$.

Although satisfying the error probability constraints,the SPRT for the hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ does not yield an optimal test in this case (in the Wald and Wolfowitz sense), the reason being that this test does not take into account the possibility that the true θ falls in $[\theta_0, \theta_1]$. In such case, the SPRT may give rise to an expected stopping time which is unnecessarily large. In fact, a desired formulation of the problem would be minimize the expected stopping time taken with respect to the true parameter θ, under the constraints (3.3.1) and (3.3.2). The problem is that, in general, the true hypothesis is assumed to be unknown. Another appealing formulation is to try to solve the Kiefer-Weiss problem of minimizing the maximum (over Θ) expected sample size under the probability of error constraints.

Many attempts have been made to address this well-known criticism of Wald's SPRT, namely, the fact that the expected sample size may be large if the "true" underlying distribution is not one of the two distributions specified by the hypotheses. In this case, there is no upper bound on the sample size. Examples that will not be discussed in this work, for such attempts to find upper bounds in specific cases, can be found in [131, Section 3.6] (*The truncated SPRT*), [2], [94] (in which the *repeated likelihood ratio test* is proposed to test a simple null hypothesis against a composite alternative, by maximizing the likelihood ratio over the alternative and rejecting the null hypothesis if, at some stage, the likelihood ratio test rejects it) and in [4, Section 2.4] and [155, Chapter 7] (where the *repeated significance tests* is defined).

The main motivation to come up with the composite hypothesis testing setting and the specific test that will be described in the next section, relies on the important fact that in some cases, the Kiefer-Weiss minimax problem boils down to a simpler problem where the inner maximum is easy to calculate (e.g., in the example discussed in the beggining of the section, in which the maximum is achieved at $\frac{1}{2}(\theta_0 + \theta_1)$ or other examples [150]). In addition, in cases where the likelihood ratio is a monotonic function of the tested parameter, for any sequential test that uses the likelihood ratio as its tested statistics, it is enough to satisfy the error probability constraints on the boundary, i.e., at $\theta = \theta_0$ and $\theta = \theta_1$. An important family of density functions (or probability mass functions, for the case of a discrete distribution) that bear this property is the *one-parameter exponential family* (also called the *Koopman-Darmois* family)[1].

[1]A distribution is said to belong to a one-parameter exponential family if the probability density function (or probability mass function) can be written as $P_\theta(y) = h(y) \exp\{\eta(\theta) \cdot T(y) - A(\theta)\}$ where θ belongs to a real (vector) parameter space and h, η, T, A are some given real-valued functions

In view of the above, it is clear why it is interesting to consider the problem known as *the modified Kiefer-Weiss* problem. Kiefer and Weiss [81] considered the problem of minimizing the expected sample size $\mathbb{E}_\theta[N]$ at some given $\theta^\star$ subject to error probability constraints at θ_0 and θ_1 in a one-parameter exponential family with a parameter θ. More precisely, the modified Kiefer-Weiss problem is formulated as follows: the observations $Y_1, Y_2, \ldots$ are random variables on a sample space where the true probability is one of three measures P_0, P_1 and $P_{\theta^\star}$. Under each of these measures, the Y_i's are i.i.d. For simplicity, we assume that the probability densities exist and belong to the one-parameter exponential family with a parameter $\theta \in \Theta \subset \mathbb{R}$. We denote the densities by $f_{\theta_0}, f_{\theta_1}$ and $f_{\theta^\star}$ where $\theta_0, \theta_1, \theta^\star \in \Theta$. This assumption will be made throughout this section unless stated otherwise. As usual, the pair (N, d) is a sequential test consisting of a stopping time N and a decision rule $d : y_1^N \to \{\theta_0, \theta_1\}$. The error probabilities are then

$$\alpha = f_{\theta_0}(\{d = \theta_1\}) \ , \ \beta = f_{\theta_1}(\{d = \theta_0\}) \tag{3.3.3}$$

and the goal is to find a sequential test that, for a given pair of error probabilities (α, β), will ensure that $\mathbb{E}_{\theta^\star}[N]$ is minimized. The idea behind this mathematical formulation is that it allows to plug in $\theta^\star$ that may yield an extremely large expected stopping time. As mentioned before, in cases where the θ that achieves maximum in the Kiffer-Weiss can be calculated analytically, plugging in this maximizer will yield a solution to the Kiffer-Weiss problem. In [98], Lorden studied the structure of an optimal test for the modified Kiefer-Weiss problem in the symmetric case (where $\theta_1 = -\theta_0$) for the mean of the normal and binomial distributions.In [150] Weiss showed that the optimal test for this problem will have structure that resembles the SPRT (i.e., based on likelihood ratios) but with non-linear boundaries U_n and V_n. Following this work, Lorden pointed out the possibility of using "backward induction" to approximately determine these optimal boundaries for the case of discrete time. Lai [85] considered the case of testing the drift of a Wiener process, and reduced the problem of finding the boundaries to the problem of solving a Stefan problem (which is a boundary value problem for a partial differential equation, adapted to the case in which the boundary can move with time). However, also in this case, only asymptotic estimates for U_n and V_n were obtained.

Since there was no success in finding an exact solution of the modified Kiefer-Weiss

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[92].

problem (and even if there was such a solution, its implementation would have been nearly infeasible), researchers began to look for asymptotically optimal solutions with the hope to gain insight as well as to find simple test structures that perform well at least asymptotically. In the sequel, a few different criteria for optimality will be given, but on the other hand, the asymptotics will always be in the sense of very large sample sizes. Mathematically, this asymptotic regime is implemented by taking the upper bound constraints on the error probabilities to zero (or, if we assume a cost for taking a sample in given, then taking the limit as this cost goes to zero will give rise to an equivalent asymptotic regime, as will be explained later).

Hoeffding [71] derived a lower bound on $\mathbb{E}_{\theta^\star}[N]$ subject to error probability constraints at some θ_0 and θ_1 in the limit where the error probabilities vanish (where $\theta^\star \neq \theta_0 \neq \theta_1$ and all belong to Θ), and Lorden [95] has made a big progress in the same direction by providing an asymptotic solution to the modified Kiefer-Weiss problem. The test that achieves this asymptotic optimality property is called *the 2-SPRT*. The 2-SPRT is fully defined given the probability densities $f_{\theta_0}, f_{\theta_1}, f_{\theta^\star}$ and two constants A_0 and A_1 which will serve as the threshold values of the test. The stopping rule of the 2-SPRT takes on the form of

$$N_{_2} = \inf_{n\geq 1}\left\{\prod_{i=1}^{n}\frac{f_{\theta^\star}(Y_i)}{f_{\theta_0}(Y_i)} \geq A_0 \text{ or } \prod_{i=1}^{n}\frac{f_{\theta^\star}(Y_i)}{f_{\theta_1}(Y_i)} \geq A_1\right\} \tag{3.3.4}$$

and the decision rule is given by

$$d_{_2}\left(Y_1^{N_2}\right) = \theta_0 \quad \text{if} \quad \prod_{i=1}^{N_2}\frac{f_{\theta^\star}(Y_i)}{f_{\theta_0}(Y_i)} \leq A_0 \quad \text{and} \quad d_{_2}\left(Y_1^{N_2}\right) = \theta_1 \quad \text{if} \quad \prod_{i=1}^{N_2}\frac{f_{\theta^\star}(Y_i)}{f_{\theta_1}(Y_i)} \leq A_1. \tag{3.3.5}$$

In the case where both equalities in (3.3.5) hold, the choice between θ_0 and θ_1 can be made arbitrarily. Notice that if $\theta^\star = \theta_0$ or $\theta^\star = \theta_1$, the 2-SPRT degenerates into Wald's SPRT, and so, at least for this case, we know that Lorden's 2-SPRT is optimal (even before taking the limit of the error probabilities to zero). Moreover, it is known that the SPRT allows for a good tradeoff between constraints on error probabilities of type I and II, and the expected stopping time under both hypotheses. From this point of view, this test is a particular natural choice from the family of tests that perform two one-sided SPRTs - one for the possible rejection of θ_1 and the other for the rejection of θ_0. The alternative hypothesis, in each one-sided SPRT, is chosen to be $f_{\theta^\star}$, which is also quite sensible since the expected stopping time which we want to control is taken with respect to this measure.

The optimality of the test is established in the next theorem:

Theorem 5 (Theorem 1 in [95]) *Let $\alpha(A_0, A_1)$ and $\beta(A_0, A_1)$ denote the error probabilities of the 2-SPRT defined by the stopping rule* (3.3.4)*, and let the decision rule be*

$$d_2 = \begin{cases} H_0 & \textit{if} \quad \prod_{i=1}^{N_2} \frac{f_{\theta^\star}(Y_i)}{f_{\theta_0}(Y_i)} \leq A_0 \quad \textit{and} \quad \prod_{i=1}^{N_2} \frac{f_{\theta^\star}(Y_i)}{f_{\theta_1}(Y_i)} > A_1 \\ H_1 & \textit{if} \quad \prod_{i=1}^{N_2} \frac{f_{\theta^\star}(Y_i)}{f_{\theta_1}(Y_i)} \leq A_1 \quad \textit{and} \quad \prod_{i=1}^{N_2} \frac{f_{\theta^\star}(Y_i)}{f_{\theta_0}(Y_i)} > A_0 \\ & \textit{or} \quad \prod_{i=1}^{N_2} \frac{f_{\theta^\star}(Y_i)}{f_{\theta_1}(Y_i)} \leq A_1 \quad \textit{and} \quad \prod_{i=1}^{N_2} \frac{f_{\theta^\star}(Y_i)}{f_{\theta_0}(Y_i)} \leq A_0 \end{cases} . \tag{3.3.6}$$

Furthermore, let $n(A_0, A_1)$ denote the infimum of $\mathbb{E}_{\theta^\star}[N]$ over all tests satisfying $\alpha \leq \alpha(A_0, A_1)$ and $\beta \leq \beta(A_0, A_1)$. Under the assumption that the second moment of both $\frac{f_{\theta^\star}(Y_i)}{f_{\theta_0}(Y_i)}$ and $\frac{f_{\theta^\star}(Y_i)}{f_{\theta_1}(Y_i)}$ exist and are finite, if $A_0 > 0$ and $A_1 > 0$, then, in the limit where $\max\{A_0, A_1\} \to \infty$,

$$\mathbb{E}_{\theta^\star}[N] - n(A_0, A_1) \to 0. \tag{3.3.7}$$

In other words, this theorem implies that the difference between the expected stopping time of the 2-SPRT and the best expected stopping time, taken at $\theta = \theta^\star$, among all sequential tests satisfying the error probability constraints at θ_0 and θ_1, goes to zero, as the upper bound constraints on the error probabilities go to zero. It can be shown that the same result holds when, instead of the asymptotic regime, the two types of error probabilities go to zero (using the relations that where reviewed in the first chapter of this work).

To prove this theorem, Lorden used the Bayes risk formulation in which some target function, known as the *Risk function* is minimized (this is done by following the proof in [148] and invoking results from [27]) to determine that (3.3.4) indeed bears the structure of an optional stopping rule with respect to the risk function. Another interesting feature of the 2-SPRT is that for testing a normal family with mean θ such that $\theta_0 = -\theta_1$, the minimax problem of minimizing $\sup_\theta \mathbb{E}_\theta[N]$ reduces to a modified Keifer-Weiss problem with $\theta^\star = 0$, and in turn, the continuation region[2] of the 2-SPRT is reduced to the triangular stopping boundary introduced earlier by Anderson [2][3], and is known as "the Anderson modification to the SPRT", in which the test

[2]Let $T(\cdot)$ be the test statistic. In sequential analysis, the region in the space of $(T(y_n), n)$ in which the decision maker decides to take another observation called *the continuation region*.

[3]By triangular stopping boundary we meed that the stopping boundaries of the test form a triangle in the time-test statistic space, in a way that ensures that $\mathbb{E}[N]$ in finite.

statistic is the random walk $\sum y_i$, and the stopping boundaries are linear in the time variable n. In a later work, Anderson's modification to the SPRT has been shown by Lai [85] to be an asymptotic solution to the optimal stopping problem associated with the original Kiefer-Weiss problem. Although out of the scope of this work, it is worth mentioning that this asymptotic optimal continuation region holds for the continuous-time problem as well, if we consider instead of the i.i.d. random process a standard Wiener process [85].

More results that generalize and extend the result discussed above regarding the use of the 2-SPRT test can be found, for example, in [64] where Hoeffding's lower bounds on $\mathbb{E}_{\theta^\star}[N]$ are used to derive, for this setup, a family of tests called "minimum probability ratio tests" that include Lorden's 2-SPRT as a special case. In [76], Huffman extended Lorden's results and showed that a suitably chosen 2-SPRT also provides an asymptotically optimal solution to the minimax sequential testing problem of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta \geq \theta_1 (> \theta_0)$ for a more general family of exponential density functions but in a weaker sense than (3.3.7)[4]. Another generalization is due to Dragalin and Novikov. In [42, Section 2], the authors have shown that as $\max\{A_0, A_1\} \to \infty$, the mean time of the 2-SPRT differs from the optimal value only by $o(1)$ (that is, by a function going to zero like $(\max\{A_0, A_1\})^{-1}$ as $\max\{A_0, A_1\}$ goes to zero).

In general, the modified Kiefer-Weiss problem does come up in some cases where the ordinary (and complicated) Kiefer-Weiss problem is the reasonable model to consider. In addition, the general structure of the test, using stopping criteria that are based on one-sided SPRTs, or, more generally, on likelihood ratios, will be shown to be of essence in many other aspects of sequential testing.

3.4 Back to the Bayes Problem - A Unified Theory

As mentioned before, the main drawback of the solution to the modified Kiefer-Weiss problem, suggested in the previous section, is that ideally, $\theta^\star$ should be chosen to be the true θ which is unknown. Nevertheless, the general structure of the 2-SPRT came up also in the derivation of certain optimal sequential tests for composite hypotheses in the Bayesian formulation. The difference between the 2-SPRT and the test that

[4] Specifically, in [76] the an optimal test is defined to be a test achieving $\frac{n(A_0,A_1)}{\mathbb{E}_{\theta^\star}[N]} \to 0$ as the error probabilities go to zero.

will be presented next, is that instead of $\theta^\star$, an estimate regarding the true parameter θ at each time step is used. The estimation of the true value of θ will be based on the knowledge available up to that point, and no prior knowledge will be assumed.

Recall that, in the Bayesian formulation, we assume a known prior π on Θ. A cost $c > 0$ is assigned to each observation and, in addition, a loss function $w(\theta)$ is given, where $w(\theta')$ is the loss associated with accepting the incorrect hypothesis θ'. We will next focus on the Bayesian problem of testing a one-sided composite hypothesis of the form $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta \geq \theta_1 (> \theta_0)$ in the one-parameter exponential family, subject to the error constraints

$$\begin{cases} P_\theta(\text{reject } H_0) \leq \alpha & \text{for} \quad \theta \leq \theta_0 \\ P_\theta(\text{reject } H_1) \leq \beta & \text{for} \quad \theta \geq \theta_1. \end{cases} \tag{3.4.1}$$

A well known asymptotic approach to this problem is due to Schwartz [129] and is often referred to as *Schwarz's theory of asymptotic shapes*. The goal in [129] was to find an asymptotic solution to the Bayes problem of testing H_0 versus H_1 with the 0-1 loss function and a cost c per observation, as $c \to 0$ while θ_1 and θ_0 are kept fixed. In other words, the problem Schwarz considered was to find a sequential rule that minimizes

$$\begin{aligned} R(\pi, N, d) &= c \int_\Theta \mathbb{E}_\theta[N]\, d\pi(\theta) + \int_{\theta \leq \theta_0} P_\theta(\text{d accepts } H_1)\, d\pi(\theta) \\ &+ \int_{\theta \geq \theta_0} P_\theta(\text{d accepts } H_0)\, d\pi(\theta). \end{aligned} \tag{3.4.2}$$

Here, as in many other risk functions, c represents the ratio of between the sampling cost and the cost due to a wrong decision.Schwarz assumed $Y_1, Y_2, \ldots$ are i.i.d. random variables whose common density belongs to the one parameter exponential family. Schwarz's asymptotic theory led to a limiting continuation region in the space of $(\sum Y_i, n)$. It turns out that replacing $\theta^\star$ in the 2-SPRT by the maximum likelihood estimate at stage n, denoted by $\hat{\theta}_n$, gives rise to the same result as the one by Schwarz. The sequential test (N_s, d_s) corresponding to Schwarz's "asymptotic shape theory", is the following:

$$N_s(c) = N_s = \inf_{n \geq 1} \left\{ \max\left\{ \prod_{i=1}^{n} \frac{f_{\hat{\theta}_n}(Y_i)}{f_{\theta_0}(Y_i)}, \prod_{i=1}^{n} \frac{f_{\hat{\theta}_n}(Y_i)}{f_{\theta_1}(Y_i)} \right\} \geq \frac{1}{c} \right\} \tag{3.4.3}$$

and the decision rule is defined to be

$$d_s(c) = d_s = \begin{cases} H_0 & \text{if} \quad \prod_{i=1}^{N_s} f_{\theta_1}(Y_i) > \prod_{i=1}^{n} f_{\theta_0}(Y_i) \\ H_1 & \text{if} \quad \prod_{i=1}^{N_s} f_{\theta_1}(Y_i) \leq \prod_{i=1}^{n} f_{\theta_0}(Y_i). \end{cases} \tag{3.4.4}$$

Note that both N_s and d_s are random variables that are parameterized by c, although this will not be emphasized in the notation. Kiefer and Sacks [79] and Wong [154] proved an upper bound on N_s, and they showed a first order approximation of $\mathbb{E}_\theta[N_s]$ for every θ in terms of $|\log c|$, θ_0 and θ_1 in the limit $c \to 0$.

The asymptotic optimality of Schwarz's test was then proved in the sense that for any prior π

$$\lim_{c\to 0} \frac{R(\pi, N_s, d_s)}{\inf_{\Delta=(N,d)} \{R(\pi, N, d)\}} = 1, \tag{3.4.5}$$

where $R(\pi, N, d)$ is defined in (3.4.2).

Another important development in the area of Bayes sequential tests of composite hypotheses is Chernoff's work [22], [25] on testing $\hat{H}_0 : \theta < \theta_0$ versus $\hat{H}_1 : \theta > \theta_0$ in the case where $\{Y_i\}$ are normal with mean θ and variance 1. Instead of assuming an indifference zone, in which there are no constraints on the probabilities of error imposed, Chernoff derived different and considerably more complicated approximations to the Bayes test using a different loss function, $w(\theta)$, for making a wrong decision. For example, for $\theta_0 = 0$, the risk function of Chernoff was

$$\begin{aligned} R(\pi, N, d) &= c\int_{-\infty}^{\infty} \mathbb{E}_\theta[N]\, d\pi(\theta) + \int_{-\infty}^{0} |\theta|\, P_\theta\left(\sum_{i=1}^{N} Y_i > 0\right) d\pi(\theta) \\ &+ \int_{0}^{\infty} \theta P_\theta\left(\sum_{i=1}^{N} Y_i \le 0\right) d\pi(\theta), \end{aligned} \tag{3.4.6}$$

where the prior distribution π is assumed normal with mean 0 and variance σ^2. Moreover, while Schwarz based his approximation on simple upper and lower bounds that are associated with stopping when the posterior risk falls below c or below a constant times $c|\log c|$ (for the upper and lower bound respectively), Chernoff's theory, which deals only with the normal case, is based on replacing the discrete-time stopping problem with a continuous-time stopping problem which can in turn be reduced to a partial differential equation problem.

The fact that setting $\theta_0 = \theta_1$ in Schwarz's test (and by that eliminating the indifference zone) does not yield Chernoff's test, has troubled the Statistical Society for almost two decades. This disturbing discrepancy between the two asymptotic approximations was finally resolved by Lai [86]. Lai proposed to replace the factor $|\log c|$ in (3.4.3) by $g(cn)$ where g is a function that satisfies $g(t) \sim \log(1/t)$ as $t \to 0$, and is the boundary of an associated optimal stopping problem for the Wiener process[5].

[5]For two function ν and μ we write $\nu \sim \mu$ if $\lim \frac{\nu(t)}{\mu(t)} \to 1$ as $t \to 0$.

Formally, we get

$$N = \inf_{n \geq 1} \left\{ \max \left\{ \sum_{i=1}^{n} \log \left(\frac{f_{\hat{\theta}_n}(Y_i)}{f_{\theta_0}(Y_i)} \right), \sum_{i=1}^{n} \log \left(\frac{f_{\hat{\theta}_n}(Y_i)}{f_{\theta_1}(Y_i)} \right) \right\} \geq g(cn) \right\}. \tag{3.4.7}$$

By plugging in the function $g(\cdot)$ that is time-dependent (and hence the boundary values are adaptive in time), Lai gave a unified solution to both problems; the problem of testing H_0 versus H_1, and, by setting $\theta_0 = \theta_1$ in (3.4.7), for testing $\hat{H}_0$ versus $\hat{H}_1$.

The unified theory for composite hypothesis testing also provides a bridge between asymptotically optimal sequential and fixed sample size tests. In the fixed sample size case, the Neyman-Pearson approach replaces the likelihood ratio by the generalized likelihood ratio (GLR) (see, for example, [91]), which is also used in (3.4.7) for the sequential test. Since the accuracy of $\hat{\theta}_n$ varies with n, it is quite natural that a time-varying boundary $g(nc)$ is used, instead of the constant boundary levels used when θ is specified. The main idea is that the stopping time defined in (3.4.7) can adapt to the unknown θ by learning it during the experiment and incorporating the diminishing uncertainties in its value into the stopping boundary function g.

A natural generalization of these results for the case where multivariate exponential families are to be tested is given in [89].More results and further references can be found in [87] and [88].

3.5 The Minimax Bayesian Formulation

An important question is whether it is possible to generalize the non-asymptotic Wald-Wolfowitz result to more than two hypotheses. Darakovsky [36] proposed a new formulation of the problem of sequential discrimination of two composite hypotheses admitting a parametric description. Namely, the problem of minimization of the maximal Bayesian risk with respect to the class of a priori distributions on a parameter space is posed. The class of a priori distributions comprises all probabilistic distributions over the parametric set for which the a priori probability of validity of one of the composite hypotheses is equal to a given value. It is only natural to call this problem the *minimax-Bayesian* problem.

Formally, Darakovsky considered the following problem: Let Θ be a parametric set in a finite-dimensional space, with $\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_0 \cap \Theta_1 = \emptyset$. The two families of probabilistic distributions that are considered are $\mathcal{P}_i = \{P_\theta(y)\}_{\theta \in \Theta_i}$, $i = 0, 1$ where

$P_\theta(y)$ is a density parameterized by θ. It is assumed that if $\theta_1 \neq \theta_2$ then $P_{\theta_1}(\mathcal{Y}) \neq P_{\theta_2}(\mathcal{Y})$ on some set $\mathcal{Y}$ (with non-zero measure). The two hypotheses to be tested are (3.1.1) and (3.1.2).

We assume that the parameter θ has a prior distribution $F(\cdot) \in \mathscr{F}_q$ where $\mathscr{F}_q = \{F(\cdot)\}$ is the family of distributions on Θ such that $\int_{\Theta_0} dF(\theta) = q$ and $0 < q < 1$. Let $\Delta = (N, d)$ be a sequential test for this problem, and define the error probabilities to be

$$\alpha_0(\Delta, \theta_0) = P_{\theta_0}(d = 1) \text{ for } \theta_0 \in \Theta_0 \tag{3.5.1}$$

$$\alpha_1(\Delta, \theta_1) = P_{\theta_1}(d = 0) \text{ for } \theta_1 \in \Theta_1 \tag{3.5.2}$$

and the conditional expectation of the stopping time:

$$T_0(\Delta, \theta_0) = \mathbb{E}_{\theta_0}[N] \text{ for } \theta_0 \in \Theta_0 \tag{3.5.3}$$

$$T_1(\Delta, \theta_1) = \mathbb{E}_{\theta_1}[N] \text{ for } \theta_1 \in \Theta_1. \tag{3.5.4}$$

Take $w_0 > 0$ and $w_1 > 0$ to be the losses due to a false decision and $c > 0$ be the cost of one observation. Then the Bayesian risk defined earlier takes on the form of

$$R(F, N, d) = \int_{\Theta_0} [w_0 \alpha_0(d, \theta_0) + cT_0(d, \theta_0)]\, dF(\theta) \tag{3.5.5}$$

$$+ \int_{\Theta_1} [w_1 \alpha_1(d, \theta_0) + cT_1(d, \theta_0)]\, dF(\theta). \tag{3.5.6}$$

We say that a test procedure is *minimax-Bayesian* if it minimizes the maximal risk with respect to the given class of prior distributions. In other words, a test is minimax-Bayesian if it attains $\min_{\Delta} \sup_{F \in \mathscr{F}_q} R(F, \Delta)$.

Note that in using this particular setting and definitions, we obtain the definition of the Bayesian criterion for the classical Wald problem in the case where each parametric set is a singleton.

One of the main results in [36] is the following: Define for $n = 1, 2, \ldots$

$$L^\star(n) = \frac{\sup_{\theta_0 \in \Theta_0} \prod_{i=1}^n P_{\theta_0}(Y_i)}{\inf_{\theta_1 \in \Theta_1} \prod_{i=1}^n P_{\theta_1}(Y_i)} = \sup_{\theta_0 \in \Theta_0} \sup_{\theta_1 \in \Theta_1} \frac{\prod_{i=1}^n P_{\theta_0}(Y_i)}{\prod_{i=1}^n P_{\theta_1}(Y_i)}, \tag{3.5.7}$$

$$L_\star(n) = \frac{\inf_{\theta_0 \in \Theta_0} \prod_{i=1}^n P_{\theta_0}(Y_i)}{\sup_{\theta_1 \in \Theta_1} \prod_{i=1}^n P_{\theta_1}(Y_i)} = \inf_{\theta_0 \in \Theta_0} \inf_{\theta_1 \in \Theta_1} \frac{\prod_{i=1}^n P_{\theta_0}(Y_i)}{\prod_{i=1}^n P_{\theta_1}(Y_i)}. \tag{3.5.8}$$

Let $q^\star(n)$ and $q_\star(n)$ be the following upper and lower posterior probabilities of the hypothesis H_0 being true after n observations:

$$q^\star(n) = \frac{qL^\star(n)}{1 - q + qL^\star(n)}, \quad q_\star(n) = \frac{qL_\star(n)}{1 - q + qL_\star(n)} \tag{3.5.9}$$

and let $Z^\star(n) = \log L^\star(n)$ and $Z_\star(n) = \log L_\star(n)$. Theorem 1 in [36] states the following: if Θ isa compact set on which $P_{\theta_0}(Y_i)$ and $P_{\theta_1}(Y_i)$ are defined, positive and continuous for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$, then there exist numbers $0 < C < 1 < D$ such that the optimal strategy, in the minimax-Bayesian sense, has the form:

- If $Z_\star(n) > \log D$ then we stop sampling and accept H_0
- If $Z^\star(n) < \log C$ then we stop sampling and accept H_1
- Otherwise continue sampling.

Note that in the case of simple hypotheses the classical Wald's SPRT rule follows from the above because $L^\star(n) = L_\star(n)$ and they are both equal to the likelihood ratio. Also, an equivalent representation of this test procedure can be given in terms of $q^\star(n)$ and $q_\star(n)$, that is, it can be shown that there exist two constants $0 < \gamma_0 < \gamma_1 < 1$ such that the following test is optimal in the minimax-Bayesian sense:

- If $q^\star(n) < \gamma_0$ then we stop sampling and accept H_1
- If $q_\star(n) > \gamma_1$ then we stop sampling and accept H_0
- Otherwise continue sampling.

This representation (given in [36]) resembles Wald's SPRT. These expressions can be understood intuitively by examining $q^\star(n)$. Notice that $q^\star(n)$ can be written in the following way:

$$q^\star(n) = \frac{qL^\star(n)}{1-q+qL^\star(n)} = \frac{qP_{\theta_0^\star}(Y^n)}{qP_{\theta_0^\star}(Y^n) + (1-q)P_{\theta_1^\star}(Y^n)}, \tag{3.5.10}$$

where $\theta_0^\star$ and $\theta_1^\star$ are the values that achieve the supremum in (3.5.7). If we assume that whenever hypothesis $\theta \in \Theta_0$ is true, then $\theta_0^\star$ is, in some sense, "dominating" the other θ's in the space Θ_0 and the same for θ_1, then $q^\star(n) \approx \Pr(\Theta_0 \mid Y_1 \ldots Y_n)$ and the test stops and accepts H_1 as soon as that $\Pr(\Theta_0 \mid Y_1 \ldots Y_n) \lesssim \gamma_0$, i.e, when H_0 is rejected. The acceptance of H_0 can be understood in the same way, with the positions of Θ_0 and Θ_1 reversed. As mentioned before, this interpretation, using the posteriors, sheds light on the resemblance to Wald's SPRT. The ideas in the proof of the theorem are based on Markov optimal stopping principles as was described in Chapter 2.

The question still to be answered is whether this test is optimal in the Wald-Wolfowitz sense (generalized to the case of composite hypotheses, of course)? The proof of this claim appeared in [36] and was further refined in [37]. To state the result, let $C, D \in \mathbb{R}$ such that $0 < C < 1 < D$ and define $\Delta_{C,D}$ to be the sequential test just described. Consider the following class of sequential procedures

$$K_{a,b} = \left\{ \Delta : \sup_{\theta \in \Theta_0} \alpha_0 (\Delta, \theta_0) \leq a, \sup_{\theta \in \Theta_1} \alpha_1 (\Delta, \theta_1) \leq b \right\}. \tag{3.5.11}$$

In [37], the following theorem was proved under some technical conditions which will not be repeated here: For any $\epsilon > 0$, there exists $C_0, D_0 \in \mathbb{R}$ such that the inequalities $C^{-1} > C_0^{-1}$ and $D > D_0$ imply that the procedure $\Delta_{C,D}$ is optimal in the following sense: for any procedure $\Delta \in K_{a,b}$ such that $\sup_{\theta \in \Theta_0} T_0 (d, \theta_0) < \infty$ and $\sup_{\theta \in \Theta_1} T_0 (d, \theta_1) < \infty$, the inequalities

$$\sup_{\theta \in \Theta_0} T_0 (\Delta_{C,D}, \theta_0) \leq \sup_{\theta \in \Theta_0} T_0 (\Delta, \theta_0) + \epsilon \tag{3.5.12}$$

$$\sup_{\theta \in \Theta_1} T_1 (\Delta_{C,D}, \theta_1) \leq \sup_{\theta \in \Theta_1} T_1 (\Delta, \theta_1) + \epsilon \tag{3.5.13}$$

hold. When Θ is a finite set, ϵ can be set to zero. Equations (3.5.12) and (3.5.12) provides the non-asymptotic generalization of the classical result. Moreover, in the case of two simple hypotheses sequential testing problem, the result gives rise to the classical Wald-Wolfowitz theorem. In addition, for any $\theta_0^\star \in \Theta_0$ and $\theta_1^\star \in \Theta_1$,

$$\lim_{D \to \infty} \frac{T_0 (\Delta_{C,D}, \theta_0^\star)}{\log (D)} = \max_{\theta_0 \in \Theta_0, \theta_1 \in \Theta_1} \left\{ \mathbb{E}_{\theta_0^\star} \left[\log \left(\frac{dP_{\theta_0}}{dP_{\theta_1}} \right) \right] \right\}^{-1}, \tag{3.5.14}$$

$$\lim_{C \to 0} \frac{T_1 (\Delta_{C,D}, \theta_1^\star)}{|\log (C)|} = \max_{\theta_0 \in \Theta_0, \theta_1 \in \Theta_1} \left\{ \mathbb{E}_{\theta_1^\star} \left[\log \left(\frac{dP_{\theta_1}}{dP_{\theta_0}} \right) \right] \right\}^{-1}.. \tag{3.5.15}$$

These relations are analogous those of two simple hypotheses. For example, in the SPRT, the test follows the random walk $\sum \log \left[\frac{P_1(Y_i)}{P_0(Y_i)} \right]$ and stops when it passes predetermined boundary values, $\log A$ or $\log B$ (where $A < B$). If P_1 is assumed true, then the drift of this random walk is given by

$$\mathbb{E}_{P_1} \left[\log \left[\frac{P_1 (Y_i)}{P_0 (Y_i)} \right] \right], \tag{3.5.16}$$

and so, at least intuitively, for large sample sizes, one would expect that

$$\frac{\log (A)}{\mathbb{E}_{P_1} [N]} \approx \mathbb{E}_{P_1} \left[\log \left[\frac{P_1 (Y_i)}{P_0 (Y_i)} \right] \right] \tag{3.5.17}$$

. Equations (3.5.14) and (3.5.15) imply that a similar phenomenom occurs in the case of two simple hypotheses as well, except that the limiting behavior of, say,

$$\frac{\log (D)}{T_0 (\Delta_{C,D}, \theta_0^\star)} \tag{3.5.18}$$

is determined by the values of θ_0 and θ_1 that minimize the expected value of

$$\log\left[\frac{P_{\theta_0}(Y_i)}{P_{\theta_0}(Y_i)}\right] \tag{3.5.19}$$

taken at $\theta_0^\star$.

Chapter 4

Multiple Hypothesis Testing

4.1 Introduction

In this chapter, the problem at hand will be still to classify a sequence of observations into $M > 2$ different simple hypotheses. Although most of the research on sequential hypothesis testing has been restricted to two hypotheses, there are several situations, particularly in engineering applications, where it is natural to consider more than two hypotheses. Examples include, among a multitude of others, target detection in multiple-resolution radar [101] and infrared systems [49], signal acquisition in direct sequence code-division multiple access systems [140], statistical pattern recognition [55], decentralized detection in sensor networks [21] and more. In the second part of this work, another example in which analysis of multiple hypothesis testing can be of aid, will be presented - variable length coding in the presence of feedback.

Several topics will be addressed in this chapter. First, some of the classical test procedures, in the multiple hypotheses setup, will be reviewed. These tests, covered in section 4.2, will be defined, and general advantages and disadvantages will be explained. Next, a modern approach for dealing with multiple hypotheses will be discussed, and asymptotic properties of some specific (more practical but suboptimal) sequential tests will be analyzed. The motivation for looking at an asymptotic regime (for large sample size) is the fact that, unlike the binary case, where Wald's SPRT is optimal, for more than two hypotheses, it is not clear if there even exists a test that minimizes the expected sample size for all hypotheses, and even if so, it would be very difficult to find its structure. In addition, in this section, a generalization to non-i.i.d. cases (where log-likelihood ratios are no longer random walks) is presented briefly.

In the last section of this chapter, yet another problem of multiple hypothesis testing is posed, where that some control over the observation sequence is considered. In other words, in addition to the $M > 2$ hypotheses, the decision maker can choose one out of $K > 1$ control actions (or "experiments") and hence, to adaptively manage and control multiple degrees of freedom and exert control over the samples' "information content". We refer to this generalization as the *controlled sensing* problem (or the *active sequential hypothesis problem*). In this setting, the goal is to design a sequential test to achieve the optimal tradeoff between reliability in terms of probability of error, and delay (or *cost*), in terms of the expected sample size needed for decision making. Both classical and state-of-the-art results will be reviewed, focusing again on the asymptotic regime and several notions of optimality will be presented. As in the case where no control is available, a few particular, suboptimal (yet simple), controlled sensing tests will be defined, and the analysis of their performance will be discussed.

4.2 The Founding Fathers of Sequential Multiple Hypothesis Testing

A. The Sobel-Wald Test

The Sobel-Wald test [133] is one of the first tests proposed for $M > 2$, and it is perhaps the simplest one. The idea is to combine two SPRTs between different pairs of hypotheses in the following way: Assume $M = 3$ and let the distribution related to each hypothesis be i.i.d. The construction of the Sobel-Wald test begins with two SPRTs: one for testing H_1 versus H_2, and another one for H_2 versus H_3. Denote the stopping times and decision rules of the two tests by (N_1, d_1) and (N_2, d_2), respectively. Throughout the derivation, Sobel and Wald assumed that the event $\{d_1 = H_1\,,\, d_2 = H_3\}$ is impossible, and hence this test should only be considered in problem that can be formalized such that this assumption holds[1]. Now define the the stopping rule of the complete test as

$$N^{\star} = \max\{N_1, N_2\} \tag{4.2.1}$$

and the decision function

[1]Note that this is a constraint on the function d which is up to the observer to choose. In the coding problem discussed in Part II this constraint is on the structure of the decoder, and is easily implemented

$$d^{\star} = \begin{cases} H_1 & \text{if } d_1 = H_1 \\ H_3 & \text{if } d_2 = H_3 \\ H_2 & \text{if } d_1 = H_2 \text{ or } \ d_2 = H_2 \end{cases}$$

The motivation is clear: it is a generalization of a test which is known to bear some optimal characteristics for the fixed sample size case in a sense of controlling the error probability (see, for example, [91]). Furthermore, Hoel and Peterson [72] presented an optimal test for multiple simple hypotheses problem for fixed sample size in the following sense: denoting their decision rule by d_{HP} then there is no other test d' with $\Pr(d' = H_i \mid H_i) \geq \Pr(d_{\mathrm{HP}} = H_i \mid H_i)$ for all i and at least one inequality is strict. Hoel and Peterson's test for three hypotheses it is exactly the Sobel-Wald test with a fixed sample size. This, of course, does not guarantee optimality of the Sobel-Wald test, but good performance may be expected. Another reason, that most certainly motivated Sobel and Wald to come up with their test, is the fact that it is constructed using two SPRTs which are known to be optimal in the sense of Theorem 4.

The Sobel-Wald test allows control of the correct decision probabilities. To control these probabilities, one must control the error probability of the SPRTs. The latter is, of course possible (to an extent described in the previous chapter) using (2.2.13). The bounds (2.2.13) also imply that by selecting the boundary values of the two SPRTs, the error probabilities can be made arbitrary small. Another important feature is the fact that in order for a decision to be reached, both SPRTs must stop, and so the stopping time of the test can be written as $N = \max\{N_1, N_2\}$ where N_1 and N_2 are the stopping times of the two component SPRTs used. The latest expression indicates that a bound on the expected stopping time is given in terms of the SPRTs stopping times by $\max\{\mathbb{E}[N_1], \mathbb{E}[N_2]\} \leq \mathbb{E}[N]$. This simple bound will be found useful in the sequel.

The Sobel-Wald test was considered in the case a of normal distribution with unknown mean [133], a Bernoulli distribution [151], and the exponential family case [59]. In all these examples, Walds approximation was used in order to estimate the error probabilities.

There are a few obvious drawbacks in the Sobel-Wald test:

- It is designed for three hypotheses only.
- There is an assumption that some relation between the three hypotheses holds,

namely that $d_1 = H_1$ cannot occur together with $d_2 = H_3$ (this is also the reason why only two SPRT components are used and are sufficient).

- The Sobel-Wald test does not use all the information available at the stopping time. To see why, notice that d_1 and d_2 do not necessarily terminate at the same time. This is significant since an SPRT can decide on one hypothesis at a certain time, but as more observations are available, this decision can be reversed.

B. The Armitage Test

The next test, suggested by Armitage in 1950 [5], was aimed to solve, some of the major problems of the Sobel-Wald test. Like the Sobel-Wald test, the Armitage test, sometimes referred to as the *matrix SPRT*, combines a number of SPRTs, but unlike the former, it is defined for any (finite) number of simple hypotheses. Another main difference is that the decision in the Armitage test is based on all the information available up to the stopping time. This is done by using the so called "extended" SPRTs, which means that each SPRT component is extended until a decision is made.

The structure of the Armitage test is the following: for the M hypotheses to test, $\{H_0, H_1 \ldots, H_{M-1}\}$, the test uses $M\left(M-1\right)/2$ SPRTs between hypotheses H_i and H_j for all $i < j$. The stopping time is defined as the first time instant at which all $(M-1)$ SPRTs involving H_j simultaneously lead to the decision H_j. Let $\lambda_n^{i,j}$ denote the likelihood ratio between the hypotheses H_i and H_j, and let the boundary values of the SPRT between H_i and H_j be chosen to be A_{ij} for the upper boundary value and $B_{ij} = -A_{ij}$ for the lower boundary value. Since $\lambda_n^{i,j} = -\lambda_n^{j,i}$, the stopping time of the Armitage test is given by

$$N = \min_{0 \leq j \leq M} \inf_{n \geq 1} \left\{ \lambda_n^{i,j} \geq A_{ij} \text{ for all } i \neq j \right\}. \tag{4.2.2}$$

The decision function $d = d\left(Y^N\right)$ is given by

$$d = H_j \text{ for which } \lambda_n^{i,j} > A_{ij} \ \forall i \neq j. \tag{4.2.3}$$

The control over the error probabilities is obtained by the free parameters of the test, $\{A_{ij}\}$. For example, the correct decision probabilities $\alpha_{ii} = \Pr\left(d = H_i \mid H_i\right)$ can be upper bounded by using the error probabilities $\alpha_{ij} = \Pr\left(d = H_j \mid H_i\right)$ and the relation $\alpha_{ij} \leq A_{ij}^{-1}$, which can be shown to hold just as the counterpart bound for the

SPRT. One also has that

$$\alpha_{ii} = 1 - \sum_{i \neq j} \alpha_{ij} \geq 1 - \sum_{i \neq j} A_{ij}^{-1} \tag{4.2.4}$$

so that by using the Armitage test one can control the whole matrix of error probabilities (as well as the vector of correct decision probabilities). In [5] Armitage studied a multiple-hypothesis testing problem with three hypotheses corresponding to three Binomial probabilities. In this specific setup, the Armitage and Sobel-Wald tests take on a similar form. In general, since the basic assumptions in the two tests are different, it is not possible to compare them in any "fair" manner but in the case where the hypotheses satisfy the constraint of Sobel-Wald tests, that is, $\Pr\{d_1 = H_1\,,\, d_2 = H_3\} = 0$. For this setup, by taking the same boundary values for the SPRTs, it is clear that the Sobel-Wald stopping time is at most the Armitage stopping time. Moreover, it has a higher probability of accepting the true hypothesis. Simulation results supporting this claim were obtained in [47] for the case of testing three hypotheses under which the observations are i.i.d. and normally distributed with means $\mu, -\mu$ and zero, and variance 1.

C. The Lorden Test

The Sobel-Wald and Armitage tests take a "positive" approach. They stop as soon as some hypothesis is preferred over all others. In this section, a different type of a multiple-hypothesis test will be presented - the *Lorden test* [93]. This test rules in favor of H_j when all other hypotheses can be rejected. The rejections are not necessarily in favor of H_j; they are just rejections of H_i for all $i \neq j$. Two other important features are the fact that this test applies also to composite hypotheses, and, that it is based on the GLRT that was briefly discussed in the previous chapter. In [93], Lorden considered the an i.i.d. sequence $Y_1, Y_2, \ldots$ whose density belongs to the exponential family $f_\theta(y) = \exp\{\theta y - b(\theta)\}$, where $\theta \in \left(\underline{\theta}, \overline{\theta}\right) \subset \mathbb{R}$ and $b(\theta)$ is a convex and infinitely differentiable in $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$. Lorden defined the log-likelihood function up to time n to be $L_\theta(n) = \theta S_n + nb(\theta)$, where $S_n = Y_1 + Y_2 + \ldots + Y_n$ and the maximum likelihood estimator based on $(Y_1, \ldots, Y_n)$ to be $\hat{\theta}_n$. The original statistical problem is specified by M intervals ($M \in \mathbb{N}$), $(\theta_0, \theta_1], [\theta_1, \theta_2] \ldots [\theta_{M-1}, \theta_M)$, $k \geq 2$ decisions that can be made, and a set of constants $\{\alpha_{ij}, 1 \leq i \leq s+1, 1 \leq j \leq k\}$ representing the constraints on the error probabilities:

$$P_{\theta}\left(\text{j'th decision}\right) \leq \alpha_{ij}\ ,\ \forall \theta \in H_i\,,\, i \neq j = 0, 1, \ldots, M-1\,,\, j = 1, 2, \ldots, k. \quad (4.2.5)$$

Some additional conditions were assumed to hold regarding the structure of $\{\alpha_{ij}\}$ [93]. Instead of considering this general model, we concentrate on a specific case (that also appears in [48]), in which M hypotheses $H_i\colon \theta = \theta_i\ ,\ i = 0, \ldots, M-1$ are to be tested about the parameter θ of the one-parameter exponential distribution that was introduced before. The decision function will then take values in $\{0, \ldots, M-1\}$, and the error probability constraints will be:

$$P_{\theta_i}\left(\text{j'th decision}\right) \leq \alpha_{ij}\ ,\ \forall \theta \in H_i\,,\, i \neq j = 0, 1, \ldots, M-1. \quad (4.2.6)$$

Lorden defined a likelihood ratio test $\left(\hat{N}, \hat{d}\right)$ as follows:

$$\begin{aligned} \hat{N}_j &= \min_{n \geq 0} \left\{ L_{\hat{\theta}_n}\left(n\right) \geq \max_{i \neq j} \left[L_{\theta_i}\left(n\right) - \log \alpha_{ij} + c_{ij} \right] \right\}, & (4.2.7) \\ \hat{N} &= \min_{j} \hat{N}_j, & (4.2.8) \end{aligned}$$

where α_{ij} are the specified constraints on the error probabilities and c_{ij} are constants, which are chosen such that the constraints on the error probability are met. The decision function $\hat{d}$ will be equal to the one hypothesis that had not been rejected, that is, $\hat{d}$ chooses the smallest j such that $\hat{N} = N_j$. The Intuitive explanation is that the chosen hypothesis is the last to be "ruled off". Specifically, the test rules in favor of a hypothesis after the log-likelihood functions of the other hypotheses are relatively small. This structure was used in other works, for example, in [116] and [122], and will be further discussed in the next section of the sequel where more modern ideas are reviewed.

In [93] two main theorems where proven regarding the performance of $\left(\hat{N}, \hat{d}\right)$. The first is that, under the proper choice of the threshold parameters c_{ij}, the difference between the expectation value of $\hat{N}$, $\mathbb{E}_{\theta}\left[\hat{N}\right]$ and the infimum of $\mathbb{E}_{\theta}\left[N\right]$ over all tests $\Delta = (N, d)$ for which (4.2.6) holds, does not exceed a given function of α_{ij} for any $\theta = \theta_i$ (Theorem 1 in [93]), and the second is a proof of the asymptotic optimality of this likelihood ratio test in the sense that there is a choice of $\{c_{ij}\}$ that guarantees that any procedure satisfying (4.2.6) has expected sample sizes which are larger than that of the likelihood ratio test as $\min_{i,j} c_{ij} \to 0$ for all $\theta_i \in \{\theta_0, \ldots, \theta_{M-1}\}$.

The main downside of this test is that Lorden does not specify how to choose the boundary values in order to achieve these asymptotically optimal results. Another setback in Lorden's test is the use of the maximum likelihood estimate. This point will be further discussed in the next chapter, but it should be noted that, although for i.i.d. observation it is reasonable that this estimate will perform well for large sample sizes, we have no guarantee regarding its performance over short random sequences. Evidently, Lorden's test may be expected to act quite poorly for short to moderate sample sizes.

4.3 MSPRT and Asymptotic Optimality

A. Two Sequential Test Procedures

In this section, we consider the Bayessian formulation. The setup is similar to the of the Armitage test. Specifically, assume there are M hypotheses $H_i \colon P = P_i$, $i \in \{0, \ldots, M-1\}$, where P_i are known distinct probability measures. We denote by $\{P_i^n \ , \ 0 \le i \le M-1\}$ the restriction of P_i to the σ-algebra $\mathcal{F}_n = \sigma(Y_1, \ldots Y_n)$, and

$$L_i(n) = \log\left[\frac{dP_i^n(Y_1, \ldots, Y_n)}{dQ^n(Y_1, \ldots, Y_n)}\right] \quad , \quad i = 0, \ldots, M-1 \tag{4.3.1}$$

denotes the log-likelihood ratio (LLR) processes with respect to a dominating measure Q^n. If $Q^n = P_j^n$ for some $0 \le j \le M-1$, the corresponding LLR process will be denoted $L_{ij}(n)$. Take $W(j,i)$ to be a given loss associated with a decision on H_i when H_j is true (without loss of generality, that the losses due to correct decisions are zero, i.e., $w(j,j) = 0$) and let $(\pi_0, \pi_1 \ldots, \pi_{M-1})$ be the prior distribution vector of the hypotheses. As before, $\Delta = (N, d)$ will represent a sequential test with a stopping rule N and a decision function d. As in other Bayesian problems, the risk associated with decision $d = i$ is defined as

$$R_i(\Delta) = \sum_{j=0, j\neq i}^{M-1} \pi_j W(j,i)\, \alpha_{ji}(\Delta) \tag{4.3.2}$$

where for $j \neq i$, $\alpha_{ji}(\Delta) = P_j(d = i)$ is the probability of accepting the hypothesis H_i when H_j is true. In the case of the $0-1$ loss function, where $W(j,i) = \delta_{j,i}$, $R_i(\Delta)$ is the same as frequentist error probability $\alpha_i(\Delta)$, which is the probability of accepting H_i incorrectly. That is, for the $0-1$ loss function

$$R_i(\Delta) = \alpha_i(\Delta) = \sum_{\substack{j=0 \\ j\neq i}}^{M-1} \pi_j \alpha_{ji}(\Delta)\,. \tag{4.3.3}$$

We now introduce the following class of tests

$$\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)=\left\{\Delta : R_i\left(\Delta\right)\leq \overline{R}_i\ ,\ i=0,1,\ldots,M-1\right\}, \tag{4.3.4}$$

where $\left(\overline{\mathbf{R}}\right)=\left(\overline{R}_0,\overline{R}_1,\ldots,\overline{R}_{M-1}\right)$ is a given vector of positive finite numbers.

A reasonable figure of merit is the minimum average observation time, $\mathbb{E}\left[N\right]=\sum_{i=0}^{M-1}\pi_i\mathbb{E}_i\left[N\right]$ among all tests in $\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)$. It turns out that in the i.i.d. case even a relatively simple test that is nearly optimal, involves a comparison of posterior probabilities with random thresholds which must be determined for each model separately ([8] and [135]). In general, it is very difficult to find the explicit form of these thresholds. Furthermore, even if one finds the boundaries of the optimal test, it would be difficult to implement this procedure in practice since it would involve the calculation of a new boundary value every time instant. In addition, while in the case of two hypotheses, Wald's SPRT minimizes not only the average observation time but both of the expectations under each possible measure, it is unclear if such a test exists for $M\geq 3$ [8]. However, in an asymptotic setting, where the risks (or the error probabilities) are sufficiently small, such tests may be found.

Next, two asymptotically optimal tests will be described. Both will be based on the likelihood ratio between the different hypotheses, and are called in general "multihypothesis SPRT" (MSPRT). The idea is to simplify the structure of the optimal test by replacing the nonlinear random thresholds with simple functions (constants in the case of the zero-one loss function):

Test Δ_a: Introduce the stopping times

$$N_i=\min_{n\geq 0}\left\{L_i\left(n\right)\geq a_i+\log\left(\sum_{j\neq i}w_{ji}\exp\left\{L_j\left(n\right)\right\}\right)\right\}, \tag{4.3.5}$$

where $w_{ji}\triangleq\frac{\pi_j W(j,i)}{\pi_i}$ and a_i are positive threshold values. The test procedure $\Delta_a=\left(N_a,d_a\right)$ is defined as follows:

$$N_a=\min_{0\leq k\leq M-1}N_k,\quad d_a=i \text{ if } N_a=N_i. \tag{4.3.6}$$

That is, we stop as soon as the threshold in (4.3.5) is exceeded for some i and decide in favor of that H_i. This test is motivated by a Bayesian framework and it was considered earlier by Fishman [52], Golubev and Khas'minskii [61], Baum and Veeravalli [8] and more. Indeed, in the special case of zero-one losses, the stopping times N_i take on the following form:

$$N_i=\min_{n\geq 0}\left\{\Pi_i\left(n\right)\geq A_i\right\}, \tag{4.3.7}$$

where

$$A_i = \frac{\exp(a_i)}{1+\exp(a_i)} \quad , \quad \Pi_i(n) = \frac{\pi_i \exp\{L_i(n)\}}{\sum_{j=0}^{M-1} \pi_j \exp\{L_j(n)\}} \tag{4.3.8}$$

that is, $\Pi_i(n)$ is the posterior of the hypothesis H_i. Note also that the problem is symmetric it terms of π_i and α_i (is the sense that it is reasonable to choose $A_i = A$ for all $0 \le i \le M-1$) the stopping time of the test is then given by

$$N_a = \min_{n\ge 0}\left\{\max_i \Pi_i(n) \ge A\right\} \quad \text{where } A = \frac{\exp\{a\}}{1+\exp\{a\}}, \tag{4.3.9}$$

i.e., we stop as soon as the largest posterior probability exceeds a threshold.

Test Δ_b: Introduce the stopping time

$$M_i = \min_{n\ge 0}\left\{L_i(n) \ge \max_{j\ne i}\left[b_{ij} + \log(w_{ji}) + L_j(n)\right]\right\} \tag{4.3.10}$$

which is the "accepted" stopping time for the hypothesis H_i, where b_{ij} are positive thresholds. The test $\Delta_b = (M_b, d_b)$ is defined as follows:

$$M_b = \min_{0\le i\le M-1} M_i \quad , \quad d_b = i \text{ if } M_b = M_i. \tag{4.3.11}$$

This is a modification of the matrix SPRT (the combination of one-sided SPRTs) by Armitage, and was discussed earlier for a specific case. It was also analyzed in Lorden's early work on asymptotically optimal tests [96] as well as in other works by Dragalin (e.g., [42]), Tartakovsky (e.g. [136]) and Verdenskaya [141]. Note also that if $b_{ij} = b_i$ and the loss function is $0-1$, then $w_{ji} = \frac{\pi_j}{\pi_i}$ and the stopping time M_i can be writen in the form

$$M_i = \min_{n\ge 0}\left\{\hat{\Pi}_i(n) \ge \exp\{b_i\}\right\}, \tag{4.3.12}$$

where

$$\hat{\Pi}_i(n) = \frac{\pi_i \exp\{L_i(n)\}}{\max_{k\in\{0,\ldots,M-1\}\setminus(i)} \pi_k \exp\{L_k(n)\}}, \tag{4.3.13}$$

i.e., $\hat{\Pi}_i(n)$ is the generalized likelihood ratio (GLR) between H_i and the remaining hypotheses.

Implementation issues: Notice that if the distribution belongs to an exponential family, which is a good model for many applications, then the test Δ_b has an advantage over the first test in that it does not require exponential transformations of the observations. This makes it more convenient for practical realizations. However, the test Δ_a has the advantage that it is easier to design the thresholds $\{a_i\}$ so as to precisely meet constraints on the risks R_i [43].

B. Bounds on the Performance of Δ_a and Δ_b

We start by stating a basic lemma that indicates that one can choose the thresholds so as to guarantee that the tests belong to the class $\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)$. It is worth emphasizing that the bounds hold under general assumptions, and require neither independence nor homogeneity of the observed data. The proof can be found, for example, in [8] and it is similar to the analysis in [5].

Lemma 6 *Let $\{Y_n\,,\, n \geq 0\}$ be an arbitrary random process observed in discrete time. For all $i = 0, 1, \ldots, M-1$*

$$R_i\left(\Delta_a\right) \leq \pi_i \exp\left(-a_i\right), \quad R_i\left(\Delta_b\right) \leq \pi_i \sum_{j=0, j\neq i}^{M-1} \exp\left(-b_{ij}\right). \tag{4.3.14}$$

Corollary 7 *Let*

$$a_i = \log\left[\frac{\pi_i}{\overline{R_i}}\right], \quad b_{ij} = b_i = \log\left[\frac{(M-1)\,\pi_i}{\overline{R_i}}\right]. \tag{4.3.15}$$

Then, both tests belong to the class $\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)$. In addition, under this choice of the threshold values, $N_a \leq M_b$

The following theorem, proved in [44] (Theorem 3.1), has a long evolution in the theory of sequential multiple hypothesis testing analysis. Its importance will be made clear in the sequel, as it can be used to obtain a bound on the potential asymptotic performance of any sequential multihypothesis test in the class $\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)$ as the risks go to zero (or as $R_{\max}$ goes to zero, where $R_{\max} \triangleq \max_{0\leq i\leq M-1} \overline{R_i}$).

Theorem 8 *Assume there exists an increasing nonnegative function $f(n)$ and positive finite constants q_{ij} $(i, j = 0, 1, \ldots M-1, i \neq j)$ such that*

$$\frac{L_{ij}(n)}{f(n)} \overset{P_i\text{-a.s}}{\longrightarrow} q_{ij} \text{ as } n \to \infty\ ,\ i, j = 0, 1, \ldots M-1, i \neq j. \tag{4.3.16}$$

In addition, assume that for all $L > 0$

$$P_i\left(\sup_{n\leq L} L_{ij}^{+}(n) < \infty\right) = 1. \tag{4.3.17}$$

Then for all $m > 0$ and $i = 0, 1, \ldots M-1$

$$\inf_{\Delta=(N,d)\in\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)} \mathbb{E}_i\left[N^m\right] \geq \left[f^{-1}\left(\frac{\left|\log \overline{R_i}\right|}{\min_{i\neq j} q_{ij}}\right)\right]^m (1 + o(1)) \tag{4.3.18}$$

as $R_{\max} \to 0$ and where $o(1) \to 0$ as $R_{\max} \to 0$.

In order to gain intuition regarding the previous result we return to the binary hypothesis case, inferring between two probability measures P and Q. Recall that in Section 2.3 we have shown that, for example, under Q

$$\mathbb{E}_Q[N]\, D(Q \parallel P) \geq \left[(1-\beta)\log\left(\frac{1-\beta}{\alpha}\right) + \beta\log\left(\frac{\beta}{1-\alpha}\right)\right]. \tag{4.3.19}$$

and that the optimal test achieves this bound. In the limit of small error probabilities, this inequality can be written as

$$\mathbb{E}_Q[N] \gtrsim \frac{-\log\alpha}{D(Q \parallel P)}. \tag{4.3.20}$$

Similarly, it can be proven that for this limit the m'th moment of the expected stopping time can be bounded by

$$\mathbb{E}_Q[N^m] \gtrsim \left[\frac{-\log\alpha}{D(Q \parallel P)}\right]^m. \tag{4.3.21}$$

Theorem 8 tells us that the bound in (4.3.21) holds for a more general case as well, at least in the case of i.i.d observations, where f is the identity function.

C. Asymptotic Optimality of Δ_a and Δ_b

So far we considered quite a general case imposing only minor restrictions on the structure of the observed process. All statements made so far have a continuous-time version as well. In this subsection, we consider the discrete-time case and assume that, under hypothesis H_i, $Y_1, Y_2, \ldots$ are i.i.d. with a known density $f_i(y)$, and that the densities do not coincide with probability one in the sense that $P_i(L_{ij}(1) = 0) < 1$ for all $j \neq i$. In general, all the forthcoming results also hold when each observation, $\mathbf{Y_i}$, is a random vector (that is, $\{\mathbf{Y_i} = (Y_{i,1}, \ldots, Y_{i,l}),\ l \in \mathbb{N}\ , i = 1, 2, \ldots\}$).

Define $\Delta L_{ij}(n)$

$$\Delta L_{ij}(n) = \log\left[\frac{f_i(Y_n)}{f_j(Y_n)}\right] \quad , \quad L_{ij}(n) = \sum_{k=1}^{n} \Delta L_{ij}(k). \tag{4.3.22}$$

The Kullback-Leibler (KL) divergence is given by

$$D(f_i \parallel f_j) \triangleq D_{ij} = \mathbb{E}_i[\Delta L_{ij}(n)]. \tag{4.3.23}$$

In addition, we define the vector $\{D_i, i = 0, 1 \ldots, M-1\}$ for which each entry is $D_i \triangleq \min_{j \neq i} D_{ij}$. Due to the aforementioned condition of distinct measures, these

KL distances are strictly positive. In fact, it is common to associate D_{ij} with a "distance" property, in spite of the fact that it is not a metric (see, for example, [30]). Using this interpretation, D_i will be said to be the minimal distance between H_i and the other hypotheses. We shall also assume that $D_{ij} < \infty$.

We consider a general asymmetric case (with respect to risk constraints) with the restriction that $\overline{R_i}$ approaches zero such that for all i, j, $0 < \frac{\log \overline{R_i}}{\log \overline{R_j}} < \infty$. In addition, due to the i.i.d. assumption, Theorem 8 implies that for any $m > 0$ and $i \in \{0, 1, \ldots, M-1\}$,

$$\inf_{\Delta=(N,d)\in\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)} \mathbb{E}_i\left[N^m\right] \geq \left(\frac{\left|\log \overline{R_i}\right|}{D_i}\right)^m (1 + o(1)) \text{ as } \overline{R}_{\max} \to 0, \tag{4.3.24}$$

where $\overline{R}_{\max} = \max_i \overline{R}_i$. The following theorem summarizes the main results on the asymptotic performance and the asymptotic optimality in the i.i.d. case. The theorem and its proof appear in [44], where the authors used results of [8], combined with classical results from the theory of stopped random walks (that are summarized in [63]) to show that both Δ_a and Δ_b, are asymptotically optimal, not only in the expected sample size, but also in any positive moment of the stopping time.

Theorem 9 *Let N_a and M_b be the stopping rule of Δ_a and Δ_b, respectively, and $0 < D_{ij} < \infty$.*

1. *For all $m \geq 1$ and $i = 1, 2, \ldots, M-1$*

$$\mathbb{E}_i\left[N_a^m\right] \sim \left(\frac{a_i}{D_i}\right)^m, \quad \text{as } a_{min} \to 0, \tag{4.3.25}$$

$$\mathbb{E}_i\left[M_b^m\right] \sim \max_{j\neq i}\left(\frac{b_{ij}}{D_{ij}}\right)^m, \quad \text{as } b_{min} \to 0, \tag{4.3.26}$$

where $a_{\min} \triangleq \min_i a_i$ and $b_{\min} \triangleq \min_{i,j} b_{ij}$.

2. *If the thresholds are determined by (4.3.15), then*

$$\inf_{\Delta=(N,d)\in\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)} \mathbb{E}_i\left[N^m\right] \sim \mathbb{E}_i\left[N_a^m\right] \sim \mathbb{E}_i\left[M_b^m\right] \sim \left(\frac{\left|\log \overline{R_i}\right|}{D_i}\right)^m \tag{4.3.27}$$

as $\overline{R}_{max} \to 0$ for all $m \geq 1$.

Everywhere above $x_\gamma \sim y_\gamma$ as $\gamma \to \gamma_0$ means that $\lim_{\gamma\to\gamma_0} \frac{x_\gamma}{y_\gamma} = 1$.

Theorems 8 and 9 are further generalized in [44] in a few directions:

The i.i.d. case: All the results stated above can be generalized to continuous-time processes if the LLRs processes have independent and stationary increments and finite first absolute moments [44]. In addition, if

$$\frac{\log \overline{R_i}}{\log \overline{R_j}} \sim 1 \quad \forall i,j,i \neq j \text{ as } \overline{R}_{\max} \to 0 \tag{4.3.28}$$

then a much stronger result is true for the expected observation time. Specifically, if the thresholds are chosen so that $R_i\left(\Delta_b\right) \sim \overline{R_i}$ and (4.3.28) is fulfilled, then

$$\mathbb{E}_i\left[M_b\right] = \inf_{\Delta=(N,d)\in\mathbf{\Delta}\left(\overline{\mathbf{R}}\right)} \mathbb{E}_i\left[N^m\right] + o\left(1\right) \text{ as } \overline{R}_{\max} \to 0 \tag{4.3.29}$$

and the same is true for Δ_a. This is derived by using the results of Lorden [96]. The authors of [44] conjecture that this characteristic is true also when (4.3.28) does not hold. Simulation results presented in [43] support this conjecture.

The non-i.i.d. case: So far the assumption of i.i.d. observations was crucial.It simplified many of the proofs since in this case the LLR process is a random walk , which has many useful properties, most importantly, the convergence due to the strong law of large numbers and its variants. To obtain similar optimality properties of Δ_a and Δ_b in the form of Theorems 8 and 9, a different notion of convergence can be used - the *r-quickly* convergence:

Definition 1 *Let $\{\zeta_t\,,\, t\in\mathbb{R}\}$ be a random process. For $h>0$, define $T\left(h\right)$ to be*

$$T\left(h\right) = \sup_{t\in\mathbb{R}}\left\{\left|\zeta_t - q\right| \geq h\right\}. \tag{4.3.30}$$

For $r>0$, ζ_t is said to converge r-quickly under the measure P if

$$\mathbb{E}_P\left[T^r\left(h\right)\right] < \infty \quad \forall h > 0. \tag{4.3.31}$$

In [44], a generalization of Theorems 8 and 9 are established, with the a.s. convergence condition (4.3.16) replaced by the r-quickly convergence condition

$$\frac{L_{ij}\left(n\right)}{f\left(n\right)} \overset{P_i\text{-r-quickly}}{\longrightarrow} q_{ij} \text{ as } n\to\infty\ ,\ i,j=0,1,\ldots M-1, i\neq j \tag{4.3.32}$$

and the optimality of Δ_a and Δ_b is proven up to the order of r (that is, for $m \leq r$ in Theorem 8 and 9) where r is the largest constant for which (4.3.32) holds.

More generalities: Woodroofe's *nonlinear renewal theory* [155] comprises powerful techniques that allow taking into account the "overshoot" over the boundary of the test statistics. For i.i.d. observations, asymptotic approximations (up to a vanishing term, as the risks go to zero) for the expected sample size of N_b and M_b can be made using tools from the *nonlinear renewal theory*. We will not elaborate more on these subjects but good references to this type of calculation are [155] and [43], in addition to simulation results that show that for some simple examples (such as testing the mean of an i.i.d. Gaussian sequence) the approximations are fairly accurate, not only for large, but also for moderate sample sizes.

4.4 Multiple Hypothesis Testing With Control

A. Multiple Hypothesis Testing Via Controlled Sensing

Generally speaking, the topic of controlled sensing for inference deals primarily with adaptively managing and controlling multiple degrees of freedom in an information-gathering system. Unlike in traditional control systems, where the control primarily affects the evolution of the state, in controlled sensing, the control affects only the observations. In other words, the goal is not to drive the state to some desired level, but for the decision maker to infer the state accurately by shaping the quality of the observations.

This section focuses on sequential hypothesis testing as a controlled sensing problem in which the controller can adaptively decide, based on past observations and controls, whether to continue collecting new observations, or to stop and make the final decision (that is, prior to making a decision about the hypothesis, the decision maker can choose among different actions to shape the quality of the observations).

The discussion in this section will concentrate on a fundamental controlled sensing test for hypothesis testing and some generalization of it. Two related models will be considered: one for two composite hypotheses and the other for simple hypothesis testing of multiple hypotheses, both with observation control. In particular, in the composite setup we assume that two disjoint sets, Θ_0, Θ_1 on some parameter space, are given and define $\Theta = \Theta_0 \cup \Theta_1$. The goal is then to test

$$H_0 : \theta \in \Theta_0 \quad \text{Vs.} \quad H_1 : \theta \in \Theta_1, \tag{4.4.1}$$

where $\theta \in \Theta$ is a parameter of some density function, as in the setup introduced in

Section 3.1. In the second model, we will dwell on an M-hypothesis testing problem, with a set of simple hypotheses $\{H_0, H_1, \ldots, H_{M-1}\}$, similar to the case described in Section 4.3. In both cases, at each time step k, the observation Y_k takes values in $\mathcal{Y}$. In addition, we now assume control is present and denote the control sequence by $\{U_1, U_2, \ldots\}$, where each U_k is a control variable taking values in a finite alphabet $\mathcal{U}$. Under each hypothesis $i \in \{0, 1, \ldots, M-1\}$ or $\theta \in \Theta$, conditioning on $U_k = u$, Y_k is assumed conditionally independent of $\left(Y^{k-1}, U^{k-1}\right)$.

In general, two classes of control policies are possible: the first is the open-loop control policy where the (possibly randomized) control sequence $\{U_1, U_2, \ldots\}$ is assumed independent of $\{Y_1, Y_2, \ldots\}$. The second policy, which will be the one discussed, is the causal control policy, where at time k, U_k can be any (possibly randomized) function of past observations and controls. The control is described by a conditional probability mass function (pmf) $q_k\left(u_k \mid y^{k-1}, u^{k-1}\right)$ and U_1 is distributed according to a pmf $q_1(u_1)$ (If all these (conditional) pmfs are point-mass distributions, i.e., the current control is a deterministic function of past observations and past controls, then the resulting policy is often referred to as a *pure* control policy). Under the aforementioned assumption and under each hypothesis$i$, the joint probability distribution function of (Y^n, U^n), denoted by $p_i(y^n, u^n)$, can be written as

$$p_i(y^n, u^n) = q_1(u_1) \prod_{k=1}^{n} p_i^{u_k}(y_k) \prod_{k=2}^{n} q_k\left(u_k \mid y^{k-1}, u^{k-1}\right) \quad \text{for any } n > 1 \tag{4.4.2}$$

and in the composite setup, the density function, for any $\theta \in \Theta$, is given by

$$p_\theta(y^n, u^n) = q_1(u_1) \prod_{k=1}^{n} p_\theta^{u_k}(y_k) \prod_{k=2}^{n} q_k\left(u_k \mid y^{k-1}, u^{k-1}\right) \quad \text{for any } n > 1 \tag{4.4.3}$$

where we have denoted, for all $u \in \mathcal{U}$, $i \in 0, \ldots, M-1, \theta \in \Theta$:

$$p_i(y_k \mid u_k) = p_i^{u_k}(y_k) \quad \text{and} \quad p_\theta(y_k \mid u_k) = p_\theta^{u_k}(y_k) \tag{4.4.4}$$

to be the density or mass function of Y_k under control $u_k \in \mathcal{U}$.

Since the controlled problem differs in many aspects from the problem discussed so far, a few refinements of the notation are in place: in this section we will denote by $\mathcal{F}_k$ the σ-algebra generated by $\left(Y^k, U^k\right)$. A sequential test $\Delta = (\phi, N, d)$ consists of a causal observation control policy ϕ, which is described by the pmfs $\left\{q(u_1), q\left(u_k \mid y^{k-1}, U^{k-1}\right)_{k=2}^{\infty}\right\}$, an $\mathcal{F}_k$-stopping time N representing, as before, the (random) number of observations before the final decision, and a decision rule $d = d\left(Y^N, U^N\right)$.

Chernoff's Test

The problem of sequential binary composite hypothesis testing with observation control was considered by Chernoff [24], and an asymptotically optimal sequential test was presented.

While Wald's SPRT is optimal in the sense that it minimizes the expected values of the stopping time among all tests for which the probabilities of error do not exceed predefined thresholds, a weaker notion of optimality is adopted in [24], which is similar in spirit to the notion of the asymptotic optimality presented in the previous section. This optimality criterion will be presented via Theorem 10 that will follow.

The proof of the asymptotic optimality of this test requires the following technical assumptions: for all $\theta \in \Theta$ and $\theta \neq \phi \in \Theta$:

$$\begin{aligned} D\left(p_\theta^u \parallel p_\phi^u\right) &> 0, && (4.4.5)\\ \mathbb{E}_{p_\theta^u}\left[\left(\log\left[\frac{p_\theta^u(Y)}{p_\phi^u(Y)}\right]\right)^2\right] &< \infty. && (4.4.6)\end{aligned}$$

The Chernoff test for binary composite hypothesis testing admits the following sequential description:

Define the sets $h(\theta)$ and $a(\theta)$ to be

$$h(\theta) = \begin{cases} \Theta_0 & \text{if} \quad \theta \in \Theta_0 \\ \Theta_1 & \text{if} \quad \theta \in \Theta_1 \end{cases}, \quad a(\theta) = \begin{cases} \Theta_1 & \text{if} \quad \theta \in \Theta_0 \\ \Theta_0 & \text{if} \quad \theta \in \Theta_1. \end{cases} \tag{4.4.7}$$

Having fixed the control policy up to time k, and obtaining the first k observations and control values $\left(Y^k, U^k\right)$, if the controller decides to collect more observations, then at time $k+1$, a randomized control policy is adopted wherein U_{k+1} is drawn from the following distribution

$$q(u) = q\big(u \mid \hat{\theta}_k\big) = \underset{\bar{q}(u)}{\operatorname{argmax}} \min_{\phi \in a\left(\hat{\theta}_k\right)} \sum_{u \in \mathcal{U}} \bar{q}(u)\, D\left(p_{\hat{\theta}_k}^u \parallel p_\phi^u\right), \tag{4.4.8}$$

where $\hat{\theta}_k = \underset{\theta \in \Theta}{\operatorname{argmax}}\, p_\theta\left(y^k, u^k\right)$ is the ML estimate of the hypothesis at time k. The stopping time N_{ch} is defined to be

$$N_{\text{ch}} = \min_{n \geq 1}\left\{S_n \triangleq \sum_{i=1}^{n} \log\left[\frac{p_{\hat{\theta}_n}^{U_i}(Y_i)}{p_{\tilde{\phi}_n}^{U_i}(Y_i)}\right] \geq -\log c\right\}, \tag{4.4.9}$$

where c is a positive parameter that will be selected to approach zero in order to drive the error probabilities to zero, and $\tilde{\phi}_k$ is the ML estimator at time k restricted to $a(\hat{\theta}_n)$. At the stopping time, the decision rule, denoted by d_{ch}, is the ML decision rule, i.e., accept the hypothesis $h(\hat{\theta}_n)$ if $S_n > -\log c$.

The asymptotic optimality of Chernoff's test was proven in [24] with respect to the Bayes-risk formulation. Let $r(\theta)$ represent the (positive) loss function due to making the wrong decision (in other words, if $\tilde{\theta} \in \Theta_i$ is true, the loss due to choosing $\theta \in \Theta_{1-i}$, is $r\left(\tilde{\theta}\right)$). Define the risk under hypothesis θ to be

$$\hat{R}(\theta) = \mathbb{E}\left[r(\theta)\,\mathbb{I}\left\{d_{\text{ch}} \to \text{error}\right\} + cN_{\text{ch}}\right], \tag{4.4.10}$$

where the event $\left\{d_{\text{ch}} \to \text{error}\right\}$ stands for the event that the Chernoff test errs and the expectation taken with respect to the true hypothesis. The next theorem will formally establish the optimality of Chernoff's test is the sense that it achieves the optimal value of the risk (up to an order of magnitude) for all $\theta \in \Theta$.

Theorem 10 (Theorem 14.1 in [26] and Theorems 1 and 2 in [24]) *For the case in which Θ_0, Θ_1 and $\mathcal{U}$ are all finite and* (4.4.5) *and* (4.4.6) *are satisfied, for any given $\epsilon > 0$, there exists a $c^\star = c^\star(\epsilon)$ such that $\hat{R}(\theta)$ of the Chernoff test satisfies:*

$$\hat{R}(\theta) \leq -[1+\epsilon]\frac{c\log c}{\max\limits_{\bar{q}(u)}\ \min\limits_{\phi\in a(\theta)} \sum_{u\in\mathcal{U}} \bar{q}(u)\, D\left(p_\theta^u \parallel p_\phi^u\right)} \quad \textit{for } c < c^\star \textit{ and for all } \theta \in \Theta \tag{4.4.11}$$

and any procedure $\Delta^\star$ for which $\hat{R}(\theta) = O(-c\log c)$ for all $\theta \in \Theta$, the following holds: for any $\epsilon > 0$, there is $c^{\star\star} = c^{\star\star}(\epsilon)$ such that

$$\hat{R}(\theta) \geq -[1+\epsilon]\frac{c\log c}{\max\limits_{\bar{q}(u)}\ \min\limits_{\phi\in a(\theta)} \sum_{u\in\mathcal{U}} \bar{q}(u)\, D\left(p_\theta^u \parallel p_\phi^u\right)} \quad \textit{for } c < c^{\star\star} \textit{ and for all } \theta \in \Theta. \tag{4.4.12}$$

A straightforward generalization of the test procedure defined by (4.4.8) and (4.4.9) to the controlled multiple simple-hypothesis testing was presented by Bessler in [12]. Similarly to the assumptions (4.4.5) and (4.4.6), Bessler assumed that for every $u \in \mathcal{U},\ 0 \leq i < j \leq M-1$:

$$D\left(p_i^u \parallel p_j^u\right) \ > \ 0, \tag{4.4.13}$$

$$\mathbb{E}_{p_i^u}\left[\left(\log\left[\frac{p_i^u(Y)}{p_j^u(Y)}\right]\right)^2\right] \ < \ \infty \tag{4.4.14}$$

and his test procedure is the following: after fixing the control policy obtain the first k observations; if the controller decides to take more observations, then at time $k+1$ randomly draw $U_{k+1} \in \mathcal{U}$ from $q_B(u)$ which is defined to be:

$$q_B(u) = q_B\big(u \mid \hat{i}_k\big) = \underset{\bar{q}(u)}{\operatorname{argmax}} \min_{j \in \{0,\dots,M-1\} \setminus \hat{i}_k} \sum_{u \in \mathcal{U}} \bar{q}(u)\, D\left(p^u_{\hat{i}_k} \parallel p^u_j\right), \tag{4.4.15}$$

where $\hat{i}_k = \underset{i \in \{0,\dots,M-1\}}{\operatorname{argmax}} p_i\left(y^k, u^k\right)$. The stopping rule is defined as the first time n for which

$$\log\left(\frac{p_{\hat{i}_n}(y^n, u^n)}{\max_{j \neq \hat{i}_n} p_j(y^n, u^n)}\right) \geq -\log c, \tag{4.4.16}$$

where c is as in (4.4.9). The decision rule is again an ML decision rule, that is, $d(y^n, u^n) = \hat{i}_n$ for the n first satisfying (4.4.16). This test, which is also refereed to as the Chenoff test (this time, for multiple simple hypotheses), admits the same form of asymptotic optimality as the parameter c goes to zero. Specifically, the proposed test is shown to achieve optimal expected values of the stopping time subject to the constraints of vanishing probabilities of error under each hypothesis. The complete statement is given in [12] and it is a natural generalization of Theorem 10.

A major shortcoming of the Chernoff test is the "separation" requirement of Θ_0 and Θ_1 (condition (4.4.5) or similarly (4.4.13) for multiple hypotheses). The necessity of this constraint is that the instantaneous control picked in (4.4.15) for example, is a function of the ML estimate of the hypothesis (and not of the reliability of the estimate). When the ML estimate is incorrect, the instantaneous control can be quite bad. This can happen with large probability especially when only a few observations are collected. Condition (4.4.13) or (4.4.5) essentially ensures that when the ML hypothesis is incorrect, the control value will not be too bad. Consequently, this control policy leads to a fast convergence of the ML estimate of the hypothesis to the true one. Without these conditions convergence can be very slow or may not happen at all (note that this phenomenon is analogous to another known phenomenon which occurs in a somewhat more exacerbated form, in stochastic adaptive control [84] illustrating the failure of ML identification in closed-loop [14]). Another drawback is the substantial dependence of the optimality criterion in the asymptotic nature of the problem. Note that Chernoff's approach involves pretending that the current estimate $\hat{\theta}_n$ of the true parameter θ is correct in deciding what control action to select next. No attempt is made to distinguish between an imprecise estimate of θ based on little evidence and a very precise estimate. As a result, the behavior of this procedure may be poor for problems where moderate sample sizes are anticipated. Nevertheless, the tests that were defined above have some fundamental properties that are quite frequent in asymptotic design and analysis of controlled hypothesis testing. Noticeable are the following basic elements:

1. In order to understand the basic structure of the tests from an intuitive point of view, it is instructive to consider the following toy problem: say an experimenter is trying to sequentially decide between two hypotheses H_0 and H_1 and he is given a choice of one out of two experiments E_1 and E_2 to conduct, but once an experiment is chosen, it is used exclusively until a final decision is made. Assume that c is the cost of taking an observation and also that available to the experimenter are four figures of merit $D_i\left(E_j\right)$ regarding the "amount of information" one can gain from conducting experiment j when hypothesis i is true. If $D_0\left(E_1\right) > D_0\left(E_2\right)$ and $D_1\left(E_1\right) > D_1\left(E_2\right)$ then it makes sense to select E_1. If, on the other hand, $D_0\left(E_1\right) > D_0\left(E_2\right)$ and $D_1\left(E_1\right) < D_1\left(E_2\right)$, then E_1 would be preferable if H_0 were true and E_2 otherwise. In all cases, if c is small, it always pays off to take an additional observation, unless the confidence of one of the hypotheses is very strong. The Chernoff tests are, in a way, a natural generalization of the proposed solution to this artificial problem. In our problem, the control action plays the role of the experiments, c is still the cost of taking an observation, and the "information numbers" are the Kullback-Leibler divergences. Of course, in the original problem, the true hypothesis is not known and so the natural thing to do is, presumably, to replace the role of the true hypothesis by its ML estimate, which is exactly the way the Chernoff tests were constructed.

2. Another way to understand the intuition behind the Chernoff tests is the following: denote $d^n\left(i,j\right) = \log\left[\frac{p_i(y^n,u^n)}{p_j(y^n,u^n)}\right]$ and notice that, given $\mathcal{F}_n$, $d^n\left(i,j\right)$ is a constant. Specifically, given $\mathcal{F}_n$, one can calculate both $\hat{i}_n$ and $d^n\left(\hat{i}_n,j\right)$ for all $j \neq \hat{i}_n$. In addition, given $\mathcal{F}_n$ and the policy q, $d^{n+1}\left(\hat{i}_n,j\right), j = \{1,\ldots,M-1\} \setminus \hat{i}_n$ are random variables, the expected values of which are $\mathbb{E}\left[d^{n+1}\left(\hat{i}_n,j\right)\right] = \sum_u q\left(u\right) D\left(p^u_{\hat{i}_n} \parallel p^u_j\right)$ for $j = \{1,\ldots,M-1\} \setminus \hat{i}_n$. The Chernoff test can be interpreted in the following way:

- At time n, estimate the true hypothesis using the maximum likelihood decoder, denoted by $\hat{i}_n$.

- Calculate the (deterministic) "distances" $d^n\left(\hat{i}_n, j\right)$ for all $j \in (0, \ldots, M-1)$.
- If $\min_{j \neq \hat{i}_n} d^n\left(\hat{i}_n, j\right) > \log c$, that is, if the most likely hypothesis is far apart for all other hypothesis in the sense of the general likelihood ratio sense, stop the process and declare $\hat{i}_n$ as the estimate of the true hypothesis.
- Otherwise, choose the control policy q that maximizes the minimum expected distance $\sum_u q(u) D\left(p^u_{\hat{i}_n} \parallel p^u_j\right)$

In this sense, at each time step, the Chernoff scheme chooses a control policy that moves apart the probability measures from the most likely one (in the sense of the averaged KL divergence $\sum_u q(u) D\left(p^u_i \parallel p^u_j\right)$).

3. Randomization is used in the causal control policy discussed above. This facilitates the simultaneous minimization of the expected stopping time under the M hypotheses as the error probability goes to zero. An instructive example similar to the one given in the previous item, that illuminates the need for randomization in controlled hypothesis testing problems, is given in Chapter 13 of [26].

4. This sequential test relies on the well known separation principle between estimation and control (e.g., [121] and [16]), with the distinction that the stationary mapping from the posterior distribution of the hypothesis to the control value is now randomized.

Stronger results and modifications

Chernoff's original work was aimed at designing an optimal structure of a sequential experiment. This challenge was dealt with in numerous different fields and research areas ranging from a design of clinical trials and medical diagnosis (e.g., [7], [10]), Multi-Armed Bandit Problems (e.g. [100]) sensor managenemt (e.g. [70]), underwater inspection (e.g., [74]) and more. Here we focus on recent work that will be relevant to the second part of this paper, where the aim is to refine the results of optimality in the asymptotic problem of multiple (simple) hypothesis testing. Most of the results can be found in [115] and [114]. Other relevant works that are subsequent extensions of [24] are [1],[13], [78],[80], [90]. Another recent look at the controlled sensing problem will be discussed in the next section.

In [115], a "modified Chernoff test" with a control policy, that is slightly different than (4.4.15), is defined. Specifically, instead of using t (4.4.15) at all times, the modified controller will occasionally sample from the uniform control, independent of the index of the ML hypothesis; Precisely, for some $a > 0$, at times $k = \lceil a^l \rceil$, $l = 0, 1, \ldots$, we let U_{k+1} be uniformly distributed on $\mathcal{U}$. At all other times, we still follow the control policy in (4.4.15). The stopping rule is still as in (4.4.16) with the same c therein, and the final decision is still ML.

Two main advantages of the modified Chernoff test are that the constraint (4.4.13) is no longer necessary in order to prove asymptotic optimality and the ability to prove asymptotic optimality in a stronger sense then described earlier. In order to formally present the statement establishing the stronger asymptotic optimality of the modified Chernoff test, recall first the definition of the probability of incorrectly deciding i, $R_i(\Delta) \triangleq R_i$, in (4.3.3), and notice that for each $i \in \{0, \ldots, M-1\}$

$$R_i \leq \max_{k \in \{0,\ldots,M-1\}} P_k(d \neq k), \tag{4.4.17}$$

where $P_k(\mathcal{A})$ is the probability of the event $\mathcal{A}$ under the k'th hypothesis. Therefore, the condition $\max_{k \in \{0,\ldots,M-1\}} P_k(d \neq k) \to 0$ implies that $\max_{k \in \{0,\ldots,M-1\}} R_k \to 0$. The following theorem establishes the asymptotically optimal nature of the test:

Theorem 11

1. *The modified Chernoff test satisfies*

$$\lim_{c \to 0} \max_{i \in \{0,\ldots,M-1\}} P_i\left(d\left(Y^N, U^N\right) \neq i\right) = 0 \tag{4.4.18}$$

and for every $i \in \{0, \ldots, M-1\}$ and $\epsilon > 0$, there is a $c^\star$ such that for any $c < c^\star$

$$\begin{aligned} \mathbb{E}_i[N] &\leq \frac{-\log\left(\max\limits_{k \in 0,\ldots,M-1} P_k(d \neq k)\right)}{\max\limits_{\bar{q}(u)} \min\limits_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)}(1+\epsilon) &\quad (4.4.19) \\ &\leq \frac{-\log R_i}{\max\limits_{\bar{q}(u)} \min\limits_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)}(1+\epsilon) &\quad (4.4.20) \end{aligned}$$

2. *Any sequence of tests with vanishing maximal risk, i.e., $\max_{k \in 0,\ldots,M-1} R_k \to 0$, and for every $i \in \{0, \ldots, M-1\}$ and $\epsilon > 0$, there there is a $c^{\star\star}$ such that for any*

$c < c^{\star\star}$

$$\mathbb{E}_i[N] \geq \frac{-\log R_i}{\max\limits_{\bar{q}(u)} \min\limits_{j \in \{0,\dots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)} (1-\epsilon). \tag{4.4.21}$$

A question that naturally arises is whether one can utilize the tests presented in Section 4.3 in order to construct a controlled multiple hypothesis test that would be optimal within $\boldsymbol{\Delta}\left(\overline{\mathbf{R}}\right)$ (defined in (4.3.4)), as was done for non-controlled hypothesis testing. The first step towards that end is re-defining the Log-Likelihood-Ratios (LLR) for the controlled case. Let

$$L_i(n) = \log\left[\frac{dP_i(Y^n, U^n)}{dQ(Y^n, U^n)}\right] \quad , \quad i = 0, \dots, M-1 \tag{4.4.22}$$

where we have assumed that $Q(y^n, u^n)$ is some dominating measure. If $Q = P_j$, the corresponding LLR process will be denoted by $L_{ij}(n)$ i.e.,

$$L_{ij}(n) = \sum_{k=1}^{n} \log\left[\frac{p_i^{U_k}(Y_k)}{p_j^{U_k}(Y_k)}\right] \quad , \quad j, i = 0, \dots, M-1, j \neq i. \tag{4.4.23}$$

In addition, define the generalized likelihood ratio (GLR) to be

$$\hat{\Pi}_i(n) = \frac{\pi_i \exp\{L_i(n)\}}{\max\limits_{k \in \{0,\dots,M-1\} \setminus \{i\}} \pi_k \exp\{L_k(n)\}}. \tag{4.4.24}$$

Next, a new controlled test (φ_c, N_c, d_c) will be presented which is a combination between the sequential non-controlled test Δ_b defined in Section 4.3 (using the GLRT as a stopping criterion) and the Chernoff test (using a randomized control policy and an ML decision maker):

Control Policy (φ_c): same as in (4.4.15)

Stopping rule: The stopping time N_c is

$$N_c = \min_{i \in \{0,\dots,M-1\}} \min_{n \geq 0} \left\{\hat{\Pi}_i(n) \geq \exp(b_i)\right\}, \tag{4.4.25}$$

where b_i , $i \in \{0, \dots, M-1\}$ are the threshold values and are chosen to be $b_i = \frac{\pi_j(M-1)}{\bar{R}_i}$

Decision rule: $d_c = i$ if $\hat{\Pi}_i(N_c) = \max\limits_{j \in 0,\dots,M-1} \hat{\Pi}_j(N_c)$

The following theorem summarizes the important properties of the test (φ, N, d) and establishes its asymptotic optimality, this time, with respect to the class $\boldsymbol{\Delta}\left(\overline{\mathbf{R}}\right)$ as $\bar{R}_{\max} = \max\limits_{j \in 0,\dots,M-1} \bar{R}_j$ goes to zero. For brevity, the notation $o(1)$ is used, where $o(1) \to 0$ as $\bar{R}_{\max} \to 0$

Theorem 12 (Lemma 3, Theorem 1 and 2 in [114])

1. *(Lower bound) Let* $\boldsymbol{\Delta}\left(\overline{\mathbf{R}}\right)$ *denote the class of tests defined in* (4.3.4) *where* $\Delta = \Delta(\varphi, N, d)$ *is a controlled multiple hypothesis test. Then, for all* $i \in \{0, \ldots, M-1\}$, *the expected stopping time satisfies*

$$\inf_{\Delta \in \boldsymbol{\Delta}\left(\overline{\mathbf{R}}\right)} \mathbb{E}_i[N] \geq \frac{-\log \bar{R}_i}{\max\limits_{\bar{q}(u)} \min\limits_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)} (1 + o(1)) \tag{4.4.26}$$

as $\bar{R}_{\max} \to 0$.

2. *The test* (ϕ_c, N_c, d_c) *belongs to the class* $\boldsymbol{\Delta}\left(\overline{\mathbf{R}}\right)$.

3. *(Upper bound) The expected stopping time of* N_c *satisfies*

$$\mathbb{E}_i[N_c] \leq \frac{-\log \bar{R}_i}{\max\limits_{\bar{q}(u)} \min\limits_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)} (1 + o(1)) \text{ as } \bar{R}_{\max} \to 0. \tag{4.4.27}$$

Using the notation in Section 4.3, we have established the asymptotic optimality since

$$\mathbb{E}_i[N_c] \sim \frac{-\log \bar{R}_i}{\max\limits_{\bar{q}(u)} \min\limits_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)} \tag{4.4.28}$$

as $\bar{R}_{\max} \to 0$.

The proof of item 1 in Theorem 12 follows the exact footsteps of [24]. On the one hand, a main difficulty arises when attempting to prove the third part of the theorem in "classical" ways. The reason is that, unlike the traditional sequential hypothesis testing problem, the LLRs are no longer i.i.d.. On the other hand, we have already mentioned the notion of r-quickly convergence (Definition 1) for non-i.i.d. likelihood ratios. In [114], the authors show that indeed $\frac{1}{n} L_{ij}(n)$ converges under the i'th measure r-quickly to positive constants l_{ij}. This implies that the r'th moment of the sample size required for the LLRs to cross predefined thresholds is governed by the minimum constant $\min\limits_{i \neq j} l_{ij}$, Specifically, it was shown that for all $i \neq j$

$$\frac{1}{n} L_{ij}(n) \overset{P_i\text{-1-quickly}}{\rightarrow} \max_{\bar{q}(u)} \min_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right) \tag{4.4.29}$$

and the claim follows.

So what makes the test (φ_c, N_c, d_c) asymptotically optimal and adjustable to meet hard constraints on the risks? The answer is related to the two main differences between this test and the two former ones:

- Note that although the calculation of the probability of incorrectly deciding in favor of i, R_i ($i \in 0, \ldots, M-1$), involves the prior distribution of the hypothesis, the Chernoff test and it's modification do not use knowledge of the prior distribution at all, whereas the new test depends on this knowledge.
- Another key to this new test is the use of different thresholds for the peak of the posterior distribution depending on the index of the ML hypothesis instead of a single threshold as in (4.4.16).

B. Controlled Sensing Analysis Using Dynamic Programming

In this section, a different point of view will be presented, which will give rise to yet another generalization to the Chernoff test. As before, we assume that M simple hypotheses are to be tested sequentially in the Bayesian scenario, with a prior $\pi = (\pi_0, \ldots \pi_{M-1})$, and a finite set of $\mathcal{U}$ controls (sometimes referred to as *sensing actions*). Let θ be the random variable that takes the value $\theta = i$ when H_i is true (that is, $\pi_i = \Pr(\theta = i)$). A slight variation of the problem of the previous section, is that, we now assume a loss, $w > 0$, that is associated with a wrong decision, i.e., w is the penalty, independent of the underlying hypothesis, of selecting $H_j, j \neq i$ when H_i is true. The object is again to find a sequential test $\Delta = (q, N, d)$ that minimizes the total cost defined as:

$$\mathbb{E}\left[N + w\mathbb{I}\left\{d\left(U^N, Y^N\right) \rightarrow \text{ error}\right\}\right] = \mathbb{E}[N] + wP_{\text{er}}, \tag{4.4.30}$$

where the expectation is taken with respect to π as well as the distribution of the observation sequence, and $P_{\text{er}} = \mathbb{E}\left[\mathbb{I}\left\{d\left(U^N, Y^N\right) \rightarrow \text{ error}\right\}\right]$ denotes the probability of making the wrong decision. The asymptotic regime is $w \rightarrow \infty$. The problem of finding a control policy that minimizes (4.4.30) will be denoted as Problem (P). Note that in some places in the literature, the total cost is given either by $c\mathbb{E}[N] + P_{\text{er}}$ or by $c\mathbb{E}[N] + wP_{\text{er}}$, where $c > 0$ is the cost of taking one observation, and the asymptotic nature of the problem is defined as $c \rightarrow 0$. It is straightforward to show that the problems are all equivalent. The reason the objective function in this section was chosen to be (4.4.30) will be made clear in the sequel.

Next, following [108], [110] and [111], results in dynamic programming (DP) theory will be used to establish lower bounds for the optimal total cost. In order to obtain upper bounds, two heuristic controlled sensing tests will be defined and analyzed.

In general, the problem of inferring among $M > 2$ hypotheses in a sequential manner in the presence of control, is a partially observable Markov decision problem (POMDP) where the state (the true hypothesis) is static and the observations are noisy. It is known that any POMDP is equivalent to an MDP with a compact yet uncountable state space, for which the or the posterior vector (also referred to as the belief vector of the decision maker about the underlying state) becomes an information state. At each time instant, n, the information state is given by the vector $\pi^{\pi}(n)$, whose i'th element is the conditional probability of H_i, given π and all the observations and control actions up to time instance n. The full derivation can be found, for example, in Chapter 6 of [84].

In one sensing step, the evolution of the belief vector follows Bayes' rule and is given by Φ^u, a measurable function from $\triangle_M \times \mathcal{Y}$ to $\triangle_M$, where $\triangle_M$ is the M-simplex, and:

$$\Phi^u(\pi, y) = \left(\pi_0 \frac{p_0^u(y)}{p_\pi^u(y)}, \pi_1 \frac{p_1^u(y)}{p_\pi^u(y)} \ldots, \pi_{M-1} \frac{p_{M-1}^u(y)}{p_\pi^u(y)}\right); \quad \forall u \in \mathcal{U}, \tag{4.4.31}$$

where $p_i^u(y), i \in \{0, \ldots, M-1\}$ are defined in (4.4.4) and $p_\pi^u(y) = \sum_{i=0}^{M-1} \pi_i p_i^u(y)$. In other words, (π, y) gives the posteriori distribution, when control u has been taken, and y has been observed. As customary in the DP literature, we next define the operator $\mathbb{T}^u, u \in \mathcal{U}$, such that for any measurable function $g\colon \triangle_M \to \mathbb{R}$:

$$(\mathbb{T}^u g)(\pi) = \int g(\Phi^u(\pi, y))\, p_\pi^u(dz), \tag{4.4.32}$$

that is, $(\mathbb{T}^u g)(\pi)$ is the expected value of g at the posterior belief, where the computation of the posterior belief follows the Bayes rule. For example, the mutual information between $\theta \sim \pi$ and $Y \sim p_\pi^u$ can be written as $I(\theta; Y) = H(\pi) - (\mathbb{T}^u H)(\pi)$ where H is the entropy of π, defined to be

$$H(\pi) = -\sum_{i=0}^{M-1} \pi_i \log \pi_i. \tag{4.4.33}$$

The following theorem, which is a consequence of Propositions 9.1 and 9.8. in [11], characterizes the solution of Problem (P):

Theorem 13 *Let $V^\star : \triangle_M \to \mathbb{R}_+$ be the minimal solution to the following fixed point equation:*

$$V^\star(\pi) = \min\left\{1 + \min_{u\in\mathcal{U}} \left(\mathbb{T}^u V^\star\right)(\pi), \min_{j\in\{0,\ldots,M-1\}} (1-\pi_j)\, w\right\}. \tag{4.4.34}$$

Then $V^\star(\pi)$, referred to as the optimal value function, is equal to the minimum of the total cost in Problem (P) with the prior belief π.

As shown in [11, Corollary 9.12.1], this theorem provides a characterization of an optimal Markov stationary deterministic control policy[2] for Problem (P): the control $u^\star = \underset{u\in\mathcal{U}}{\operatorname{argmin}} \left(\mathbb{T}^u V^\star\right)(\pi)$ is the least costly control action, resulting in $1+\min_{u\in\mathcal{U}} \left(\mathbb{T}^u V^\star\right)(\pi)$, and is the optimal action to take, unless the penalty of wrongly declaring $H_{i^\star}$, where $i^\star = \underset{j\in\{0,\ldots,M-1\}}{\operatorname{argmin}} w(1-\pi_j)$, is even less costly. In the latter case, it is optimal to retire and declare $H_{i^\star}$.

We have the following technical assumptions:

Assumption 1. For any two hypotheses i and j, $i \neq j$, there exists a control $u \in \mathcal{U}$, such that $D\left(p_i^u \parallel p_j^u\right) > 0$.

Assumption 2. It holds that

$$\max_{i,j\in\{0,\ldots,M-1\}} \max_{u\in\mathcal{U}} \sup_{y\in\mathcal{Y}} \frac{p_j^u(y)}{p_j^u(y)} < \infty \tag{4.4.35}$$

Assumption 1 ensures the possibility of discrimination between any two hypotheses, hence ensuring Problem (P) has a meaningful solution. Assumption 2 implies that no two hypotheses are fully distinguishable using a single observation sample.

The lower bounds on the optimal value function, $V^\star$, that will be presented in this section, are based on the following lemma, proven in [111]:

Lemma 14 *Suppose there exists a functional $V^\star : \triangle_M \to \mathbb{R}_+$ such that for all belief vectors $\tilde{\pi} \in \triangle_M$*

$$V(\tilde{\pi}) \le \min\left\{1 + \min_{u\in\mathcal{U}} \left(\mathbb{T}^u V\right)(\tilde{\pi}), \min_{j\in(0,\ldots,M-1)} (1-\tilde{\pi}_j)\, w\right\}. \tag{4.4.36}$$

Then $V(\tilde{\pi}) \le V^\star(\tilde{\pi})$ for all $\tilde{\pi} \in \triangle_M$ where $V^\star$ is the optimal solution to Problem (P).

[2] A *Markov stationary control policy* q is a policy under which the probability of a control $u \in \mathcal{U}$ is selected at a belief state $\tilde{\pi}$ is given by $q(u \mid \tilde{\pi})$. A Markov stationary control policy q is referred to as *deterministic* if for each $\pi \in \triangle_M$, there exists a control $u \in \mathcal{U}$ for which $q(u \mid \tilde{\pi}) = 1$

Using Lemma 14, one can show the following:

Corollary 15 *Under Assumption 1 and for* $w > 1$, $V^\star(\pi) > \underline{V_1}(\pi)$ *where*

$$\underline{V_1}(\pi) = \left[\sum_{i=0}^{M-1} \max_{i \neq j} \frac{\log\left(\frac{1-w^{-1}}{w^{-1}}\right) - \log\left(\frac{\pi_i}{\pi_j}\right)}{\max_{u \in \mathcal{U}} D\left(p_i^u \parallel p_j^u\right)} - K'\right]^+ \tag{4.4.37}$$

and K' *is a constant independent of* w.

The next corollary provides another lower bound, which is more appropriate for large values of M, as will be explained in Section 5.6. Define $D_{\max} = \max_{i,j \in \{0,\ldots,M-1\}} \max_{u \in \mathcal{U}} D\left(p_i^u \parallel p_j^u\right)$, and $I_{\max} = \max_{u \in \mathcal{U}} \max_{\tilde{\pi} \in \triangle_M} I\left(\tilde{\pi}; p_{\tilde{\pi}}^u\right)$. The following then holds:

Corollary 16 *Under Assumption* 1 *and for* $w > 1$,

$$V^\star(\pi) \geq \left[\frac{H(\pi) - h_2(\alpha(w,M)) - \alpha(L,M)\log(M-1)}{I_{\max}} + \alpha(w,M)\, w\right]^+, \tag{4.4.38}$$

where $\alpha(w,M) = \frac{M-1}{M-1+2^{wI_{\max}}}$. *Furthermore, under Assumption* 2 *and for* $L > \frac{\log(M)}{I_{\max}}$ *and arbitrary* $\delta \in (0, 0.5]$,

$$V^\star(\pi) \geq \left[\frac{H(\pi) - h_2(\delta) - \delta\log(M-1)}{I_{\max}} + \frac{\log\left(\frac{1-w^{-1}}{w^{-1}}\right) - \log\left(\frac{1-\delta}{\delta}\right)}{D_{\max}} \mathbb{I}\left\{\max_i \pi_i \leq 1-\delta\right\} - \hat{K}'\right]^+, \tag{4.4.39}$$

where $\hat{K}'$ *is a constant independent of* w *and* M.

The two lower bounds can be understood intuitively in the following way: say we have a "measure of uncertainty" function $F : \triangle_M \to \mathbb{R}_+$. Assume we start with some $F(\pi^\pi(0))$ at time zero, and we are interested in reducing uncertainty to a level of $F(\hat{\pi})$. The number of samples required to do so has to be at least $\frac{F(\pi^\pi(0)) - F(\hat{\pi})}{F_{\max}}$, where $F_{\max}$ is the maximum amount of reduction in F associated with a single sample, i.e., $F_{\max} = \max_{f \in \mathcal{F}} \max_{\tilde{\pi} \in \triangle_M} \left\{F(\tilde{\pi}) - \left(\mathbb{T}^f F\right)(\tilde{\pi})\right\}$. The lower bound in Corollary 15 is associated with such a lower bound when taking U to be the log-likelihood function, while the lower bound in Corollary 16 is associated with setting F to be the Shannon entropy. The intuitive explanation of the second bound of Corollary 16 involves Fano's inequality [29] that states the following: Let X be a random variable taking values in the set $\{0, \ldots, M-1\}$, suppose Y is a random variable which is related to X by the conditional distribution $P(y \mid x)$ and let $\hat{X}(Y)$ be an estimate of X. Define the error probability $\delta = \Pr\left(X \neq \hat{X}\right)$. Then

$$h_2(\delta) + \delta \log(M-1) \geq H(X \mid Y). \tag{4.4.40}$$

Note that the right hand side of (4.4.40) appears in the numerator of the first term in (4.4.39). Intuitively, the bound can be interpreted in the following way: let δ be some target error probability, and denote the stopping time by T. Next, we write T as $T = T_\delta + (T - T_\delta)$, where T_δ is the time it takes in order for some estimator to reach the error probability δ. The bound in (4.4.39) implies that the total stopping time can be written as

$$\frac{F_1(\pi^\pi(0)) - F_1(\hat{\pi})}{F_{1,\max}} - \frac{F_2(\pi^\pi(0)) - F_2(\hat{\pi})}{F_{2,\max}} \tag{4.4.41}$$

where F_1 is the entropy, and it is used until the error probability drops below δ, and F_2 is the likelihood function which is used from time T_δ until the test stops. This intuition further implies a two-phase scheme that achieves this bound, using the entropy as a test statistic in the first phase and the likelihood ration in the second. We will see later that this intuitive approach helps to produce a good upper bound as well. The proofs of Corollary 15 and Corollary 16 can be found in [111].

In [111], the authors have also proposed two heuristic sequential tests, and analyzed their performance. These tests will be denoted by Δ_I and Δ_{II}. The main difference between the Chernoff test and its generalizations, discussed in the previous section, and the tests Δ_I and Δ_{II}, is that the latter have two operational phases: the first is a phase in which the belief about all hypotheses is below a certain threshold; while in the second phase, the belief about one of the hypotheses has passed that threshold and actions are selected in favor of that hypothesis. The difference between the two tests is in the actions they take in each phase.

For a set $\mathcal{A}$, define the collection of all probability distributions on elements of $\mathcal{A}$ by $\triangle_{|\mathcal{A}|}$. In order to define Δ_I and Δ_{II}, we need to specify the stopping times, N_I and N_{II}, the decision rules, d_I and d_{II}, and the control policies, q_I and q_{II}. To that end, consider a threshold value $\bar{\pi}$, $\bar{\pi} \in \left(\frac{1}{2}, 1 - w^{-1}\right)$. We define Δ_I as follows:

- The stopping time is defined as

$$N_I = \min_{n\geq 0} \max_{i\in\{0,\ldots,M-1\}} \pi^{\pi}(n) \geq 1 - w^{-1} \tag{4.4.42}$$

and d_I selects $H_{i^\star}$, where $i^\star = \underset{i\in\{0,\ldots,M-1\}}{\text{argmax}}\ \pi^{\pi}(N_I)$

- If $\pi_i^{\pi}(n) \in [\bar{\pi}, 1 - w^{-1})$ for some i, then

$$q_I(u) = \mu_{i,u}; \quad \forall u \in \mathcal{U}, \tag{4.4.43}$$

where $\boldsymbol{\mu}_i = \underset{\boldsymbol{\lambda}\in\triangle_{|\mathcal{A}|}}{\text{argmax}} \min_{j\neq i} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \parallel p_j^u\right)$.

- If, for all $i \in \{0,\ldots,M-1\}$, $\pi_i^{\pi}(n) \in [0,\bar{\pi})$, then

$$q_I(u) = \mu_{0,u}; \quad \forall u \in \mathcal{U}, \tag{4.4.44}$$

where $\boldsymbol{\mu}_0 = \underset{\boldsymbol{\lambda}\in\triangle_{|\mathcal{A}|}}{\text{argmax}} \min_{i\in\{0,\ldots,M-1\}} \min_{j\neq i} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \parallel p_j^u\right)$.

In its first phase, the control function q_I selects actions in a way that all pairs of hypotheses can be distinguished; while in the second phase, that coincides with Chernoff's scheme, only the pairs including the most likely hypothesis are considered. Among other advantages of two phase schemes, that will be further discussed in the second part of this work, one can notice that they allow to relax condition (4.4.13) without adding additional randomization, as was done in the modified Chernoff test.

The second test proposed in [111], denoted by Δ_{II}, is similar to Δ_I, except vectors $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_i$ are replaced by $\boldsymbol{\eta}_0$ and $\boldsymbol{\eta}_i$ where

$$\begin{aligned}
\boldsymbol{\eta}_0 &= \underset{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}}{\text{argmax}} \min_{i\in\{0,\ldots,M-1\}} \min_{\hat{\pi}\in\triangle_M} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \middle\| \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u\right), && (4.4.45)\\
\boldsymbol{\eta}_i &= \underset{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}}{\text{argmax}} \min_{\hat{\pi}\in\triangle_M} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \middle\| \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u\right). && (4.4.46)
\end{aligned}$$

The idea behind this test is similar to that of Δ_I, only this time, instead of considering the minimizing $j \in \{0,\ldots,M-1\}$ of $D\left(p_i^u \parallel p_j^u\right)$, the minimum is now taken with respect to all the measures which are mixtures of p_j^u, that are in the form of $q_{i,j} = \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u$ (and again, the object is to maximize, over all randomized control policies, the minimum "distance" in terms of the Kullback-Leibler divergence)

By analyzing the performance of tests Δ_I and Δ_{II}, upper bounds on the optimal value function can be attained, as was done in [111]. Nevertheless, the main contribution of [111], to the field of multiple hypothesis testing with control, boils down to the following theorems regarding the asymptotical optimality of these two tests:

Theorem 17 *Let $V^{\Delta_I}(\pi)$ denote the value function for test Δ_I, i.e., the expected total cost achieved by test Δ_I when the initial belief is π. Then for fixed M, test Δ_I satisfies*

$$\lim_{w\to\infty}\frac{V^{\Delta_I}(\pi)-V^{\star}(\pi)}{V^{\Delta_I}(\pi)}<1 \tag{4.4.47}$$

for all $\pi\in\triangle_M$.

From the definition of $V^{\star}(\pi)$, for any test Δ, $\lim_{w\to\infty}\frac{V^{\star}(\pi)}{V^{\Delta}(\pi)}\geq 0$, whereas Theorem 17 implies that $\lim_{w\to\infty}\frac{V^{\star}(\pi)}{V^{\Delta_I}(\pi)}>0$, that is, asymptotically, for large w (i.e., large sample sequences), $V^{\Delta_I}(\pi)$ grows proportionally to $V^{\star}(\pi)$.

In order to state the asymptotic performance of Δ_{II}, we denote the vectors attaining $\boldsymbol{\eta}_0$ and $\boldsymbol{\eta}_i$, by $\boldsymbol{\eta}_0^{\star}$ and $\boldsymbol{\eta}_i^{\star}$ respectively, that is:

$$\begin{aligned}
\boldsymbol{\eta}_0^{\star} &= \max_{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}}\min_{i\in\{0,\ldots,M-1\}}\min_{\hat{\pi}\in\triangle_M}\sum_{u\in\mathcal{U}}\lambda_u D\left(p_i^u\middle\|\sum_{i\neq j}\frac{\hat{\pi}_j}{1-\hat{\pi}_i}p_j^u\right), && (4.4.48)\\
\boldsymbol{\eta}_i^{\star} &= \max_{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}}\min_{\hat{\pi}\in\triangle_M}\sum_{u\in\mathcal{U}}\lambda_u D\left(p_i^u\middle\|\sum_{i\neq j}\frac{\hat{\pi}_j}{1-\hat{\pi}_i}p_j^u\right). && (4.4.49)
\end{aligned}$$

Then the following holds:

Theorem 18 *For $w>\frac{\log M}{I_{\max}}$ and if all the elements $\boldsymbol{\eta}_0^{\star}$ are strictly positive, then Δ_{II} satisfies*

$$\lim_{M\to\infty}\lim_{w\to\infty}\frac{V^{\Delta_{II}}(\hat{\pi})-V^{\star}(\hat{\pi})}{V^{\Delta_{II}}(\hat{\pi})}<1, \tag{4.4.50}$$

where $\hat{\pi}$ is the uniform prior(that is, $\hat{\pi}_i=\frac{1}{M}$ for all $i\in\{0,\ldots,M-1\}$)

Theorem 19 *If*

$$\min_{j\neq i}\max_{u\in\mathcal{U}}D\left(p_i^u\parallel p_j^u\right)=\boldsymbol{\eta}_i^{\star}\quad\forall i\in\{0,\ldots,M-1\}, \tag{4.4.51}$$

then, for fixed M and all $\tilde{\pi}$, there exists a constant B, such that

$$V^{\Delta_{II}}(\tilde{\pi})-V^{\star}(\tilde{\pi})\leq B. \tag{4.4.52}$$

Furthermore, for $w > \frac{\log M}{I_{\max}}$ *and if* $D_{\max} = \boldsymbol{\eta}_i^\star$ *for all* $i \in \{0, \ldots, M-1\}$*, then test* Δ_{II} *satisfies*

$$\lim_{M\to\infty} \lim_{w\to\infty} \frac{V^{\Delta_{II}}(\hat{\pi}) - V^\star(\hat{\pi})}{V^{\Delta_{II}}(\hat{\pi})} = 0, \tag{4.4.53}$$

where $\hat{\pi}$ *is as above.*

We will return to these theorems and their interpretations in Section 5.6 where an important example of a controlled sensing problem will be formulated. Test Δ_{II} will then play an important role as an optimal scheme.

We close this section with a lemma connecting Problem (P) to another problem, denoted by Problem (P'), defined as

$$\text{minimize } \mathbb{E}[N] \quad \text{subject to} \quad P_{\text{er}} \leq \epsilon, \tag{4.4.54}$$

where $\epsilon > 0$ and the expectation is with respect to the true hypothesis.

Lemma 20 *Let* $\mathbb{E}[N_\epsilon^\star]$ *be the minimal expected number of samples required to achieve* $P_{er} \leq \epsilon$*, then*

$$\mathbb{E}[N_\epsilon^\star] \geq (1 - \epsilon w)(V^\star(\pi(0)) - 1), \tag{4.4.55}$$

where $V^\star(\pi(0))$ *is the optimal solution to Problem (P) for a prior* $\pi(0)$ *and cost for wrong decision* w*.*

The fact that there is a relation between the two problems is not surprising, since Problem (P) can be viewed as a Lagrangian relaxation[3] of Problem (P'). Moreover, Lemma 20 is a mathematical rationalization of the intuitive idea that, as $w \to \infty$, the solution of Problem (P) is related to that of Problem (P') as $\epsilon \to 0$.

[3] Recall that for the optimization problem of finding the minimum of a function $f(\mathbf{x})$ under the constraint that $g(\mathbf{x}) \leq \epsilon$, the Lagrangian relaxation problem is defined to be $\min\{f(\mathbf{x}) + wg(\mathbf{x})\}$, where $w \geq 0$ is called the Lagrange multiplier

Part II

Channel Coding With Feedback

Chapter 5

Channel Coding with Instantaneous Feedback

5.1 Introduction

The effect of feedback in communication that has been studied from the early days of information theory. In this chapter, results regarding performance of point-to-point communication systems with instantaneous feedback will be discussed. Unless stated otherwise, the feedback channel will be assumed perfect, i.e., with infinite capacity and error free. Another basic assumption is the use of block codes, that is, the information sent through the channel is about only one message at a time, and the time intervals used for sending successive messages are disjoint. In a block coding scheme with feedback, the transmitter is allowed to have fixed-length codewords, whose elements depend on previous channel outputs and the message. Another configuration of block codes is that in which the duration of each coded message is not necessarily constant. Instead, the receiver decides when to stop transmission and deciphers the coded message using the information gathered up to that point. In this chapter, we will focus on the latter problem and the goal will be to illuminate on the connections with sequential hypothesis testing. Specifically, the block error exponent, which is an important parameter of communication, is studied. This error exponent is not to be confused with the bit error exponent that is commonly discussed in non-block code analysis (for example [75] and [124].

As is well known [130], feedback does not increase the capacity of a point-to-point memoryless channel. This remains true even if variable-rate coding is allowed (see

Appendix I).

As for error exponents, Dobrushin [40] showed that for fixed block-length codes and above a critical rate, the error exponent does not increase for symmetric DMCs[1] as a result of feedback (whereas for rates under this critical rate, Dobrushin results are only upper bounds on the error exponent). It has been long conjectured, but never proved, that this is true also for non-symmetric channels. The best known upper bound for block codes with feedback, when no symmetry is assumed, is Haroutunian's bound [65], which coincides with Dobrushin's result for symmetric channels (i.e., it is equal to the sphere-packing error exponent for symmetric channels, but strictly larger for non-symmetric channels). However, there is no achievability result for this exponent. A similar result for *additive white Gaussian noise channels* (AWGNC) is given by Pinsker [117], who assumed, in addition to a constant decoding time constraint, also constant power for each message. Furthermore, if the feedback link is also an AWGNC and if there is a power constraint either on the expected power or on the power itself (in an almost sure sense) then even for two messages, the error probability decays only exponentially as has been shown by Kim *et al.* [82].

Pinsker's result on the best achievable error exponent [117] seems, at first glance, to contradict the widely known results of Schalkwijk and Kailath [125], [127], according to which the error probability can decay doubly exponentially in the block length. These contradictions however are illusive, because the models used in these papers are different: in [117], the power constraint holds with probability one, whereas in [125] and [127], the power constraint is only on the expected value. We therefore see, that the results highly depend on the model used. This is an important point and it continues to play a role throughout the rest of the sequel.

For the same reason, it is not surprising that in the case of variable length (VL) coding with feedback (i.e. for schemes where the decoding time is a random variable), that will be discussed in the following sections, the error exponent will take yet another form, which is significantly different from the fixed block length expression. As will be seen in the next section, the error exponent of systems with incorporated feedback is actually strictly better in almost all non-trivial cases.

This chapter also emphasises the relation between coding with feedback and hypothesis testing. The main idea is that in both problems there are hypotheses to be

[1]Channels with a transition probability matrix whose columns are permutations of each other, and so are the rows.

tested and the decision-maker has the freedom to decide not only in favor of some hypothesis but also on the time in which this decision is made. After formalizing this claim, we will demonstrate how one can use the results covered in the previous chapter to attain intuition and results regarding the communication problem.

5.2 Basic Model and Notation

A. Forward Channel

We will assume that the channel is stationary and memoryless. We denote the channel input and output at time n by X_n and Y_n, respectively, and assume finite input and output alphabets, $\mathcal{X} = \{1, \ldots, K\}$ and $\mathcal{Y} = \{1, \ldots, L\}$. The channel transition probabilities will be denoted by

$$p(j \mid i) = P_{Y|X}(Y_n = j \mid X_n = i), \quad i \in \mathcal{X}, \ j \in \mathcal{Y}. \tag{5.2.1}$$

It is assumed that all rows of the matrix $\{p(j \mid i)\}$ are distinct, and that $p(j \mid i) > 0$ for all $i, j \in \mathcal{X} \times \mathcal{Y}$.

B. Feedback Channel

We denote the feedback channel input at time n, by Z_n, and the output at time n by Z'_n. We only deal with noiseless feedback (that is, $\Pr(Z_n = Z'_n) = 1$) for all $Z_n, Z'_n \in \mathcal{Z}$, where $\mathcal{Z}$ is the common input and output alphabet of the feedback channel). In addition to this assumption, it will be further assumed that the feedback channel is instantaneous, that is, no delay is added to a symbol passing through it.

C. Coding Algorithm

All schemes considered here are block-coding schemes, where the transmitter is assigned one of M equiprobable messages, denoted by $\theta \in \{0 \ldots, M-1\}$.

An encoder is a sequence of functions denoted by

$$X_n(\theta) \equiv X(\theta, Z_1, \ldots Z_{n-1}), \quad \forall (Z_1, \ldots Z_{n-1}) \in \mathcal{Z}^{n-1}, \ \theta \in \{0, \ldots, M-1\}. \tag{5.2.2}$$

For the sake of simplicity we will also use the shorthand notation X_n. We denote by $\mathcal{F}_n$ "all the knowledge accumulated at the receiver up to time n", which in the case of perfect feedback is

$$\mathcal{F}_n = \sigma\left(Y_1, \ldots, Y_n, Z_1, \ldots, Z_n\right) = \sigma\left(Z_1, \ldots, Z_n\right) \tag{5.2.3}$$

where the second equality holds since under perfect feedback the knowledge of $\{Z_1, \ldots, Z_n\}$ implies the knowledge of $\{Y_1, \ldots, Y_n\}$ as well. Note that the sequence $\{\mathcal{F}_n\}$ forms a filtration of sigma-algebras (i.e., $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n, \forall n$). The overall communication system is depicted in Figure 5.2.1.

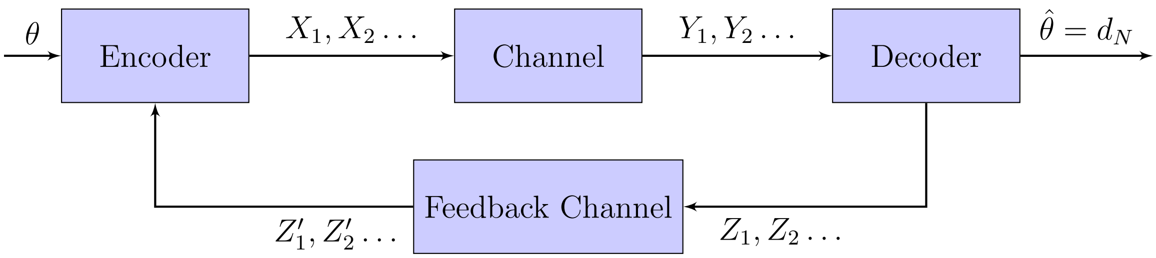

Figure 5.2.1: General Communication scheme with feedback.

Since we assume perfect feedback channel, it is possible to draw a equivalent and simpler representation as depicted in Figure 5.2.2.

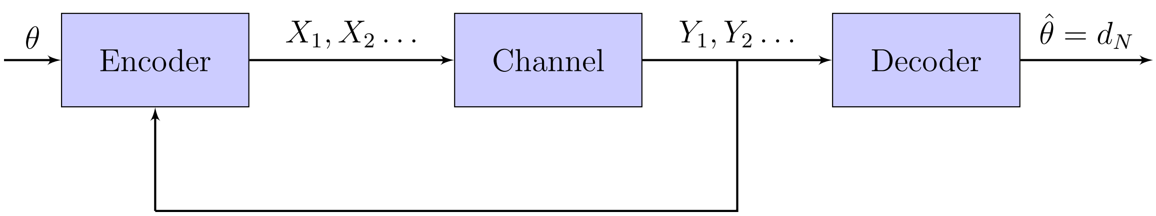

Figure 5.2.2: Communication scheme with perfect feedback.

D. Decoding Criteria

A decoding criterion is a pair (N, d_n), where N is the *decoding time* i.e., a stopping time with respect to the filtration $\{\mathcal{F}_n\}$, that indicates whether the receiver continues to take samples, or it stops the process and decodes the message based on the observations available. At each time instant n, the *decision function*, denoted by d_n, takes values in $\{0, \ldots, M-1, M\}$. When $d_n \in \{0 \ldots, M-1\}$, d_n is the estimator of the message, and $d_n = M$ means continue the transmission. The stopping time N is then given by

$$N = \inf_{n \geq 0} \{d_n \neq M\} . \tag{5.2.4}$$

The probability of error, P_e, is given by:

$$P_e = \frac{1}{M} \sum_{i=0}^{M-1} P\left(e \mid \theta = i\right) \tag{5.2.5}$$

where, for $i \in \{0, \ldots, M-1\}$, $P\left(e \mid \theta = i\right) = \Pr\left(d_N \neq i \mid \theta = i\right)$, and the expected transmission time is given by

$$\mathbb{E}\left[N\right] = \frac{1}{M} \sum_{i=0}^{M-1} \mathbb{E}\left[N \mid \theta = i\right], \tag{5.2.6}$$

where the expectation is taken over all channel realizations and messages. In other words, the observation space can be represented as the leaves of a complete $|\mathcal{Y}|$- ary tree $\mathcal{T}$, (complete in the sense that each intermediate node has $|\mathcal{Y}|$ descendants), with expected depth $\mathbb{E}\left[N\right]$. The decision time N is the first time the sequence $Y_1, Y_2, \ldots$ of channel outputs hits a leaf of $\mathcal{T}$. Furthermore, we may label each leaf of $\mathcal{T}$ with the message decoded when that leaf is reached. This way the decoder is completely specified by the labeled tree $\mathcal{T}$. The message statistics, the code, and the transition probabilities of the channel determine a probability measure on $\mathcal{T}$.

Note that, in this formulation of the problem, the a-posteriori probability vector of the message at time n, denoted by $\pi_n = (\pi_{n,0}, \ldots, \pi_{n,M-1})$, is a random vector, as it is a function of the random observation process up to that time.[2] For this reason, any functional operating on the a posteriori probability vector, or any function of the measurements themselves, is a random element. For example, the corresponding entropy of this a posteriori distribution is a random variable, measurable with respect

[2]In our model, the prior probability of the messages is assumed to be uniform (at time zero), i.e., $\pi_0 = \left(\frac{1}{M}, \ldots, \frac{1}{M}\right)$. After the first observation of the channel output is received, the a-posteriori probability vector $\pi_1 = \pi_1\left(y_1\right)$ takes the place of π_0, where π_1 is typically different then π_0. Obviously, one can continue to calculate these a posteriori probabilities as more and more observations are accumulated.

to $\mathcal{F}_n$. This random variable will be denoted by

$$H_n = H(\pi_n) = -\sum_{i=0}^{M-1} \pi_{n,i} \log(\pi_{n,i}) \tag{5.2.7}$$

$$= H(\theta \mid \{Y_1 \ldots, Y_n\} = \{y_1 \ldots, y_n\}). \tag{5.2.8}$$

E. Rate and Performance Measure

Since the block lengths are random, new definitions forcoding rate and error exponents are needed, which are consistent with those of fixed-length codes. We define the rate as

$$R = \frac{\log M}{\mathbb{E}[N]}, \tag{5.2.9}$$

and the error exponent as

$$E(R) = \limsup_{P_e \to 0} \frac{-\log P_e}{\mathbb{E}[N]} \tag{5.2.10}$$

where the asymptotic regime is such R is kept fixed. Since in order for P_e to approach zero, we must have $\mathbb{E}[N] \to \infty$, another interpretation of (5.2.10) is

$$E(R) = \limsup_{\mathbb{E}[N]\to\infty, R \text{ fixed}} \frac{-\log(P_e)}{\mathbb{E}[N]}. \tag{5.2.11}$$

These definitions are not only consistent with the definitions for fixed-length codes and with the definition used in the literature of variable-length coding, they also correspond the average stopping time, to which the system converges after many successive uses.

Unless stated otherwise, it will be assumed throughout the sequel that the input letters sent through the forward channel are random. More specifically, we will assume that the inputs are drawn according to the PMF P_X on the input alphabet. We consider the case where at time n, the PMF can depend on n, but once P_X is determined X_n is stochastically independent of $X_1, \ldots, X_{n-1}$ given $Y_1, \ldots, Y_{n-1}$. The probability mass functions on the output alphabet will be denoted by P_Y.

5.3 The Basic Lemmas of VL Coding

Before stating the famous result by Burnashev [18] regarding the error exponent function of VL coding, using perfect feedback, we quote four lemmas, which are fundamental and (are the core of the proof by Burnashev), and are interesting and instructive in general. The proofs of all these lemmas can be found in [18].

Lemma 21 (Generalized Fano Inequality) *For any coding algorithm and decoding rule such that* $\Pr(N < \infty) = 1$*,*

$$\mathbb{E}[H_N] \leq h_2(P_e) + P_e \log(M-1), \tag{5.3.1}$$

where $h_2(p) \triangleq -p\log(p) - (1-p)\log(1-p)$ *for* $p \in [0,1]$ *denotes the binary entropy function, and* $\mathbb{E}[H_N]$ *is the expected value of the random process* H_n *defined in* (5.2.8)*, evaluated at the random stopping time* N*, and the expectation is taken over the PMF of* N*.*

This is a generalization of the Fano inequality (see, for example, [29]), for the case where N is a stopping time. The general idea in the proof by Burnashev, is to use the average error probability in order to upper bound the expected entropy over decoding instances. Then, given a threshold value for the entropy, lower bounds on the expected time to reach this threshold are established.

The following lemmas deal with the change of entropy at each sampling step:

Lemma 22 *For all* $n \geq 0$*, we have the inequality,*

$$\mathbb{E}[H_n - H_{n+1} \mid \mathcal{F}_n] \leq C \quad a.s. \tag{5.3.2}$$

where C *is the channel capacity, given by*

$$C = \max_{P_X} I(X;Y) \tag{5.3.3}$$

Note that the expected decrease we are bounding here is averaged over different possible messages using $\Pr(\theta = i \mid \{Y_1 \ldots, Y_n\} = \{y_1 \ldots, y_n\})$. In other words, at a specific a posteriori probability distribution on the messages, one may be able to propose a coding method under which the entropy will decrease, at every time step, on average for a specific source message, by more than C. However, this method will have a poorer performance in the case where one of the other messages is sent. If one weights these cases with the corresponding probabilities of the messages, then the weighted sum is less then C. This result appears almost obvious for the viewpoint of information theory.

Lemma 23 *For all* $n \geq 0$*, we have the inequality,*

$$\mathbb{E}[\log H_n - \log H_{n+1} \mid \mathcal{F}_n] \leq C_1 \quad a.s. \tag{5.3.4}$$

where

$$C_1 = \max_{i,k} \sum_{j=1}^{L} p(j \mid i) \log \left[\frac{p(j \mid i)}{p(j \mid k)} \right] = \max_{i,k} D\left(p(\cdot \mid i) \parallel p(\cdot \mid k)\right). \tag{5.3.5}$$

The proof of the next lemma is quite technical, and the lemma itself is brought here for completeness.

Lemma 24 *For all $n \geq 0$, and given the event that $Y_{n+1} = l$,*

$$\log H_n - \log H_{n+1} \leq \max_{i,k} \log \left[\frac{p(l \mid k)}{p(l \mid i)} \right] \leq \max_{i,k,l} \log \left[\frac{p(l \mid k)}{p(l \mid i)} \right] \triangleq F \tag{5.3.6}$$

where the first inequality holds almost surely. Note that both Lemmas 22 and 23 consider the average over the difference between information quantities; the entropy and logarithm respectively. Lemma 22 appears almost obvious for the viewpoint of information theory. It states that, in average, the entropy cannot be decreased by more that C. In other words, after n channel uses we can expect that a good code would be able to increase the decoders knowledge regarding the underlying message nC bits. This is, of course, a well known fact that is strongly related to the coding theorem. It is also known that constructing a code using the test statistic $I(X_n; Y_n)$ achieves capacity even without making use of feedback. Note also that the expected decrease we are bounding here is averaged over different possible messages. In other words, at a specific a posteriori probability distribution on the messages, one may be able to propose a coding method under which the entropy will decrease, at every time step, on average for a specific source message, by more than C. However, this method will have a poorer performance in the case where one of the other messages is sent. If one weights these cases with the corresponding probabilities of the messages, then the weighted sum is less then C. This is also true for Lemma 23, where the average of the difference between the difference in log of the entropy. In the sequel we show that Lemma 23 comes in handy when the entropies are small. When this occurs, it makes sense to use a non-linear function such as log in order to gain a lager difference between the entropies in successive time instants. Indeed, it is easy to show that $C < C_1$, and so this lemma gives an even stronger upper bound in terms of difference between two functions of the entropy.

5.4 The Converse of the Error Exponent Theorem

In this section, the converse of the theorem of the error exponent for VL coding with feedback will be given, as well as an outline of the proof by Burnashev [18], using Lemmas 21 - 24.

Theorem 25 *For any transmission method over a DMC with feedback, for all $P_e > 0$, and for all $M > e^B$, the expected number of observations $\mathbb{E}[N]$ satisfies the inequality:*

$$\mathbb{E}[N] \geq \frac{\log M}{C} - \frac{\log P_e}{C_1} - \frac{\log(\log M - \log P_e + 1)}{C_1} - \frac{P_e \log M}{C} + K \tag{5.4.1}$$

where K and B are constants determined by the channel transition probabilities, satisfying $B < F + 1$, and F is defined in (5.3.6)*.*

The constants K and B can be computed using only the channel parameters.

In order to see the relation between the error exponent function and Theorem 25, recall that $E(R)$ is defined in the limit as P_e tends to zero. For $M > 2$ and $P_e > 0$, we can divide both sides by $\mathbb{E}[N]$, and, after some algebra, get:

$$\frac{\log P_e}{C_1 \mathbb{E}[N]}\left(1 + \frac{\log(1 - \log P_e)}{\log(P_e)}\right) \leq 1 - \lim_{P_e \to \infty} \frac{\log M}{C\mathbb{E}[N]}\left(1 - P_e + \frac{CC_1 K - C\log(1 + \log M)}{C_1 \log M}\right). \tag{5.4.2}$$

Taking the limit as $P_e \to 0$, and using (5.2.9),

$$E(R) \leq C_1\left(1 - \frac{R}{C}\right) \triangleq E_B(R). \tag{5.4.3}$$

It will be shown that $E_B(R)$ is achievable. This means that (5.4.3) is the reliability function. This is in contrast to the reliability function without feedback which is known exactly only for rate zero and rates above the critical rate. For positive rates below the critical rate, only upper and lower bounds are known.

The proof of Theorem 25 is quite lengthy, and only a very brief review of its outline will be given. The core of the proof is the analysis of the following random process:

$$\xi_n = \begin{cases} C^{-1} H_n + n & \text{if } H_n \geq B \\ C_1^{-1} \log(H_n) + a + n & \text{if } H_n < B \end{cases} \tag{5.4.4}$$

$$= n + C^{-1} H_n \mathbb{I}\{H_n \geq B\} + \left\{C_1^{-1}\log(H_n) + a\right\}\mathbb{I}\{H_n < B\} \tag{5.4.5}$$

where B and a are some given constants. The first step in the proof is showing that $(\xi_n, \mathcal{F}_n)$ is a submartingale. To show $\mathbb{E}[|\xi_n|]$ is bounded, Lemma 24 is used, in addition to the trivial bound $H_n \leq \log(M)$. To show the condition on the difference process holds, i.e., $\mathbb{E}[\xi_{n+1} - \xi_n \mid \mathcal{F}_n] > 0$, notice that from Lemma 22 and Lemma 23 if follows that $(C^{-1} H_n + n, \mathcal{F}_n)$ and $\left(C_1^{-1} \log(H_n) + a + n, \mathcal{F}_n\right)$ are both submartingales. Then, these submartingales are used to show the process ξ_n, which is a process that "patches" together the two, is too a submartingale, if B and a are chosen properly. The next step of the proof is to use the fact that if $(x_n, \mathcal{F}_n)$ is a submartingale, then so is $(x_{n \wedge N}, \mathcal{F}_n)$ [58], and the following holds:

$$\xi_0 \leq \mathbb{E}[\xi_{n \wedge N} \mid \mathcal{F}_n] \leq \lim_{n \to \infty} \mathbb{E}[\xi_{n \wedge N} \mid \mathcal{F}_n]. \tag{5.4.6}$$

Plugging in (5.4.5), using the non-negativity of the entropy ,and invoking Jensen's inequality (for the concave function $\log(\cdot)$), one gets:

$$\xi_0 \leq \lim_{n \to \infty} \mathbb{E}\left[(n \wedge N) + C^{-1} H_{n \wedge N} \mid \mathcal{F}_0\right] + C_1^{-1} \log(\mathbb{E}[H_{n \wedge N} \mid \mathcal{F}_0]) + |a| + \frac{\log(B)}{C_1} \tag{5.4.7}$$

It is now not hard to show, using the fact that $H_0 = \log(M) > B$, Lemma 21 and taking the limit as $n \to \infty$[3], that the result in Theorem 25 holds.

5.5 The Direct Part

In this section, we present and discuss a coding theorem by Burnashev [18] that asserts that the lower bound of Theorem 26 is essentially achievable.

Define $\mathcal{J}$ as the set of pairs of input symbols $(i, k) \in K^2$ that achieve C_1, i.e., for any $(i, k) \in \mathcal{J}$,

$$C_1 = D(p(\cdot \mid i) \parallel p(\cdot \mid k)), \tag{5.5.1}$$

and define

$$\bar{C}_1 = \max_{(i,k) \in \mathcal{J}} D(p(\cdot \mid i) \parallel p(\cdot \mid k)). \tag{5.5.2}$$

Theorem 26 *For a DMC with noiseless feedback and zero delay, the following holds:*

[3]The fact that $\mathbb{E}[n \wedge N] \to \mathbb{E}[N]$ as $n \to \infty$ is since $\Pr(N < \infty) = 1$ was assumed. If this does not hold, the result in Theorem 25 is trivial. Also, $\mathbb{E}[H_{n \wedge N} \mid \mathcal{F}_0] \to \mathbb{E}[H_N \mid \mathcal{F}_0]$ as $n \to \infty$, due to the Bounded Convergence Theorem.

- *If $\bar{C}_1 > C$, a coding algorithm exists such that*

$$\mathbb{E}[N] < \frac{\log M}{C} - \frac{\log P_e}{C_1} + K' \tag{5.5.3}$$

where K' is a constant determined by the transition probabilities.

- *For all values of $\bar{C}_1$ and $\epsilon > 0$, encoder and decoders exist such that*

$$\mathbb{E}[N] < \frac{1}{1-\epsilon}\left(\frac{\log(M)}{C} - \frac{\log(P_e)}{C_1} + \frac{(C_1 - C)\log(\epsilon)}{CC_1}\right) + K'' \tag{5.5.4}$$

where K'' is a constant determined by the transition probabilities.

The coding schemes used to prove Theorem 26, for the complementary cases where $\bar{C}_1 > C$ and $\bar{C}_1 \leq C$, are different but similar. We will review only the scheme that is used in the former case, which is also somewhat simpler.

The converse proof section, relied mostly on the fact that one can distinguish between two phases in the behavior of an information quantity, H_n. This fact was used in order to motivate the construction of the submartingale process $\{\xi_n, \mathcal{F}_n\}$. This process is a patch between the submartingale processes $\xi'_n = C^{-1}H_n + n$ and $\xi''_n = C_1^{-1}\log H_n + a + n$, in a way that as long H_n is relatively large, $\xi_n = \xi'_n$, and $\xi_n = \xi''_n$ otherwise.

The direct part of the proof makes use of two different phases of an information quantity, but this time, instead of H_n, the log likelihood ratio function is used. In some sense, the proposed coding scheme is a generalization of the sequential test Δ_a defined in Section 4.3.A.; however, the prior is uniform and, since feedback channel is used, the likelihood ratio at time n is a function, not only of $\{Y_1 \ldots, Y_n\}$, but also of $\{X_1(Y_1), \ldots, X_{n-1}(Y_1 \ldots, Y_{n-1})\}$. In other words, the log-likelihood ratio of the m'th message at time n is now defined as

$$\Lambda_m(n) \triangleq \Lambda_m(Z_1, \ldots, Z_{n-1}, Y_n) = \log\left[\frac{\pi_m(n)}{1 - \pi_m(n)}\right] = \log\left[\frac{\pi_m(n)}{\sum_{m' \neq m}\pi_{m'}(n)}\right] \tag{5.5.5}$$

where $\pi_m(n) = \Pr(\theta = m \mid Z_1, \ldots, Z_{n-1}, Y_n)$ is the posteriori. The stopping time is then given by

$$N(\delta) = \min_{n \geq 0}\left\{\max_j \Lambda_j(n) \geq \ln\left(\frac{1}{\delta}\right)\right\}, \tag{5.5.6}$$

for some predetermined positive constant δ. Next, the coding algorithm will be defined. The novel approach in this coding algorithm is the fact that the two-phase concept, used in the converse statement, influences also on the coding strategy itself. The idea is the following: at each time instant n, $\Lambda_j(n)$, $j = 0 \dots M-1$, are calculated at the receiver (and, due to the feedback, the result of this computation is known also to the transmitter). Then, while all Λ_j's are less than a predetermined threshold p_0, a randomized strategy will be chosen, drawing the the $(n+1)$'st input letter in a way that the probability of the k'th input letter, given $\theta = m, Z_1, \dots, Z_{n-1}$ and Y_n is $\phi_k(\theta = m, Z_1, \dots, Z_{n-1}, Y_n)$, where

$$\begin{aligned} \sum_{k=1}^{K} \phi_k(\theta = m, Z_1, \dots, Z_{n-1}, Y_n) &= 1 \quad \forall m, Z_1, \dots, Z_{n-1}, Y_n, && (5.5.7) \\ \sum_{m=0}^{M-1} \phi_k(\theta = m, Z_1, \dots, Z_{n-1}, Y_n)\,\pi_m(n) &= \pi_k^C \quad \forall k, Z_1, \dots, Z_{n-1}, Y_n. && (5.5.8) \end{aligned}$$

where π^c is the capacity-achieving prior. In other words, $\phi_k(\theta = m, Z_1, \dots, Z_{n-1}, Y_n)$ will be such that posterior given $(m, Z_1, \dots, Z_{n-1}, Y_n)$ is equal to the capacity-achieving distribution. When the a posteriori probability of one of the messages, say $\hat{\theta}$, is over the threshold p_0, then if $\theta = \hat{\theta}$, the input symbol $i^\star$ will be assigned, and otherwise $k^\star$ will be assigned, where $i^\star$ and $k^\star$ are defined in (5.3.5). This is carried out until $n = N(\delta)$.

The proof that this scheme achieves the error exponent (5.4.3) (in the case at hand) is based on the following three main steps:

1. Define the stopping time

$$\bar{N}(\delta) = \min_{n \geq 0} \left\{ \Lambda_\theta(n) \geq \ln\left(\frac{1}{\delta}\right) \right\}, \tag{5.5.9}$$

for any $\delta > 0$. Note that for our coding algorithm, $N(\delta) \leq \bar{N}(\delta)$ holds almost surely, and so $\mathbb{E}[N(\delta)] \leq \mathbb{E}\left[\bar{N}(\delta)\right]$. In [18], the stopping time $\bar{N}(\delta)$ is analyzed in order to derive an upper bound on $\mathbb{E}\left[\bar{N}(\delta)\right]$.

2. The second step of the proof consists of the analysis of the difference process $\Lambda_m(n+1) - \Lambda_m(n)$. More precisely, it was proved in [18] that the sequence $\Lambda_m(n)$ forms a submartingale, and

$$\mathbb{E}\left[\Lambda_\theta(n+1) - \Lambda_\theta(n) \mid Z_1, \dots, Z_{n-1}, Y_n, \theta\right] \geq \begin{cases} C & \text{if} \quad \Lambda_\theta(n) < \log\left[\frac{p_0}{1-p_0}\right] \\ C_1 & \text{if} \quad \Lambda_\theta(n) \geq \log\left[\frac{p_0}{1-p_0}\right]. \end{cases} \tag{5.5.10}$$

In words, this means that, independent of the phase of the coding, the expected increase in the log-likelihood ratio process of the true message is greater than or equal to C, if the a posteriori probability of the true message is less than some threshold value. In addition, when the a posteriori probability of the true message is greater than this threshold, the expected increase in the log-likelihood of the true message is C_1.

3. It can be shown that for the coding scheme defined above, the probability of error is less than or equal to δ. Applying a result from [19] to the submartingale $\Lambda_\theta(n)$ yields the following:

$$\mathbb{E}\left[\bar{N}(\delta)\right] < \frac{\log(M)}{C} - \frac{\log(P_e)}{C_1} + K \tag{5.5.11}$$

where K is a constant determined by the transition probabilities. Hence from (5.5.6) and (5.5.9) it follows that $\mathbb{E}[N(\delta)] < \frac{\log(M)}{C} - \frac{\log(P_e)}{C_1} + K$ and the direct part is proven. This result holds in the case where $\bar{C}_1 > C$. In case $\bar{C}_1 \leq C$ holds, a similar coding scheme can be designed. Its structure is quite similar to the one described above, and will be omitted in this work.

Both the converse and the direct proofs by Burnashev are important for two reasons: first, his lemmas are useful in many other setups, andmoreover, they are building blocks for the later proofs, and the second is that it shows how the theory of martingales can be of aid in information-theoretic problems. In addition, we will see in the next section, that the direct part of Burnashev that is closely related to the hypothesis testing problem discussed in Section 4.4.B.

5.6 Alternative Proofs of the Error Exponent

In this section, two alternative proofs will be reviewed; one of the direct part, due to Yamamoto and Itoh [156], and the other for the converse, due to Berlin *et al.* [9]. The two proofs may be found, in some sense, more intuitive than the proofs by Burnashev, and help to shed light on the each element in the reliability function and on the special nature of the problem at hand. In addition, unlike the cumbersome proofs by Burnasev, the alternative proofs are simpler, and can be extended to other channels (for example Markov channels [28]) and to more complex communication setups (for example, cases with cost constraints [112], feedback and belief propagation decoding and noisy feedback).

A. Yamamoto & Itoh's Proof of the Direct Part

The Yamamoto-Itoh coding scheme [156] is a generalization of an earlier scheme [126], for the AWGNC with perfect feedback and a peak power constraint. Like Burnashev's scheme [18], this is a two-phase scheme, but instead of taking the log-likelihood ratios (or the conditioned entropy) to be the test statistics,here, the scheme uses of the error probability itself.

Yamamoto and Itoh's coding is done in cycles. Each transmission cycle has two modes of operation (or *phases*): the first is a *message mode* ("phase I") and the second is a *control mode* ("phase II"). Throughout the transmission time, the number of channel uses per operation phase (and, hence, per transmission cycle) is kept fixed. Let $L \in \mathbb{N}$ denote the total transmission time of each cycle, and let $0 < \gamma < 1$ be such that γL and $(1-\gamma) L$ are the lengths of the first and second phase, respectively. Next, the coding algorithm in each phase will be described, and the general idea of the analysis will be given.

<u>*Phase I:*</u>

In this phase, the transmitter sends one of the M messages using a code at rate $C(1-\epsilon)$, where $\epsilon \in (0,1)$ is arbitrarily small. It follows from the coding theorem [29], that given $P_{e,1} \in (0,1)$, and all sufficiently large M, there exists a code of block-length $\frac{\log M}{C(1-\epsilon)}$, for which the probability of error is at most $P_{e,1}$. In other words, there exists a code of rate $C(1-\epsilon)$ for which the error probability is upper bounded by $P_{e,1}$ and its block length is bounded by

$$\gamma L < \frac{\log(M)}{C(1-\epsilon)} + K(\epsilon, P_{e,1}) \tag{5.6.1}$$

where $K(\epsilon, P_{e,1})$ is a function of the transition probabilities ϵ and $P_{e,1}$.

At the end of this phase, the receiver can produce a good estimate θ_1, of the message.

<u>*Phase II:*</u>

Keeping in mind that the encoder is aware of the tentative estimate θ_1, the second phase is used to transmit an acknowledgment (ACK) message, in the case where $\theta_1 = \theta$, or a rejection message (NACK) otherwise. The receiver decodes this control signal, and if the decoded result is an ACK, the receiver accepts θ_1, as the decoded message. If the output is NACK, the receiver discards the message, and waits for a retransmission. The transmitter is again informed which of the control signals was received. If the receiver's decision is NACK, the transmitter retransmits in the next message mode. Otherwise, it sends the next message.

In this phase, a simple repetition code is used, mapping ACK to $(i^\star, i^\star \ldots, i^\star)$, and NACK to $(k^\star, k^\star \ldots, k^\star)$, where $i^\star, k^\star$ are as defined in (5.5.1). Let ϵ be some positive constant, $0 < \alpha_0 < 1$ and $0 < \beta_0 < 1$. Using the Chernoff-Stein Lemma (see, for example, [33], p. 19, Corollary 1.2) and a little algebra, it can be shown that there exists a (fixed length) binary hypothesis test with block length $(1-\gamma) L$ that satisfies both

$$\begin{aligned} \alpha \triangleq \Pr(\text{Choose ACK} \mid \text{NACK sent}) &\leq \alpha_0, & (5.6.2)\\ \beta \triangleq \Pr(\text{Choose NACK} \mid \text{ACK sent}) &\leq \beta_0 & (5.6.3)\end{aligned}$$

and

$$\begin{aligned} (1-\gamma) L &\leq \frac{-\log \alpha_0}{(1-\epsilon) \max_{i,k} D(p(\cdot \mid i) \parallel p(\cdot \mid k))} + K'(\beta_0, \epsilon) & (5.6.4)\\ &= \frac{-\log \alpha_0}{(1-\epsilon) C_1} + K'(\beta_0, \epsilon) & (5.6.5)\end{aligned}$$

where $K'(\alpha_0, \epsilon)$ is a function of β_0 and ϵ.

Combining (5.6.1) and (5.6.5), and using the fact that the observations at each cycle are independent, one obtains that for any $\delta, \epsilon > 0$:

$$\begin{aligned} \mathbb{E}[N] &\leq \frac{1}{1-2\delta}\left[\frac{\log M}{C} - \frac{\log P_e}{C_1} + \frac{\delta}{(1-2\delta) C_1} + \hat{K}(\delta, \epsilon)\right] & (5.6.6)\\ &\leq \frac{1}{1-2\delta-\epsilon}\left(\frac{\log M}{C} - \frac{\log P_e}{C_1} + \hat{K}(\delta, \epsilon)\right) & (5.6.7)\end{aligned}$$

where $\hat{K}(\delta, \epsilon)$ is a function of δ and ϵ. This proves the achievability of Burnashev's exponent.

It is worth mentioning that in [156], Yamamoto and Itoh did not realize that their scheme achieves the optimal error exponent. The reason was that they compared with Horstein's scheme [75], which achieves higher convergence rates. The reason for this contrariety, is that Horstein's decoding scheme is not done in blocks, and hence it is unfair to compare.

B. Modified Yamamoto & Itoh Scheme and Sequential Analysis

Instead of going through the analysis of the upper bound (5.6.7), we propose a novel modification to the Yamamoto and Itoh coding scheme and analyze it. The motivation for this analysis is that it connects the performance of communication links with feedback and sequential hypothesis testing, and it provides insight by clarifying the role played by the quantities that appear in the bound.

In the modified scheme, instead of a fixed-length code, a multiple hypothesis test procedure will be used in the first communication phase in order to make a tentative decision. Furthermore, instead of a fixed-length binary hypothesis test in the second phase, an SPRT is performed. The rest of the coding algorithm remains similar. Intuitively, the new scheme is not be worse than Yamamoto and Itoh's one. The reason is that we know that sequential procedures allow a reduction in the transmission length at the expense of the use of feedback (in the sense that the expected number of transmissions needed to reduce the error to a certain level is less than the number of transmissions needed under the fixed block length constraint). Specifically, the modified scheme is as follows:

<u>*Phase I':*</u>

In this phase, a randomly selected codebook is used, by assigning each message with an infinite random sequence, generated using the capacity-achieving distribution.

Let the codeword of message i be denoted by $\boldsymbol{x}^{(i)}$ and the channel output by $\boldsymbol{y}$. For each $i \in \{0, \ldots, M-1\}$, define the following two hypotheses:

$$H_0^i: \quad \Pr\left(\boldsymbol{x}^{(i)}, \boldsymbol{y}\right) = p\left(\boldsymbol{y} \mid \boldsymbol{x}^{(i)}\right) \Pr\left(\boldsymbol{x}^{(i)}\right) \tag{5.6.8}$$

$$H_1^i: \quad \Pr\left(\boldsymbol{x}^{(i)}, \boldsymbol{y}\right) = \Pr\left(\boldsymbol{y}\right) \Pr\left(\boldsymbol{x}^{(i)}\right), \tag{5.6.9}$$

In words, for each message i, we define H_0^i as the hypothesis that $\boldsymbol{x}^{(i)}$ was transmitted and H_1^i as the hypothesis that $\boldsymbol{x}^{(j)}, j \in \{0, \ldots, M-1\} \setminus \{i\}$ is chosen (and so $\boldsymbol{y}$ and $\boldsymbol{x}^{(i)}$ are independent).

For any infinite sequence $\boldsymbol{w}$ and $n \in \mathbb{N}$, let $[\boldsymbol{w}]_n$ be the first n symbols of $\boldsymbol{w}$. For each communication cycle k and any $\epsilon > 0$, define the following M stopping times, each stopping time corresponding to a single one-sided SPRT, testing H_0^i versus H_1^i:

$$N_{I,k}^i = \inf_{n\geq 0}\left\{\log\left[\frac{p\left([\boldsymbol{y}]_n \mid [\boldsymbol{x^{(i)}}]_n\right)}{\Pr\left([\boldsymbol{y}]_n\right)}\right] \geq (1+\epsilon)\log M\right\} \tag{5.6.10}$$

$$= \inf_{n\geq 0}\left\{\sum_{j=1}^{n}\log\left[\frac{p\left(y_j \mid x_j^{(i)}\right)}{\Pr\left(y_j\right)}\right] \geq (1+\epsilon)\log M\right\} \tag{5.6.11}$$

where (5.6.11) holds since the channel is memoryless.

Next, define the stopping time of the first phase at the k'th cycle to be:

$$N_{I,k} = \min_{i\in\{0,\ldots,M-1\}} N_{I,k}^i \tag{5.6.12}$$

$$= \min_{i\in\{0,\ldots,M-1\}}\left[\inf_{n\geq 0}\left\{\sum_{j=1}^{n}\log\left[\frac{p\left(y_j \mid x_j^{(i)}\right)}{\Pr\left(y_j\right)}\right] \geq (1+\epsilon)\log M\right\}\right]. \tag{5.6.13}$$

The operational meaning of $N_{I,k}$ is that, at each cycle, the decoder constructs M one-sided SPRTs with a threshold $(1+\epsilon)\log M$. At the end of each such a phase, the decoder chooses, as a tentative decision, the i'th message that corresponds to the i'th one-sided SPRT first to exceed the boundary value.

To understand the intuition and motivation of such a test, assume that $\boldsymbol{x^{(0)}}$ is transmitted through the channel. Then the one-sided SPRT (corresponding the the 0'th message) follows a random walk with a positive drift $\mathbb{E}\left[\log\left[\frac{p\left(y_j|x_j^{(0)}\right)}{\Pr(y_j)}\right]\right] = C > 0$, while the other $M-1$ random walks have a negative drift. Since $N_{I,k} \leq N_{I,k}^0$,

$$\mathbb{E}\left[N_{I,k}\right] \leq \mathbb{E}\left[N_{I,l}^0\right]. \tag{5.6.14}$$

By Wald's first equation, it is clear that the expected time for a random walk with drift C to pass the threshold $(1+\epsilon)\log M$ is approximately

$$\mathbb{E}\left[N_{I,k}^0\right] \approx \frac{(1+\epsilon)\log M}{C}, \tag{5.6.15}$$

and also, for large enough M, the error probability would meet any constraint. The full derivation is given in Appendix A.

Phase II':

In this phase, an infinite repetition code is used[4], mapping ACK to $(i^\star, i^\star \ldots)$, and

[4]By an infinite repetition code we mean that we do not restrict the codewords to be of finite size. In particular, each codeword is taken to be an infinite sequence, and it is the decoder that informs the encoder when to stop transmitting and move on to the next message.

NACK to $(k^\star, k^\star \ldots)$. Define the two hypotheses regarding the output sequence at this phase by:

$$H_{ACK}: \quad \Pr(Y_1^n) = \prod_{i=1}^n P_A(Y_i) \tag{5.6.16}$$

$$H_{NACK}: \quad \Pr(Y_1^n) = \prod_{i=1}^n P_N(Y_i), \tag{5.6.17}$$

where $P_A(y) = p(y \mid i^\star)$ and $P_N(y) = p(y \mid k^\star)$ for $y \in (1, \ldots, L)$. In this phase, at each cycle k, the decoder runs an SPRT with a stopping time denoted $N_{II,k}$ and a decision function denoted $d_{II,k} = d_{II,k}(N_{II,k})$. Note that, given the input sequence, the observations are indeed i.i.d. since a DMC is assumed, and $\Pr(Y_1^n)$ has in a product form. Let $0 < \alpha_0, \beta_0 < 1$ be the chosen bounds on the error probabilities α and β, respectively. By the theory developed in Part I, setting the boundary values of the SPRT to be $A = \frac{1}{\alpha_0}$ and $B = \beta_0$ will assures that

$$\alpha \le \alpha_0, \quad \beta \le \beta_0. \tag{5.6.18}$$

Recall that the (random) length of the second phase of the k'th cycle is denoted by $N_{II,k}$ and let $K \in \mathcal{N}$ be the total number of cycles until decoding. Note that the total transmission time needed for each message is given by

$$N = \sum_{j=0}^{K} [N_{I,k} + N_{II,k}] = \sum_{k=0}^{K} N_{I,k} + \sum_{k=0}^{K} N_{II,k}. \tag{5.6.19}$$

The idea of the proof is the following: As will be proven in Appendix A, one can choose M large enough, such that the probability that NACK would be sent in the second phase, denoted by π_N, is arbitrarily small. Specifically, we take it to be upper bounded by $P_{e,1} \in (0, 1/3)$. For this reason, the probability of sending ACK, $\pi_A = 1 - \pi_N$, can be made very close to one. Moreover, we know from Part I that both $\mathbb{E}_{P_A}[N_{II,k}]$ and $\mathbb{E}_{P_N}[N_{II,k}]$ are finite and therefore, for all $k = 1, 2, \ldots$,

$$\begin{aligned} \mathbb{E}[N_{II,k}] &= \pi_A \mathbb{E}_{P_A}[N_{II,k}] + \pi_N \mathbb{E}_{P_N}[N_{II,k}] && (5.6.20) \\ &\approx \mathbb{E}_{P_A}[N_{II,k}]. && (5.6.21) \end{aligned}$$

It is now left to upper bound $\mathbb{E}_{P_A}[N_{II,k}]$. Note that under H_{ACK}, the SPRT decodes a message according to whether a random walk, $\sum_n \log\left[\frac{P_A(Y_n)}{P_B(Y_n)}\right]$ passes one of the boundary values, set to be $-\log\alpha_0 (> 0)$ and $\log\beta_0 (< 0)$. Since $\mathbb{E}_A\left(\log\left[\frac{P_A(Y_n)}{P_B(Y_n)}\right]\right) = C_1 > 0$, and $P_e \le \alpha_0 P_{e,1}$ it is intuitively clear that

$$\mathbb{E}_{P_A}[N_{II,k}] \lesssim \frac{-\log\alpha_0}{C_1} \lesssim \frac{-\log P_e}{C_1}. \tag{5.6.22}$$

If we take large M and small β_0, the overall probability of retransmission is small. Taking $K \approx 1$ yields that

$$\mathbb{E}[N] \approx \mathbb{E}[N_{I,1}] + \mathbb{E}[N_{II,1}] \lesssim \frac{\log M}{C} - \frac{\log P_e}{C_1} \qquad (5.6.23)$$

which is the leading term in Burnashev's bound.

To conclude, in this section and in the former one, two alternative achievability proofs, based on simple schemes, were outlined. In both, the general idea was to alternate between a communication phase and a confirmation phase, until the receiver estimates the codeword sent by the transmitter, and gets an acknowledgement that the message estimate is correct. The main advantage of the scheme described in Section 5.6.B., is that it provides some insight, by clarifying the role played by the quantities that appear in Burnashev's bound. Specifically, from the channel coding theorem, we expect it to take roughly $\frac{\log(M)}{C}$ channel uses in order to reduce the probability of error to a sufficiently low level, so that it is safe to make a tentative decision in a reliable manner. In the sequential scheme, the fact that we know that approximately nC channel uses are needed to establish reliable communication, motivated the construction of a multiple hypothesis scheme that is build out of M one-sided SPRTs, one for each (equiprobable) possible message, in a way that the test statistics of the test following the correct message, will be a random walk with drift equal to the channel capacity, and setting the boundary value to be such that the average time for the random walk to hit it would be roughly $\frac{\log(M)}{C}$. After deciding on the primary guess regarding the message, the encoder and decoder have to agree on a test procedure that will allow the encoder to gather information about a binary message (ACNK or NACK) in the best way, in terms of error probability and expected stopping time. It is quite intuitive that the SPRT would be a good candidate for such a mission, using the two "most distinct" codewords for the given channel. Doing the analysis, we showed that the expected time for such a test to stop is given roughly by $\frac{-\log(P_e)}{C_1}$.

C. An Alternative Converse

The same intuition gained in Section B. was also used to prove a simple converse in [9]. The proof shows that the optimal coding algorithm (or any other coding algorithm using feedback for that manner) can be artificially divided into two phases: a coding phase and a binary hypothesis test phase, each having the same role as in the direct proof. Specifically, in order to lower bound the stopping time of a general feedback scheme, another coding scheme is constructed which bears the two-phase structure, has the same stopping time, but it has a better error probability. Next, the proof will be outlined, emphasizing the two-phases approach.

Assume N and P_e are the stopping time and error probability of the optimal decoder with respect to the of error exponent. Let $P_e\left(y^n\right)$ be the error probability of the maximum a posteriori probability (MAP) decoder after having received the output $\{y_1, \ldots, y_n\}$ (or, in other words, $P_e\left(y^n\right) = 1 - \max_{0 \le m \le M-1} P_{M|Y^n}\left(m \mid y^n\right)$) and define the stopping time $\tau \in \mathcal{F}_n$ to be:

$$\tau = \min_{n \ge 0} \left\{P_e\left(y^n\right) \le \delta \text{ or } n = N\right\}, \tag{5.6.24}$$

for some $\delta > 0$. Using Lemmas 21 and 22, the following result can be proven:

Lemma 27 (Lemma 2 in [9]) *For any* $0 < \delta < \frac{1}{2}$,

$$\mathbb{E}\left[\tau\right] \ge \left(1 - \delta - \frac{P_e}{\delta}\right) \frac{\log M}{C} - \frac{h_2\left(\delta\right)}{C}. \tag{5.6.25}$$

The key idea is the following: in the event $\{\tau \ne N\}$, which is shown to occur with high probability, it is possible to artificially divide the decoding time N into two parts: $[0, \tau)$, which is a (random) time interval at the end of which at least one message has high probability (as implied by the fact that $P_e\left(y^\tau\right) \le \delta$), and the remaining interval $[\tau, N]$. Assume that at time τ, the message set is divided into two sets, $\mathcal{G}$ and $\mathcal{G}^c$. Given access to the decoder estimate at time N, denoted by $\hat{\theta}$, consider a second decoder, working from time τ to time N, that declares $\theta \in \mathcal{G}$ if $\hat{\theta} \in \mathcal{G}$ and $\theta \notin \mathcal{G}$ otherwise. The following facts hold regarding the new decoder and the sets $\mathcal{G}$ and $\mathcal{G}^c$:

- If $\hat{\theta} = \theta$ then both the original decoder and the new one are correct, and hence the error probability of the new decoder does not exceed the error probability of the original one, conditioned on Y^τ.

- For any realization of Y^τ, the message sets $\mathcal{G}$ and $\mathcal{G}^c$ can be chosen to assure that both have posterior probabilities greater than $\lambda\delta$, where $\lambda \triangleq \min_{i,j \in \mathcal{X} \times \mathcal{Y}} p\left(j \mid i\right) (> 0)$. The way to construct these groups, given Y^τ, is described in detail in [9] and will not be repeated here.

It is now clear that in order to find a lower bound on the error probability, one can lower bound the error probability of the new decoding scheme, both conditioned on Y^τ. This is done using the following lemma (Lemma 1 in [9], which is closely related to results obtained in Section III is [137] and in Section 2.2 in [35]):

Lemma 28 *The error probability of a binary hypothesis test performed across a DMC with feedback and variable-length codes is lower-bounded by*

$$P_e \geq \frac{\min\{\pi_A, \pi_N\}}{4} \exp\left(-C_1 \mathbb{E}\left[N\right]\right), \tag{5.6.26}$$

where π_A and π_N are the prior probabilities of the hypotheses.

This lemma, proven by applying Doob's optional stopping theorem on the bounded supermartingale $\left(\log\left[\frac{P_A(Y_1,\ldots,Y_n)}{P_N(Y_1,\ldots,Y_n)}\right] - nC_1, \mathcal{F}_n\right)$ is of importance to the analysis due to the fact that the new decoder performs a binary hypothesis test from time τ to time N. In addition, the construction of the two (possibly composite) hypotheses, after observing T^τ, is done such that the conditional probabilities of the two hypotheses are greater than $\lambda\delta$ and hence, by Lemma 28,

$$\Pr\left(\hat{\theta} \neq \theta \mid Y^\tau\right) \geq \frac{\lambda\delta}{4} \exp\left(-C_1 \mathbb{E}\left[N - \tau \mid Y^\tau\right]\right). \tag{5.6.27}$$

Using the convexity of the exponential function and Jensen's inequality, we obtain

$$P_e \geq \frac{\lambda\delta}{4} \exp\left(-C_1 \mathbb{E}\left[N - \tau\right]\right) \tag{5.6.28}$$

and hence

$$\mathbb{E}\left[N - \tau\right] \geq \frac{-\log P_e - \log(4) + \log \lambda\delta}{C_1}. \tag{5.6.29}$$

Plugging in (5.6.25) yields that

$$\mathbb{E}\left[N\right] \geq \frac{\log M}{C} - \frac{\log P_e}{C_1} - \left(\delta + \frac{P_e}{\delta}\right)\frac{\log M}{C} - \frac{h_2(\delta)}{C} + \frac{\log \lambda\delta - \log(4)}{C_1} \tag{5.6.30}$$

which is essentially Burnashev's lower bound (since δ is arbitrarily small).

5.7 Variable-Length Coding with Cost Constraints

The mathematical setup so far assumed that all the input letters have equal cost. In other words, we have not been taking into account the possibility that some letters of the set $\{0, \ldots, |\mathcal{X}| - 1\}$ are "preferred" over other ones (e.g., require less power to be used by the transmitter). In this section this restriction will be relaxed, and a cost criteria will be imposed on the transmitted codewords. The motivation for this extra constraint is the fact that in practice, such a restriction is inherent to the communication problem (such as transmitted energy constraint), in addition to the desire to separate the effect of cost constraints from that of infinite alphabet size, thus allowing a better understanding of channels such as additive Gaussian noise, where these effects are combined. in other words, the work that is presented in this section isolated one of the elements that differ the setup debt with so far and the one in practice and analyses its effect on the error. Mathematically, in order to introduce a cost constraint, we consider the same DMC with feedback model that was proposed in Section 5.2, with the exception that for each input letter $k \in \mathcal{X}$, there is nonnegative transmission cost $\rho_k \geq 0$ and at least one ρ_k is zero. The cost $\mathcal{S}_N$ of transmitting a codeword of (random) length N is the sum of the costs of the N symbols in the codeword. A cost constraint $\mathcal{P}$ means that $\mathbb{E}\left[\mathcal{S}_N\right] \leq \mathcal{P}\mathbb{E}\left[N\right]$. If this relation holds, the codeword satisfies a power constraint of $\mathcal{P}$. With this definition, $\mathcal{P}$ can also be seen as an upper bound of the long term time average cost per symbol over a long string of independent successive message transmissions. Having set the mathematical setup, the question is whether an optimal error exponent, as defined in Section 5.2, can be found for a given pair $(R, \mathcal{P})$. In [112] the authors found the answer to be affirmative. Moreover, a closed-form optimal error exponent function was found for any set of costs $\{\rho_k\}$, any power constraint and any rate below capacity. By doing so, the Burnashev error exponent result was generalized to include the case above.

A good way to understand the result in [112] is to follow the direct part of the proof of main theorem. This part of the proof is a generalization of the Yamamoto-Itoh scheme that fits the cost constraint setup, and that results in the best error exponent possible for this case. As in Yamamoto-Itoh's scheme, the transmission of each codeword is in two phases. In the first, a capacity-achieving code is used and a tentative decision is made. In the second, an ACK/NACK message is sent to boost the reliability of the decision. The transmitter and receiver then act just as in Yamamoto-Itoh's scheme, that is, the message is sent again in case a NACK is decoded and an estimate is generated otherwise. Two key features of the scheme, which also simplifies the analysis, are the following:

a. The code of the first phase has rate less than the capacity.

b. The binary hypothesis test of the second phase is designed such that the probability of retransmission vanishes as the expected stopping time goes to infinity.

For these reasons, and just as in Yamamoto-Itoh's scheme, one can analyze the performance of the code as if it was a fixed-block-length code with a block length of $\mathbb{E}[N]$[5]. There is, however, a main difference between Yamamoto-Itoh's scheme and the one to be presented in this section, related to the cost constraint. Specifically, the total power $\mathcal{P}$ is divided between the two phases such that the power allocated to the first phase is $\mathcal{P}_1$, and the power allocated to the second phase is $\mathcal{P}_2$. The two new power constraints are chosen such that

$$\mathcal{P} = \gamma\mathcal{P}_1 + (1-\gamma)\,\mathcal{P}_2, \tag{5.7.1}$$

where γ is the time sharing parameter, defining the fraction of time the transmitter uses each communication phase (see Section 5.6). The way these power constraints are incorporated in the new proposed scheme is as follows:

- In the first phase, the fixed-block-length code is taken to be a code that achieves the capacity $C(\mathcal{P}_1)$ defined as

$$C(\mathcal{P}_1) = \max_{P_X : \sum_k P_X(k)\rho_k \leq \mathcal{P}_1} I(X;Y) \tag{5.7.2}$$

- Recall that in the original Yamamoto-Itoh scheme, the ACK and NACK codewords used in the second phase were repetition codewords composed of the two "most far apart" input letters in the KL divergence sense. These codewords, when used in a binary hypothesis testing problem are optimal in that they achieve the best error exponent, as can be seen from the Chernoff-Stein Lemma. Nevertheless, we have no guarantee that these two codewords will satisfy the cost constraint and hence we cannot always use them.[6]. In order to include the cost constraint, we need the following natural extension of the Chernoff-Stein Lemma:

Lemma 29 *Define the maximum single-letter divergence for the input letter $k \in$*

[5]The mathematical reason for this outcome is the fact that the probability of retransmission vanishes with an exponential rate as $\mathbb{E}[N]$ goes to infinity and hence, effectively, the code length can be said to be "almost deterministic" and highly concentrated around the value $\mathbb{E}[N]$.

[6]Note that the only codeword that should satisfy the power constraint is the ACK codeword. The reason is that the binary hypothesis test in our case, as in Yamamoto-Itoh's scheme, will be designed so that the probability that a NACK message will be sent vanishes (exponentially) with the expected stopping time. For this reason any exceeding from the power constraint caused by this codeword will, effectively, make no difference asymptotically.

$\mathcal{X}$ as

$$D_k \triangleq \max_{m \in \mathcal{X}} \sum_j p(j \mid k) \log \left[\frac{p(j \mid k)}{p(j \mid m)} \right] \tag{5.7.3}$$

and let m_k be the input letter achieving this maximum. If the ACK codeword contains $P_X(k)\,\mathbb{E}[N]$ occurrences of letter k, and the NACK codeword is chosen to contain the letter m_k whenever the ACK codeword contains k then the following result holds: for any $\delta > 0$ there is an $\epsilon \equiv \epsilon(\delta) > 0$ such that

$$\alpha \leq \exp\left[-(1-\gamma)\,\mathbb{E}[N] \left(\sum_k P_X(k)\, D_k - \delta \right) \right] \tag{5.7.4}$$

$$\beta \leq \exp\left[-(1-\gamma)\,\mathbb{E}[N]\,\epsilon \right], \tag{5.7.5}$$

where α and β are as defined in (5.6.2) - (5.6.3).

From (5.7.4), it is clear that in order to maximize the error exponent we must choose the ACK codeword to maximize $\sum_k P_X(k)\, D_k$ subject to the power constraint. Thus, for a power constraint $\mathcal{P}_2$ in phase II, define $D(\mathcal{P}_2)$ as

$$D(\mathcal{P}_2) \triangleq \max_{P_X :\, \sum_k P_X(k)\rho_k \leq \mathcal{P}_2} \sum_k P_X(k)\, D_k. \tag{5.7.6}$$

Following the aforementioned results, the new proposed scheme can be interpreted as providing a nominal rate of $R = \gamma C(\mathcal{P}_1)$, a nominal power constraint of $\mathcal{P} = \gamma\mathcal{P}_1 + (1-\gamma)\,\mathcal{P}_2$, and a nominal exponent of error probability $(1-\gamma)\,D(\mathcal{P}_2)$. Altogether, we have demonstrated the existence of variable length block codes for which the actual average rate, power, and exponent approach these values arbitrarily closely as $\mathbb{E}[N]$ becomes large. The error exponent achieved by this scheme is then given by

$$E(R, \mathcal{P}) \triangleq \sup\{(1-\gamma)\, D(\mathcal{P}_2)\} \tag{5.7.7}$$

subject to

$$\gamma C(\mathcal{P}_1) = R \tag{5.7.8}$$

$$\gamma\mathcal{P}_1 + (1-\gamma)\,\mathcal{P}_2 = \mathcal{P} \tag{5.7.9}$$

$$\mathcal{P}_1, \mathcal{P}_2 \geq 0, \quad \gamma \in [0,1]. \tag{5.7.10}$$

where the supremum is taken over $\mathcal{P}_1, \mathcal{P}_2$ and γ.

In [112] it was shown that there is a certain interval of γ for which these constraints hold, and that $\mathcal{P}_1$ and $\mathcal{P}_2$ are essentially uniquely defined as a function of γ in that interval. Thus (5.7.7) is simply a concave maximization over an interval. The resulting function is then also concave in $(R, \mathcal{P})$, and thus also as a function of R for any given $\mathcal{P}$. It can be shown that $E(R, \mathcal{P})$ is strictly decreasing in R from $D(\mathcal{P})$ at $R = 0$. The main theorem, proven in [112], is the following:

Theorem 30 *Assume ideal feedback for a DMC with all $p(j \mid k) > 0$. Then for all $\mathcal{P} > 0, R \leq C(\mathcal{P}), \delta > 0$, and all sufficiently large l there is a variable-length block code of expected length $\mathbb{E}[N], l \leq \mathbb{E}[N] \leq l+1$ with $M \geq \exp[\mathbb{E}[N](R-\delta)]$ messages such that for each message $\theta \in \{1, \ldots, M\}$, the probability of error P_e and the expected energy $\mathbb{E}[S_N]$ satisfy:*

$$P_e \leq \exp\{-\mathbb{E}[N][E(R, \mathcal{P}) - \delta]\} \tag{5.7.11}$$

$$\mathbb{E}[S_N] \leq \mathcal{P}\mathbb{E}[N] + \epsilon(\delta) \tag{5.7.12}$$

where $\epsilon(\delta) > 0$ for each $\delta > 0$, and the probability that the codeword length exceeds l is at most δ. Furthermore, for sufficiently large $\mathbb{E}[N]$, all variable length block codes with

- *expected energy $\mathbb{E}[S_N] \leq \mathcal{P}\mathbb{E}[N] + \delta$*
- *$M \geq \exp\{-\mathbb{E}[N][E(R, \mathcal{P}) + \delta]\}$ equiprobable massages*

must satisfy

$$P_e \geq \exp\{-\mathbb{E}[N][E(R, \mathcal{P}) + \delta]\}. \tag{5.7.13}$$

The proof of the converse part of Theorem 30 can be found in [112], where the authors generalized the basic lemmas of VL coding (see Section 5.3) to fit into the framework discussed here, and then used them to upper bound the random time of both communication phases, akin to Burnashev's proof. To conclude, Theorem 30 specifies the reliability function for the class of variable-length block codes for DMCs with cost constraints where the transition probabilities of the DMC are all positive, and the feedback channel is ideal.

A few notes are in place:

1. Assume that at least one transition probability is zero, say $p(j \mid m) = 0$ for some $j \in \mathcal{Y}$ and $m \in \mathcal{X}$. Without loss of generality, assume also that for each output

j, $p(j \mid k) > 0$ for at least one input k and thus $D_k = \infty$. Suppose that the ACK codeword of phase II uses all k's, the NACK message all m's, and that the receiver decodes ACK only if it receives one or more j's. In this case, no errors can ever occur for the corresponding VL block code. Since, asymptotically, phase II can occupy a negligible portion of the total communication duration, designing the first phase such that the total energy satisfies the cost constraint will yield that the total energy also satisfies $\mathbb{E}[S_N] \leq \mathbb{E}[N]\mathcal{P} + \delta$ (for any $\delta > 0$). Following this idea and after a little analysis, one can find that for the case where at least one transition probability is zero, there is a VL code satisfying

$$M \geq \exp\{\mathbb{E}[N](R - \delta)\}, \qquad P_e = 0, \qquad \mathbb{E}[S_N] \leq \mathcal{P}\mathbb{E}[N] + \epsilon(\delta), \quad (5.7.14)$$

and hence probability of error equal to zero is achievable at all rates up to the cost constrained capacity.

2. The rate and the error exponent are specified in terms of the expected block length. By looking at a long sequence of successive message transmissions, it is evident from the law of large numbers that the rate corresponds to the average number of bits transmitted per unit time. In the same way, the cost constraint is satisfied as an average over both time and channel behavior. The theorems then say that essentially the probability of error, P_e, for the best variable-length block code of given rate R, power constraint $\mathcal{P}$, and expected stopping time $\mathbb{E}[N]$ satisfies

$$\frac{-\log(P_e)}{\mathbb{E}[N]} \to E(R, \mathcal{P}) \quad \text{as} \quad \mathbb{E}[N] \to \infty. \quad (5.7.15)$$

A similar result is true, for the same reason, also for the case where no constraints are posed on the power of transmission.

5.8 Variable-Length Coding and Controlled Sensing

One of the main motivations behind this work is to establish a better understanding of the connection between the sequential hypothesis testing problem and variable length coding. The issue considered in this part of the work, i.e., the VL coding problem, is of the kind of sequential discrimination of multiple hypotheses with control of the observations. Nevertheless, it is important to note that the controlled sensing problem, that was considered in Part I, is fundamentally different. Unlike in the setup in Section 5.2, where the control actions are functions of the underlying hypothesis, the control actions in Section 4.4 cannot be functions of the unknown hypothesis. Knowledge of he hypothesis simplifies both the optimization of control policies and their performance analysis. When the hypothesis is unknown, the controller has to base its actions on estimates of the hypothesis. Another important difference between the two mathematical setups is the asymptotic regime. In the sequential hypothesis testing problem, we took interest in the asymptotic regime where $\mathbb{E}[N]$ goes to infinity (or, equivalently,the probability of error goes to zero), while the number of hypotheses is kept fixed, whereas in the communication problem, this number grows exponentially fast, to keep the rate, $\frac{\log(M)}{\mathbb{E}[N]}$, fixed.

Despite the differences in the information structure, it is not hard to see that the channel coding problem can be treated as a special case of the sequential hypothesis testing with control. We start by considering the zero rate case. In other words, the controlled hypothesis testing problem will be formalized as a "legitimate" VL coding problem with perfect feedback, while a fixed number of messages (which play the role of the hypotheses) will still be assumed. As an example, it will be shown, following [114], that $E_B(0)$ is the optimal error exponent at zero-rate. This is done by invoking Theorem 12 and the controlled hypothesis testing scheme (ϕ_c, N_c, d_c), defined in Section 4.4.A. To that end, consider the $M \times 1$-dimension vector $(X_n(0), \ldots, X_n(M-1)) \in \mathcal{X}^M$ to correspond to the control $u_n \in \mathcal{U}$. Particularly, for the equivalent sequential hypothesis testing problem, we take the control set $\mathcal{U} = \mathcal{X}^M$. For any specific control action $u = (x_n(0), \ldots, x_n(M-1))$ the observation model is simply $p_i^u(j) = p(j \mid i)$, where $p(j \mid i)$ is the channel transition matrix defined in (5.2.1). Assume, as in most communication models, equiprobable messages, i.e., $\pi_i = \frac{1}{M}$ for all $i \in \{0, \ldots, M-1\}$ and consider a feedback communication system as illustrated in Figure 5.2.2. The description above can be made more rigorous in the following way: let $\mathcal{E} = \{e(\cdot) : \{0, \ldots, M-1\} \to \mathcal{X}\}$ be the set of all mappings from the set of messages $\{0, \ldots, M-1\}$ to $\mathcal{X}$. In [109], using the results from [99], it was proven that without loss of generality, a fictitious agent can be added to the communication system depicted in Figure 5.2.2, who has access to past channel outputs, and is responsible for selecting actions from the set $\mathcal{E} \cup \{\mathcal{D}\}$, where $\mathcal{D}$ stands for the decision-making action, i.e, action $\mathcal{D}$ marks the termination of the transmission phase at the stopping time, while the choice of encoding function e_n at time n determines the input to the channel at time n. In other words, $X_n = e_n(\theta)$. It can be shown that the two definitions of the problem, with or without the the addition of the fictitious agent, are equivalent. The reason is that the decision of the fictitious agent at any time instant n relies solely on $\{Y_1, \ldots, Y_{n-1}\}$, which is fully observable by both the transmitter and receiver and hence are easily replicated at transmitter and receiver in isolate. The new and equivalent representation can be schematically illustrated in Figure 5.8.1.

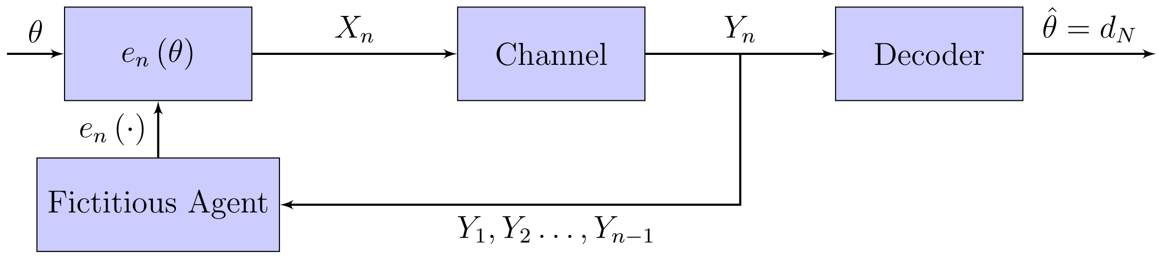

Figure 5.8.1: Variable length coding with feedback from the point of view of the fictitious agent.

From this point of view, variable-length coding with noiseless feedback is closely related to a special case of Problem (P') defined in Section 4.4.B. where the fictitious agent plays the role of the Bayesian decision maker whose available actions coincide with the set $\mathcal{E}$, and whose observation kernels are given by $p_i^u(j) = p(j \mid e(i))$.

Given that we have posed the channel coding problem as a hypothesis testing problem, we can simply invoke Theorem 12 to compute an upper bound on the expected coding length, subject to a constraint on the error probability $P_e \leq \epsilon$ (for arbitrarily small values of ϵ), and evaluate the performance of (ϕ_c, N_c, d_c) in order to obtain a lower bound. Note also that, for the equivalent hypothesis testing problem,

$$\sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right) = \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p(\cdot \mid i) \parallel p(\cdot \mid j)\right). \tag{5.8.1}$$

Following (4.4.26), we are interested in $\max_{\bar{q}(u)} \min_{j \in \{0,\ldots,M-1\} \setminus \{i\}} \sum_{u \in \mathcal{U}} \bar{q}(u) D\left(p_i^u \parallel p_j^u\right)$. From now on we will assume transition probability for all channel inputs bear a certain symmetry such that the inner minimization over j is superfluous. Consequently, the outer maximization over the control policy $q(u)$ is achievable by a point mass distribution at the two most distinguishable channel input symbols, i.e.

$$\max_{\bar{q}(u)} \min_{j\in\{0,\ldots,M-1\}\setminus\{i\}} \sum_{u\in\mathcal{U}} \bar{q}\left(u\right) D\left(p_i^u \parallel p_j^u\right) = C_1, \quad \forall i \in \{0,\ldots,M-1\} \tag{5.8.2}$$

Note that for this symmetric setup, $\bar{R}_i = \frac{\epsilon}{M}$ for all i, and so, it follows that $\mathbb{E}\left[N\right] \sim \frac{-\log P_e}{C_1}$ or

$$\frac{-\log P_e}{\mathbb{E}\left[N\right]} \sim C_1, \tag{5.8.3}$$

which is Burnashev's exponent, defined in (5.4.3), at zero rate.

As was mentioned in the previous part, the main drawback of the feedback schemes covered in Section 4.4.A., including (ϕ_c, N_c, d_c), is that in the asymptotic optimality notion, the complementary role of the number of the hypotheses, M, was neglected. In particular, the notion of asymptotic optimality in the sense of vanishing error probability with a fixed number of hypotheses, falls short in showing the tension between an asymptotically large number of samples to discriminate among a few hypotheses with asymptotically high accuracy, or, alternatively, an asymptotically large number of hypotheses with a lower degree of accuracy. This is, of course, why we could obtain results that are restricted to zero rate, i.e., potentially unbounded number of samples are used to acquire $\log M$ bits of information. It is also worth mentioning that the scheme of Chernoff, if specialized to channel coding with feedback, coincides with the second phase of Burnashev's scheme. However, while the first phase of Burnashev's scheme can achieve any information rate up to capacity Chernoff's one-phase scheme has a zero rate. The fact that in order to obtain asymptotically optimal performance a two-phase scheme was used, one to obtain reliability and another to assure non zero rate, is not surprising.

In what follows, we use the test Δ_{II} in Section 4.4.B., in order to prove the achievability of optimality for positive rates. Specifically, in order to prove asymptotic optimality, following [111], upper and lower bounds on $\mathbb{E}\left[N_\epsilon^\star\right]$, defined as the minimal expected number of samples required to achieve $P_\mathrm{e} \leq \epsilon$ in problem (P'), will be established. For the upper bound, Lemma 20 is used, setting $w = \frac{1}{\epsilon\log(M/\epsilon)}$, combined with

Corollary 16, with $\delta = \frac{1}{\log(M/\epsilon)}$. This yields that:

$$\mathbb{E}\left[N_\epsilon^\star\right] \geq \left(1 - \frac{1}{\log\left(\frac{M}{\epsilon}\right)}\right)\left[\frac{H\left(\pi\right) - \frac{\log(M-1)}{\log\left(\frac{M}{\epsilon}\right)}}{I_{\max}}\right.$$

$$\left. + \frac{\log\left(\frac{1}{\epsilon}\right) - 2\log\left(\log\left(\frac{M}{\epsilon}\right)\right)}{D_{\max}} \mathbb{I}\left\{\max_i \pi_i \leq 1 - \frac{1}{\log\left(\frac{M}{\epsilon}\right)}\right\} - O\left(1\right)\right]^+ \quad (5.8.4)$$

$$\geq \frac{\log\left(M\right)}{C} + \frac{\log\left(\frac{1}{\epsilon}\right)}{C_1} \mathbb{I}\left\{\max_i \pi_i \leq 1 - \frac{1}{\log\left(\frac{M}{\epsilon}\right)}\right\} - O\left(\log\left(\log\left(\frac{M}{\epsilon}\right)\right)\right), \quad (5.8.5)$$

where the last inequality holds for the specific hypothesis testing depicted in Figure 5.8.1 which was shown to be equivalent to the channel coding problem with perfect feedback.

For the lower bound, notice that for $w > \frac{1}{\epsilon}$,

$$\mathbb{E}\left[N_\epsilon^\star\right] \leq V^\star\left(\pi\right). \quad (5.8.6)$$

Moreover, for test Δ_{II}, it can be shown (as in [111]) that the following bound on the value function holds:

$$V^{\Delta_{II}}\left(\pi\right) \leq \frac{H\left(\pi\right)}{\max_{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}} \min_{i\in\{0,\ldots,M-1\}} \min_{\hat{\pi}\in\triangle_M} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \parallel \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u\right)} \quad (5.8.7)$$

$$+ \sum_{i=0}^{M-1} \frac{\pi_i \log\left(\frac{1-w^{-1}}{w^{-1}}\right)}{\max_{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}} \min_{\hat{\pi}\in\triangle_M} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \parallel \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u\right)} + O\left(1\right). \quad (5.8.8)$$

Applying this bound to the problem at hand, and using the fact that under these conditions (Claim 1 in [111]):

$$\max_{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}} \min_{i\in\{0,\ldots,M-1\}} \min_{\hat{\pi}\in\triangle_M} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \middle\| \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u\right) \geq C \quad (5.8.9)$$

$$\max_{\boldsymbol{\lambda}\in\triangle_{|\mathcal{U}|}} \min_{\hat{\pi}\in\triangle_M} \sum_{u\in\mathcal{U}} \lambda_u D\left(p_i^u \middle\| \sum_{i\neq j} \frac{\hat{\pi}_j}{1-\hat{\pi}_i} p_j^u\right) = C_1, \quad (5.8.10)$$

it follows that

$$\mathbb{E}\left[N_\epsilon^\star\right] \leq \frac{\log M}{C} + \frac{\log\left(\frac{1}{\epsilon}\right)}{C_1} + O\left(1\right). \quad (5.8.11)$$

By combining (5.8.5) and (5.8.11), and using the correct asymptotic regime, Burnashev's result is recovered for all rates up to capacity.

Chapter 6

Communication Systems With Limited Feedback

6.1 Introduction

In the previous chapter, we have characterized and discussed the maximum error exponent that can be achieved using a DMC in the presence of a perfect feedback channel, otherwise known as Burnashev's exponent. One of the main and obvious drawbacks in the setting considered so far, is the unrealistic assumptions regarding the feedback channel, i.e., instantaneous and error-free. Another undesired property of the coding schemes discussed in the previous chapter is the fact that their decoding time is not necessarily bounded. In other words, at least theoretically, the time one may wait until a final decision is made can be by far larger than its expected value. The rest of this work will be devoted to studying how the error exponent function is affected when some of these assumptions are relaxed.

In this chapter, a different mathematical setup will be considered, which imposes a more realistic constraint on the feedback channel, and hence on the coding schemes. In particular, we consider a DMC with a noiseless binary feedback channel[1] where only a single use of this channel per message is allowed, that is, for each codeword transmitted, the receiver can use a binary, instantaneous, error-free feedback channel only once. As an example, consider the case where the receiver can analyze the observation sequence, and signal to the transmitter to stop the transmission when it has high confidence regarding the message sent. Note that the most significant difference between

[1]Using the notation defined at the beginning of Chapter 5, this assumption means that $|\mathcal{Z}| = 2$.

this setup and the one considered in Chapter 5, is that now, one cannot assume that the transmitter has access to all the information obtained by the receiver. Therefore, the transmitter cannot determine whether the message was decoded correctly nor can it send new symbols in a manner that depends on the observation sequence. Mathematically, given $\theta \in \{0, \ldots, M-1\}$, the new constraint implies that the symbol sent by the encoder at time instant n, denoted in (5.2.2) by $X(\theta, Z_1, \ldots Z_{n-1})$, is no longer a function of the received observation sequence $\{Y_1 \ldots, Y_{n-1}\}$ (and hence of $\{Z_1 \ldots, Z_{n-1}\}$), and so, we can redefine $X(\theta, Z_1, \ldots Z_{n-1}) \triangleq X_n(\theta)$ for any $\{Z_1 \ldots, Z_{n-1}\}$.

6.2 Forney's Error Exponent

In this section, an achievable error exponent will be presented for the communication setup at hand via a particular scheme due to Forney [53]. Forney proposed a decoding scheme based on the erasure-decoding principle to be defined later. The most important feature of this error exponent function is that it provides a proof that even a feedback of one bit, as the one described in the Introduction, increases the error exponent significantly in comparison to the error exponent without feedback.

A. Erasure and Undetected Error Exponents

Before delving into the scheme itself, a few related results, obtained in [53] are needed. We start by considering fixed block-length coding, where, at the end of transmission, the decoder has an additional option of not deciding, i.e., of rejecting all messages. The resulting output is called an *erasure*. Under this setup, only if the decoder estimates, the message incorrectly, we have an undetected error. It is clear that by allowing the erasure probability to increase, the undetected error probability can be reduced.

Mathematically, consider the same forward DMC that was defined in Section 5.A. A a rate-R block code of length n consists of $M = \exp(nR)$ vectors of length n, $\{x_m \in \mathcal{X}^n, m \in \{0, \ldots, M-1\}\}$, which represent M different messages. As before, we assume that all messages are a-priori equiprobable. A decoder with an erasure option is a partition of the observation space $\mathcal{Y}^n$ into $(M+1)$ regions, $\mathcal{R}_0, \ldots, \mathcal{R}_M$. Such a decoder works as follows: if the output sequence $y \in \mathcal{Y}^n$ falls into $\mathcal{R}_m, m \in \{0, \ldots, M-1\}$, then a decision is made in favor of message number m. If, on the other hand, $y \in \mathcal{R}_M$, no decision is made and an erasure is declared. We will refer to the event $\{y \in \mathcal{R}_M\}$ as the erasure event. A graphical illustration of the erasure decoding decision regions is given in Figures 6.2.1 and 6.2.2.

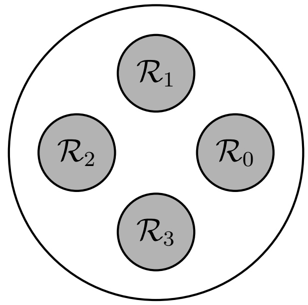

Figure 6.2.1: Typical partition of the observation space in erasure-decoding schemes.

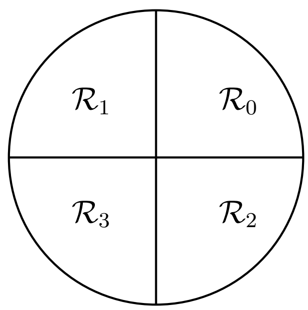

Figure 6.2.2: Typical partition of the observation space in classical decoding schemes.

Following Forney [53], we next define two additional undesired events. The event $\mathcal{E}_1$ is the event of not making the right decision. This event is the disjoint union of the erasure event and the event $\mathcal{E}_2$, which is the undetected error event, namely, the event of making the wrong decision. The probabilities of these events are given by

$$\Pr(\mathcal{E}_1) = \sum_{m=0}^{M-1} \sum_{y \in \mathcal{R}_m^c} \Pr(x_m, y) = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{y \in \mathcal{R}_m^c} p(y \mid x_m), \tag{6.2.1}$$

$$\Pr(\mathcal{E}_2) = \sum_{m=0}^{M-1} \sum_{y \in \mathcal{R}_m} \sum_{m' \neq m} \Pr(x_{m'}, y) = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{y \in \mathcal{R}_m} \sum_{m' \neq m} p(y \mid x_{m'}). \tag{6.2.2}$$

If $\mathcal{E}_1$ occurs, either an undetected error or an erasure must ensue. Therefore, the probability of erasure is given by

$$\Pr(\mathcal{R}_M) = \Pr(\mathcal{E}_1) - \Pr(\mathcal{E}_2) \tag{6.2.3}$$

In [53], using the Neyman-Pearson theorem, Forney showed that the best trade-off between $\Pr(\mathcal{E}_1)$ and $\Pr(\mathcal{E}_2)$ (or, equivalently, between $\Pr(\mathcal{R}_M)$ and $\Pr(\mathcal{E}_2)$) is attained by the following decision regions:

$$\mathcal{R}_m^\star = \left\{ y \in \mathcal{Y}^n : \frac{p(y \mid x_m)}{\sum_{m' \neq m} p(y \mid x_m)} \geq \exp\{nT\} \right\}, \quad m \in \{0, \ldots, M-1\}, \tag{6.2.4}$$

$$\mathcal{R}_M^\star = \bigcap_{m=0}^{M-1} (\mathcal{R}_m^\star)^c \tag{6.2.5}$$

where $T \geq 0$ is a parameter that controls the balance between $\Pr(\mathcal{E}_1)$ and $\Pr(\mathcal{E}_2)$. Note that, if $T > 0$, there can be at most one $m \in \{0, \ldots, M-1\}$ satisfying the condition; clearly, as T increases, $\Pr(\mathcal{E}_1)$ increases while $\Pr(\mathcal{E}_2)$ decreases, since the decision regions $\{R_m\}$ shrink. In addition, note that

$$\frac{p(y \mid x_m)}{\sum_{m' \neq m} p(y \mid x_m)} = \frac{\Pr(y, x_m)}{\sum_{m' \neq m} \Pr(y, x_m)} = \frac{\Pr(x_m \mid y)}{1 - \Pr(x_m \mid y)}, \tag{6.2.6}$$

and so, an equivalent test statistic, which will produce the exact same estimate, is to choose the message $m \in \{0, \ldots, M-1\}$ according to

$$y \in \mathcal{R}_m \Leftrightarrow \Pr(x_m \mid y) \geq \eta \tag{6.2.7}$$

where

$$\eta \triangleq \frac{\exp(nT)}{1 + \exp(nT)} \geq \frac{1}{2}. \tag{6.2.8}$$

From this relation, one can read off the idea behind optimal decoding with the erasure option. Recall that without erasure, the optimal decoding algorithm, in the sense of minimum error probability, is to choose the codeword x_m for which $\Pr(x_m \mid y)$ is the largest among all other codewords. On the other hand, in the erasure case, the optimal decoding algorithm is to choose the codeword x_m for which $\Pr(x_m \mid y)$ is the largest among all other codewords, but only as long as $\Pr(x_m \mid y) \geq \eta, \eta \geq \frac{1}{2}$, and otherwise, declare an erasure. In other words, the decoder has the option to be "more careful" with its decision, and it sends an erasure if the confidence after receiving n output symbols is not large enough.

For the error events $\mathcal{E}_1$ and $\mathcal{E}_2$, define the error exponents $e_i(R,T), (i = 1, 2)$, to be the exponents associated with the average probabilities of error $\overline{\Pr}(\mathcal{E}_i)$, where the average is taken with respect to the ensemble of randomly selected codes, drawn independently according to an i.i.d. distribution $\mathbf{p}(x) = \prod_{i=1}^n p(x_i)$, that is

$$e_i(R,T) \triangleq \limsup_{n \to \infty} \left[-\frac{1}{n} \log \left(\overline{\Pr(\mathcal{E}_i)} \right) \right]. \tag{6.2.9}$$

The main contribution of [53] is the following upper bounds on both the erasure and undetected error probabilities (or, equivalently, lower bounds on the error exponents $e_1(R,T)$ and $e_2(R,T)$):

Theorem 31 (Theorem 2 in [53]) *There is a block code of length n and rate $R = \frac{\log M}{n}$ such that when the likelihood ratio criterion of* (6.2.4) *and* (6.2.5) *is used with some threshold $T \geq 0$, one can simultaneously obtain*

$$\overline{\Pr(\mathcal{E}_1)} \leq \exp\{-nE_1(R,T)\}, \tag{6.2.10}$$

$$\overline{\Pr(\mathcal{E}_2)} \leq \exp\{-nE_2(R,T)\} \tag{6.2.11}$$

where $\Pr(\mathcal{E}_1)$ and $\Pr(\mathcal{E}_2)$ are as defined in (6.2.1) *and* (6.2.2)*, and $E_1(R,T)$ is given at high rates by*

$$E_1(R,T) = \max_{0<s<\rho<1,\mathbf{p}} \{E_0(s,\rho,\mathbf{p}) - \rho R - sT\}, \tag{6.2.12}$$

$$E_0(s,\rho,\mathbf{p}) = -\log\left[\sum_{y\in\mathcal{Y}}\left(\sum_{x\in\mathcal{X}} p(x)\,p^{1-s}(y\mid x)\right)\cdot\left(\sum_{x\in\mathcal{X}} p(x)\,p^{s/\rho}(y\mid x)\right)^{\rho}\right] \tag{6.2.13}$$

and at low rates by

$$E_1(R,T) = \max_{0<s<1,\rho\geq 1,\mathbf{p}} \{E_x(s,\rho,\mathbf{p}) - \rho R - sT\}, \tag{6.2.14}$$

$$E_x(s,\rho,\mathbf{p}) = -\rho\log\left[\sum_{x\in\mathcal{X}}\sum_{x_1\in\mathcal{X}} p(x)\,p(x_1)\left(\sum_{y\in\mathcal{Y}} p^{1-s}(y\mid x_1)\,p^{s}(y\mid x)\right)^{1/\rho}\right] \tag{6.2.15}$$

where $\mathbf{p} = \{p(x), x\in\mathcal{X}\}$ denotes the input probability distribution. Since

$$E_0(s,1,\mathbf{p}) = E_x(s,1,\mathbf{p}), \tag{6.2.16}$$

it holds that the two bounds are connected at intermediate rates by a straight line of slop -1.

In addition, $E_2(R,T)$ is given by:

$$E_2(R,T) = E_1(R,T) + T. \tag{6.2.17}$$

Note that the terms *high rates* and *low rates* can be defined using (6.2.16), that is, the rates lower than the point at which (6.2.16) holds will be referred to as low rate, while the rates above the point at which (6.2.16) holds will be denoted as high rates.

These bounds (6.2.10) and (6.2.11) are merely a generalization of the known bounds on the error exponent of fixed-length block codes. Specifically, for block codes without erasure, it is well known that the decoder that operates using decision regions according to the likelihood ratios, is optimal in the sense of average error probability. Analyzing the performance of this decoder led to the celebrated Gallager's lower bound on the error exponent, for high rates, and to the expurgation bound, for low rates. The derivation of both bounds can be found, e.g., in [56, Chapter 5] or [143, Chapter 3]. In order to prove (6.2.10) and (6.2.11), similar techniques to those used in the derivation of the Gallager's lower bound (for high rates) and the expurgation bound (for low rates) were used by Forney. Moreover, note that by setting $T = 0$, we get the Gallager error exponent. Hence, as mentioned in [53], not only Gallager bound is a special case of the above bounds, but furthermore, these bounds prove that the random coding error exponent is attainable, not only under the ML decoder, but also under the decoding rule of (6.2.4) and (6.2.5) with $T = 0$. In order to emphasize how to apply the classical bounding techniques to a decoder using the decision regions (6.2.4) and (6.2.5), the main steps of the derivation of (6.2.10) for high rates will be outlined.

Since, for $y \in \mathcal{R}_m^c$ and $s > 0$,

$$1 = \exp\left(snT\right)\exp\left(-snT\right) \leq \exp\left(snT\right)\left(\sum_{m' \neq m} \frac{p\left(y \mid x_{m'}\right)}{p\left(y \mid x_m\right)}\right)^s, \tag{6.2.18}$$

it follows that

$$\begin{aligned}
\Pr\left(\mathcal{E}_1\right) &= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{y \in \mathcal{R}_m^c} p\left(y \mid x_m\right) && (6.2.19)\\
&\leq \frac{1}{M}\sum_{m=0}^{M-1}\sum_{y \in \mathcal{Y}^n} p\left(y \mid x_m\right)\exp\left\{snT\right\}\left(\sum_{m' \neq m} \frac{p\left(y \mid x_{m'}\right)}{p\left(y \mid x_m\right)}\right)^s && (6.2.20)\\
&= \frac{\exp\left\{snT\right\}}{M}\sum_{m=0}^{M-1}\sum_{y \in \mathcal{Y}^n} p^{1-s}\left(y \mid x_m\right)\left(\sum_{m' \neq m} p\left(y \mid x_{m'}\right)\right)^s. && (6.2.21)
\end{aligned}$$

Taking the expectation with respect to the ensemble of codes, and using the fact $\mathbf{p}\left(x\right)$ is in a product form, we get

$$\overline{\Pr}\left(\mathcal{E}_1\right) \leq \frac{\exp\left\{snT\right\}}{M}\sum_{m=0}^{M-1}\sum_{y \in \mathcal{Y}^n} \mathbb{E}\left[p^{1-s}\left(y \mid x_m\right)\right] \cdot \mathbb{E}\left[\left(\sum_{m' \neq m} p\left(y \mid x_{m'}\right)\right)^s\right]. \tag{6.2.22}$$

The first factor of the summand, $\mathbb{E}\left[p^{1-s}\left(y \mid x_m\right)\right]$, appears in (6.2.13), using the explicit expression for the expectation. For the second factor, Forney used the fact that for any non-negative sequence $\{a_i\,, a_i > 0\}$, and for any $0 < \lambda \le 1$, $\sum_i a_i \le \left(\sum_i a_i^{\lambda}\right)^{\frac{1}{\lambda}}$ (see [143, Appendix 3A]), and so

$$\mathbb{E}\left[\left(\sum_{m' \neq m} p\left(y \mid x_{m'}\right)\right)^{s}\right] \le \mathbb{E}\left[\left(\sum_{m' \neq m} p\left(y \mid x_{m'}\right)^{s/\rho}\right)^{\rho}\right] \tag{6.2.23}$$

for $\rho \ge s$. Then, using Jensen's inequality to insert the expectation into the brackets, which is allowed only by limiting ρ to be within the interval $[0, 1]$, yields the bound (6.2.13). Other, slightly deferent, derivation of the bound (6.2.10) can be found in [73]. In [73], the exponential behavior of $\Pr\left(\mathcal{E}_1\right)$ and $\Pr\left(\mathcal{E}_2\right)$ is bounded using *tilting measures*, a bounding technique introduced in [39], using an auxiliary tilting measure and only then invoking Jensen's inequality. Forney's bounds are then obtained as a special case of a more general bound, stated in [73]. Soon after the publication of Forney's paper [53], Viterbi showed that for the very noisy channel (VNC) defined in [143, Chapter 3.4] and for AWGNC these bounds are tight, and can be attained using orthogonal signalling [142]. Another property of the bounds on $\overline{\Pr}\left(\mathcal{E}_1\right)$ and $\overline{\Pr}\left(\mathcal{E}_2\right)$ in (6.2.10) and (6.2.11), which was proven in [53], is that they take a very simple form in the case where the function $\gamma_y\left(s\right) \triangleq -\log\left(\sum_{x\in\mathcal{X}} p\left(x\right) p^{s}\left(y \mid x\right)\right)$ is independent of y for any real s.[2] First, note that in this case, the term $\mathbb{E}\left[p^{1-s}\left(y \mid x_m\right)\right]$ is particularly easy to handle, and $\mathbb{E}\left[p^{1-s}\left(y \mid x_m\right)\right] = \exp\left\{-n\gamma_y\left(s\right)\left(1-s\right)\right\}$. Secondly, following [143], define the sphere packing exponent, $E_{\text{sp}}\left(R\right)$, to be

$$E_{\text{sp}}\left(R\right) = \max_{\rho \ge 0}\left\{E_0\left(\rho, \mathbf{p}\right) - \rho R\right\} \tag{6.2.24}$$

where $E_0\left(\rho, \mathbf{p}\right) \triangleq -\log\left[\sum_{y\in\mathcal{Y}}\left(\sum_{x\in\mathcal{X}} p\left(x\right) p^{1/1+\rho}\left(y \mid x\right)\right)^{1+\rho}\right]$ and $\mathbf{p}$ is the uniform distribution over the input set[3]. In addition, let R^{conj} be the conjugate rate of R, defined to be the rate for which the slope of the sphere-packing error exponent $E_{\text{sp}}\left(\cdot\right)$ is the reciprocal of the slope at rate R, i.e.,

$$E'_{\text{sp}}\left(R^{\text{conj}}\right) = \frac{1}{E'_{\text{sp}}\left(R\right)}. \tag{6.2.25}$$

[2] An important class of channel for which this property holds is the symmetric channels, such as the binary symmetric channel. For a discussion about channels that satisfy this condition, the reader is referred to [53] and [103].

[3] Note that a more general definition of the sphere packing exponent includes also an optimization over the input distribution $\mathbf{p}$. Since from now on we will be concerned only with totally symmetric channels, for which uniform distribution is always optimal, the discussion was restricted to this distribution and the extra optimization was omitted.

Using these definitions, we can state the following result:

Theorem 32 (Theorem 3(a) in [53]) *Assume that $\gamma_y(s)$ does not depend on y for any real s. Then there is a threshold $\hat{T} \geq 0$ such that for any rate $\hat{R}$ satisfying $R^{conj} \leq \hat{R} \leq C$ the following error exponents are achievable:*

$$E_1\left(R, \hat{T}\right) = E_{sp}\left(\hat{R}\right), \tag{6.2.26}$$

$$E_2\left(R, \hat{T}\right) = E_{sp}(R) - R + \hat{R}. \tag{6.2.27}$$

In a more recent work, some new results regarding the expressions (6.2.1) and (6.2.2) were obtained. In [103], Merhav uses a different route in order to evaluate the second factor of the summand (6.2.22), i.e., $\mathbb{E}\left[\left(\sum_{m' \neq m} p(y \mid x_{m'})\right)^s\right]$. The idea is that instead of using the inequality $\sum_i a_i \leq \left(\sum_i a_i^\lambda\right)^{\frac{1}{\lambda}}$ and Jensen's inequality, which are not, in general, exponentially tight, the relevant moments of certain distance enumerators were assessed. In order to understand the main novel steps in the derivation of the new bound, in addition to gaining some geometric insight, it will be enough to limit the discussion to the special case of the binary symmetric channel (BSC) and the uniform random coding distribution. The extension to more general DMCs and random coding distributions, as well as the general and full derivations and results can be found in [103]. Consider the case where $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, assume the crossover probability is $0 < \epsilon < \frac{1}{2}$ and the random coding distribution is $p(x) = 2^{-n}$ for all $x \in \mathcal{X}^n$. For any given $y \in \mathcal{Y}^n$, let $N_y(d)$ denote the distance enumerator relative to y, that is, $N_y(d)$ is the number of incorrect codewords $\{x_{m'}, m' \neq m\}$ at Hamming distance d from y. Defining $\alpha \triangleq \log\left(\frac{1-\epsilon}{\epsilon}\right)$, it follows that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{m' \neq m} p(y \mid x_{m'})\right)^s\right] &= \mathbb{E}\left[\left((1-\epsilon)^n \sum_{d=0}^{n} N_y(d) e^{-\alpha d}\right)^s\right] && (6.2.28)\\
&\doteq \mathbb{E}\left[\left((1-\epsilon)^n \max_d N_y(d) e^{-\alpha d}\right)^s\right] && (6.2.29)\\
&\doteq (1-\epsilon)^{ns} \sum_{d=0}^{n} \mathbb{E}\left[N_y^s(d)\right] e^{-\alpha s d} && (6.2.30)
\end{aligned}$$

where the notation $a_n \doteq b_n$ means that $\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{a_n}{b_n}\right) = 0$, and thus implies that a_n and b_n are equal to the first order in the exponent. The important point in the above exponential equalities, is that they hold even before taking the expectations, because the summation over d consists of a subexponential number of terms, as opposed to the exponential number of terms in the original summation over the codewords. The second main point in the proposed approach is the observation that $\mathbb{E}\left[N_y^s\left(d\right)\right]$ behaves differently under two distinct ranges of d, or, equivalently, of $\delta \triangleq \frac{d}{n}$. More precisely, for $0 \leq \delta \leq 1$,

$$\mathbb{E}\left[N_y^s\left(\delta n\right)\right] \doteq \begin{cases} e^{ns[R+h_2(\delta)-\log(2)]}, & \delta \in \mathcal{G}_R \\ e^{n[R+h_2(\delta)-\log(2)]}, & \delta \in \mathcal{G}_R^c \end{cases} \tag{6.2.31}$$

where $\mathcal{G}_R \triangleq \{\delta : \delta \in [\delta_{\mathrm{GV}}(R), 1-\delta_{\mathrm{GV}}(R)]\}$, and $\delta_{GV}(R)$ is the Gilbert-Varshamov (G-V) distance, i.e., solution, δ, to the equation $h_2(\delta) = \log 2 - R$ for which $\delta \in \left[0, \frac{1}{2}\right]$. Plugging in (6.2.31) into (6.2.30), and using some simple algebraic manipulations (one of which is using once again the fact that a sum of subexponential number of exponential terms is exponentially equivalent the maximum over all these terms), we obtain the following theorem:

Theorem 33 *For the BSC with crossover probability $0 < \epsilon < \frac{1}{2}$ and for the uniform random coding distribution, it holds that $e_i(R,T) \geq E_i^\star(R,T)$ where*

$$E_1^\star(R,T) \triangleq \sup_{s\geq 0}\left\{\mu(s,R) - s\log(1-\epsilon) - \log\left[\epsilon^{1-s} + (1-\epsilon)^{1-s}\right] - sT\right\} \tag{6.2.32}$$

where

$$\mu(s,R) = \begin{cases} \mu_0(s,R) & s \geq s_R \\ \alpha s \delta_{GV}(R) & s < s_R \end{cases}, \tag{6.2.33}$$

s_R being the solution of the equation $\gamma(s) - s\gamma'(s) = R$, and

$$\mu_0(s,R) = s\log(1-\epsilon) - \log\left[\epsilon^s + (1-\epsilon)^s\right] + \log 2 - R. \tag{6.2.34}$$

Furthermore $E_2^\star(R,T) = E_1^\star(R,T) + T$.

The optimal value of s in (6.2.32) has an explicit expression, given in [103], but this expression was omitted since it does not contribute much to intuition. In (6.2.32) one can also find the proof of Theorem 33, as well as the analogous theorem that applies for a general DMC, under the condition that $\gamma_y(s)$ is independent of y for any real s.

There are a few points that are worth mentioning regarding the new lower bounds on the error exponent functions:

- Using the aforementioned bounding techniques, the bounds obtained are at least as tight as Forney's bounds. The reason for this is that from the point (6.2.22) and throughout the derivation, only exponentially tight evaluations of the relevant expressions were done, as opposed to Forney's derivation.

- Under certain symmetry conditions associated with the channel (i.e., the condition on the function $\gamma_y(s)$) and the random coding distribution, the bounds are simpler than Forney's bounds, in the sense that they involve an optimization over a single parameter. This simplification is due to the fact that the inequality $\sum_i a_i \leq \left(\sum_i a_i^\lambda\right)^{\frac{1}{\lambda}}$, that introduced the second optimization parameter, ρ, was not used. In addition, in certain special cases, like the BSC, the optimum value of this parameter can be found in closed form.

- A numerical example for which the new bounds are strictly better than Forney's bound was not found yet. Nevertheless, when applying the same analysis technique to a certain universal decoder with erasures, it was demonstrated by numerical examples in [103, Part V], that significantly tighter exponential error bounds can be obtained, compared to the technique used in [53]. Note, however, that no optimality claims regarding the decision rule simulated are made. In addition, in a more recent work [105], Merhav studied lower bounds of the achievable random coding error exponent pertaining to random binning associated with Slepian-Wolf decoding[4]. Specifically, the random binning error exponent was analyzed using both the bounding techniques used by Gallager [56] and Forney [53], and the type class enumeration method described above. Interestingly, for this case, Merhav proved that the bounds produced using the type class enumeration method are strictly tighter than those produced using Gallager and Forney's technique for positive values of the threshold T. Moreover, an arbitrarily large improvement is achieved using negative values of T[5]. Using negative values of T was not tried in [103], and since the analysis in both papers is very similar, it will not be surprising if negative values of T will yield an improvement in our case as well.

In a later work [134], Somekh-Baruch and Merhav continued using distance enumerators in order to analyze random coding exponents of an optimum decoder with an erasure option. However, unlike the approach described above, in this work, the starting point was not a Gallager-type bound on the probability of error, based on the expectation of the sum of certain likelihood ratios. The novelty in this work is in

[4]More information regarding the Slepian-Wolf problem can be found in [132]. For the definition of the random binning method and its relation to the Slepian-Wolf problem see [34], [32] and [57].

[5]It was shown in [53] that negative values of T correspond to the problem of "List Decoding", in which the decoder outputs a list of messages instead of a single estimator. The parameter T determines the trade-off between the size of the list and the error exponents.

the fact that the exact expression that defines the probability of an erasure and undetected error, (6.2.1) and (6.2.2), where used in a manner that assures exponentially tight results. In other words, in [134], the authors derive *exact* single-letter formulas for the error exponents, in lieu of the lower bounds that were discussed so far. We will make do with stating $e_1(R,T)$ and $e_2(R,T)$ for the BSC case, where the expressions take on a simple and compact form.

Theorem 34 *For the BSC with crossover probability $0<\epsilon<\frac{1}{2}$ and for the uniform random coding distribution, if $R \geq \log(2) - h_2\left(\epsilon + \frac{T}{\alpha}\right)$, $e_1(R,T) = 0$ and otherwise*

$$e_1(R,T) = \min_{\nu \in \left[\epsilon, \delta_{GV}(R) - \frac{T}{\alpha}\right]} \left\{ D(\nu \parallel \epsilon) - h_2\left(\nu + \frac{T}{\alpha}\right) + \log 2 - R \right\} \tag{6.2.35}$$

and $e_2(R,T) = e_1(R,T) + T$.

As mentioned before, more general results and full derivations can be found in [134]. As in the previous result, for the BSC, the optimum value of ν can be found in closed form. The reason both Theorem 33 and 34 are given here, although the bounds in Theorem 34 are tighter, is that there is still no analytical assurance that they are strictly tighter. In fact, in both [103] and [134], the authors mention that no numerical example was found, for which either one of the bounds is strictly better than Forney's bound. This may provide an additional evidence to support Forney's conjecture that his bound is tight for the average code. In addition, for the case of the BSC with the uniform random coding distribution, several numerical calculations have been conducted, which indicate that Forney's bound coincides with the exact random coding exponent of Theorem 34.

Recently, there has been a revived interest in the errors-and-erasures decoding. Other works on the subject that will not be covered in this work, including topics like universally achievable performance [33], [106], [107], extensions to channels with side information [123], and implementation with linear block codes [73]. Note that the encoders used in these works, as well as the encoder that was reviewed in this section, do not have feedback. However, a key point is that if the transmitter can learn whether the decoded message was an erasure, it can resend the message whenever it is erased. In the next section, we will analyze the performance of such a scheme, using Forney's results stated in Theorem 31.

B. Erasure-Decoders and ARQ Feedback Schemes

In the introduction to this section, a feedback communication link was described. In that setup, only one bit per message was allowed to be sent from the receiver to the transmitter, through an instantaneous and error-free binary feedback channel. It is easy to see how decision rules with an erasure option for fixed-length block codes can be used in the case of feedback. In particular, in [53], Forney proposed the following scheme: the transmitter sends a codeword $x_m \in \mathcal{X}^n$ of length n, chosen from a codebook of rate $\tilde{R}$ where $m \in \{0, \ldots, M-1\}$. As before, $M = e^{n\tilde{R}}$ represents the total number of messages, each of which is assumed equally likely. The transmitted codeword reaches the receiver after passing through the forward channel, defined in (5.2.1). After receiving a block of n symbols, the receiver uses an erasure-decoder, which decides that the transmitted codeword was $x_m, m \in \{0, \ldots, M-1\}$, if and only if the received sequence $y \in \mathcal{Y}^n$ falls in $\mathcal{R}_m^\star$, defined in (6.2.4), where $T \geq 0$ is a controllable threshold. If, on the other hand, $y \notin \mathcal{R}_m^\star$ for all $m \in \{0, \ldots, M-1\}$, then the receiver declares an erasure and sends a NACK bit back to the transmitter. Upon receiving NACK, the transmitter repeats the same message. For obvious reasons, this type of protocol is also referred to as an automatic repeat query (ARQ) protocol, and each decision to repeat the message or decode (that is made every n symbols) is called an ARQ round. Note that in this scheme, the decoder discards the earlier received sequence and uses only the latest received n symbols for decoding (also known as memoryless decoding). In other words, after each erasure event, the decoder gathers the next n-symbol sequence y. If $y \in \mathcal{R}_M^\star$ an erasure is declared (and the decoder asks for a retransmission). When the decoder does not declare an erasure, the receiver transmits an ACK to the transmitter, and the transmitter sends the next message. Note that:

- At least theoretically, this scheme allows for an infinite number of ARQ rounds. Nevertheless, it will be argued that for the specific decoding and erasure algorithms of interest, the expected number of ARQ rounds is bounded.

- This scheme can also be implemented using only one bit for feedback per codeword by asking the receiver to only send back ACK bits, and asking the transmitter to keep repeating the current codeword repetitively until it receives an ACK. Using this modification, we see that this scheme fits the constraints on the use of the feedback channel discussed earlier.

Let N denote the total number of channel uses per message. The expected value of this stopping time (which is equivalently defined in (5.2.6)) is, of course, of interest and can be easily expressed using the erasure probability $\Pr\left(\mathcal{R}_M^\star\right)$ in the following way:

$$\begin{aligned}\mathbb{E}\left[N\right] &= n\sum_{k=1}^{\infty} k\Pr\left(\text{Transmission stops after } k \text{ rounds}\right) &(6.2.36)\\ &= n\sum_{k=1}^{\infty} k\left[\Pr\left(\mathcal{R}_M^\star\right)\right]^{k-1}\left[1-\Pr\left(\mathcal{R}_M^\star\right)\right] &(6.2.37)\\ &= \frac{n}{1-\Pr\left(\mathcal{R}_M^\star\right)}, &(6.2.38)\end{aligned}$$

which implies that the rate, defined in (5.2.9), is equal to

$$R = \frac{\log M}{\mathbb{E}\left[N\right]} = \frac{\log M}{n}\left[1-\Pr\left(\mathcal{R}_M^\star\right)\right] = \tilde{R}\left[1-\Pr\left(\mathcal{R}_M^\star\right)\right] \qquad (6.2.39)$$

where $\tilde{R} = \frac{\log M}{n}$. It is clear, using Theorem 31 with the results of (6.2.38) and (6.2.39), that as long as $E_1\left(\tilde{R},T\right) > 0$, $R \to \tilde{R}$ as $n \to \infty$.

The overall average probability of error is given by:

$$P_e = \sum_{k=1}^{\infty}\left[\Pr\left(\mathcal{R}_M^\star\right)\right]^{k-1}\Pr\left(\mathcal{E}_2\right) = \Pr\left(\mathcal{E}_2\right)\left(1+o\left(1\right)\right) \qquad (6.2.40)$$

where the second equality follows from Theorem 31 when $E_1\left(\tilde{R},T\right) > 0$. It is, therefore, clear that the error exponent function, defined in (5.2.10), achieved by Forney's memoryless decoding scheme is given by

$$E\left(R\right) = \limsup_{n\to\infty}\left[\frac{-\log P_e}{\mathbb{E}\left[N\right]}\right] \geq E_2\left(R,T\right). \qquad (6.2.41)$$

It is shown in [53] that choosing the threshold T such that $E_1\left(\tilde{R},T\right) \to 0$ maximizes, the exponent $E_2\left(\tilde{R},T\right)$ while ensuring that $R \to \tilde{R}$ as $N \to \infty$. This establishes the fact that Forney's memoryless decoding scheme achieves the feedback error exponent $E_f\left(R\right)$ defined as

$$\begin{aligned}E_f\left(R\right) &\triangleq \lim_{E_1\left(\tilde{R},T\right)\to 0} E_2\left(\tilde{R},T\right) &(6.2.42)\\ &= \max_{0\leq s\leq\rho\leq 1,\mathbf{p}}\left\{\frac{E_0\left(s,\rho,\mathbf{p}\right)-\rho R}{s}\right\} &(6.2.43)\\ &= \max_{\nu\geq 1,\mathbf{p}}\left\{E_{0f}\left(\nu,\mathbf{p}\right)-\nu R\right\} &(6.2.44)\end{aligned}$$

where

$$E_{0f}(\nu, \mathbf{p}) \triangleq \frac{d}{ds}\left[E_0(s, \nu s, \mathbf{p})\right]_{s=0} \tag{6.2.45}$$

and $E_0(s, \rho, \mathbf{p})$ is defined in (6.2.13). An important feature of the error exponent function $E_f(R)$ is that $\lim_{R\to C} E_f'(R) = -1$, in contrast to the error exponent without feedback, whose slope is typically zero at capacity[6]. This implies that near capacity, the achievement of low error probabilities is dramatically simplified by the use of feedback. In addition, it is shown in [53] that for the symmetric channel of Theorem 32,[7]

$$E_f(R) = E_{\mathrm{sp}}(R) - R + C \triangleq E_{\mathrm{Forney}}(R), \quad R_\infty \le R \le C \tag{6.2.46}$$

where R_∞ in the smallest rate at which the sphere packing bound is infinite. In addition, for any channel for the capacity-achieving input distribution,

$$E_f(R) \ge E_{\mathrm{sp}}(R) - R + C, \quad R_\infty \le R \le C. \tag{6.2.47}$$

Note that $E_f(R)$ is larger than $E_{\mathrm{sp}}(R)$, which is the exact error exponent in the absence of feedback at high rates.At least for the BSC with crossover probability $0 < \epsilon < \frac{1}{2}$, and the uniform random coding distribution, the same result regarding the error exponent of the ARQ scheme can be easily derived by analyzing $e_1(R, T)$ and $e_2(R, T)$. This derivation can be found in Appendix B.

All the results obtained hitherto were derived random coding and an optimal decoder in the sense of the erasure-error trade-off. Nevertheless, in [33], Csiszár and Körner analyzed a different coding and decoding algorithm that achieves the same error exponent as in (6.2.46), using constant composition codes, which are block codes that bear the property that all the codewords are chosen from the same type[8]. The decoding algorithm of Csiszár and Körner is a generalization of the maximum mutual information (MMI) decoder, that chooses the codeword having the highest mutual information with respect to the received output sequence. Specifically, define the

[6]For pathological examples where this does not hold, see [143, Problem 3.2].

[7]This property holds also for the very noisy channel (VNC) defined in [143, Chapter 3.4] and for AWGC [142].

[8]The type of a sequence $x \in \mathcal{X}^n$ is the distribution P_x on $\mathcal{X}$ defined by $P_x(a) = \frac{1}{n} N(a \mid x)$ for every $a \in \mathcal{X}$, where $N(a \mid x)$ denoted the number of occurrences of a in x [31]. In fact, Csiszár and Körner's bound is tighter than Foney's one for compositions that are not optimal, but this difference disappears after optimizing over the input distribution.

following decision regions:

$$\mathcal{R}_{m,\mathrm{CK}} = \left\{ y : I\left(x_m; y\right) > \hat{R} + \lambda \left[I\left(x_k; y\right) - R \right]^{+} \text{ for } k \neq m \right\}, m \in \{0, \ldots, M-1\}, \tag{6.2.48}$$

$$\mathcal{R}_{M,\mathrm{CK}} = \bigcap_{m=0}^{M-1} \left(\mathcal{R}^{\star}_{m,\mathrm{CK}} \right)^{c} \tag{6.2.49}$$

where λ, $\hat{R}$ and R are parameters satisfying $\lambda > 0$ and $\hat{R} \geq R > 0$. The decoder then decodes m if $y \in \mathcal{R}_{m,\mathrm{CK}}$ and x_m is jointly typical with y. Otherwise, it decides on an erasure. By studying the performance of this error-and-erasure decoding algorithm, achievable error and erasure exponents were proved (without indicating whether these exponents match Forney's exponent). More importantly, by embedding this decoding rule as the core of an ARQ algorithm, in the same manner as was done using Forney's error-and-erasure decoding algorithm earlier, Csiszár and Körner found that the error exponent achieved is again lower bounded by $E_{\mathrm{Forney}}\left(R\right)$. Note that although this generalized MMI decoder does not yield an improvement upon Forney's results regarding ARQ schemes, it bears the remarkable feature that it does not depend on the forward channel. In other words, an ARQ defined by the decision regions (6.2.48) and (6.2.49) derives an error exponent that is upper bounded by $E_{\mathrm{Forney}}\left(R\right)$ *universally* over all DMCs. Constant composition codes were later used also in [138] and [139], where new bounds on $E_f\left(R\right)$ were obtained which are tighter when applied to some classes of channel models, but for symmetric channels, the two bounds coincide.

A few technical notes are in order:

- **Optimality**: Note that no optimality claim was made regarding the coding and decoding algorithms in [33]. One of the reasons that must have motivated Csiszár and Körner's choice of the aforementioned decoder (which was recently generalized in a work by Moulin [107]), is the fact that it is amenable to analysis using *the method of types* (see [31] for a survey on the subject). For example, the packing lemma [33, Lemma 10.1], some basic properties of typical sets and consideration of joint typicality between the codewords and the received sequence at each ARQ round were all vastly used in the derivation of the bounds. By contrast, Forney's test statistics, i.e.,

$$\frac{p\left(y \mid x_m\right)}{\sum_{m' \neq m} p\left(y \mid x_m\right)}, \tag{6.2.50}$$

is much harder to handle using the same considerations. For a further discussion on this subject, the reader is referred to [106] and [77], where the competitive minimax approach [50] was used to bypass this difficulty, and where a lower bound the error and erasure exponents was obtained.

- **Complexity**: Note that Forney's scheme requires evaluation of all the likelihood values. Of course, Csiszár and Körner's scheme does not require less computational power. This practical inconvenience necessitates suboptimal decision schemes which provide reasonable performance with reasonable decoding complexities. Since computational issues and structured codes are not in the scope of this work, we will only mention some of the earlier works in which such suboptimal schemes were analyzed. Examples are schemes based on explicit error-detection coding, such as convolutional codes or low-density-parity-check (LDPC) codes. This type of schemes is also known as hybrid ARQ schemes. Some hybrid ARQ schemes are introduced in [69], [83], [68] and [73] (for the AWGNC model and decoders that are based on bounded distance maximum-likelihood or bounded angle maximum-likelihood see also [41] and [3]). Another technique used in order to reduce computational effort is to consider a modified version of the the decoding rule in (6.2.48) and (6.2.49), where the term

$$\log\left[\frac{p\left(y\mid x_m\right)}{\sum_{m'\neq m}p\left(y\mid x_{m'}\right)}\right], \tag{6.2.51}$$

is replaced by an approximated expression of the form

$$\log\left[\frac{p\left(y\mid x_m\right)}{\max_{m'\neq m}p\left(y\mid x_{m'}\right)}\right] \tag{6.2.52}$$

or

$$\log\left\{\frac{p\left(y\mid x_m\right)}{\left[M\sum_{x'\in\mathcal{X}^n}p\left(x'\right)p\left(y\mid x'\right)^{1/(1+\rho)}\right]^{1+\rho}}\right\}, \tag{6.2.53}$$

where in the first expression, instead of $\sum_{m'\neq m}p\left(y\mid x_{m'}\right)$, only the maximal term was taken (i.e., this approximation is good if the sum is dominated by the maximum summand), and in the second, the expression $\sum_{m'\neq m}p\left(y\mid x_m\right)$ was bounded using Gallager's technique [56], when random coding is assumed. These types of decoding rules are further addressed, for example, in [66], [67], [69] and [157].

C. Erasure-Decoding and Burnashev's Reliability Function

In Chapter 5, we have discussed the maximal attainable error exponent in communication systems with feedback. The feedback channel described in Chapter 5 was memoryless, instantaneous, and error-free. The coding scheme to be defined next, following [45], is another example of using an erasure-decoding block. The motivation behind the work of Draper *et al.* [45] is to shed light on the amount of feedback really needed in order to achieve Burnashev's error exponent.

Akin to Yamamoto and Itoh's scheme [156], we define two distinct feed-forward phases of communication, each accompanied by a different feedback strategy:

Phase I: A message $m \in \{0, \ldots, M-1\}$ is sent over the forward channel in γn channel uses (where $0 < \gamma < 1$). The decoder decodes the message using an erasure-decoder as in Figure 6.2.1. If $y \in \mathcal{R}_{\hat{m}}$, a tentative decision in favor of $\hat{m}$ is made, or else, an erasure is declared. Next, the M messages are partitioned into $M_{\text{fb}} \triangleq \exp\{nR_{\text{fb}}\}$ bins $\mathcal{B}_0 \ldots, \mathcal{B}_{M_{\text{fb}}}$. The receiver feeds back the bin index k such that $\hat{m} \in \mathcal{B}_k$. In case of erasure, the receiver feeds back an erasure symbol and the transmitter retransmits the same message. The improvement upon Forney's performance is achieved by passing the detection of decoding errors from the decoder to the encoder, as in Yamamoto and Itoh's scheme.

Phase II: The transmitter uses the forward channel $(1-\gamma)\,n$ times to send ACK in case $m \in \mathcal{B}_k$, and NACK otherwise. The ACK and NACK messages are sent using the same repetition code described in Section A.. The receiver decides whether ACK or NACK was sent, and feeds back a single bit accordingly. If ACK is detected, the receiver accepts its tentative guess $\hat{m}$ and both paties continue to a new message. If NACK is detected, a repeated attempt to communicate the current message is made.

The decoding process is illustrated through the following example: consider the error event if a codeword from the bin $\mathcal{B}_1 \triangleq \{\mathcal{R}_0, \mathcal{R}_2\}$ of Figure 6.2.3 is sent, say $m = 1$. Unless y falls in one of the other decoding regions of this bin, it has landed either in the decoding region of a codeword that belongs to a different bin, or in the erasure region. In either case NACK is sent. Undetectable errors only occur if y lies in the decoding region of another codeword in the same bin. Such an error event is depicted in Figure 6.2.4. As can be seen by comparison with Figure 6.2.1, the probability of this event is just the probability of undetected error in an erasure-decoding problem where only messages from the bin $\mathcal{B}_1$ are taken into account.

A key question yet to be answered is how should one choose the partition of the

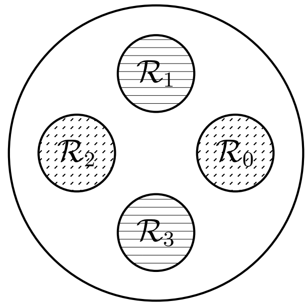

Figure 6.2.3: Typical decision regions of the in erasure-plus-binning-decoding schemes.

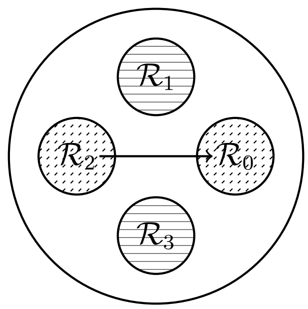

Figure 6.2.4: Decoding error event in which the observation sequence falls into a different codeword region, but in the same bin.

messages into bins. In order to understand the decision rule used in [45], we first revisit Telatar's work [138], [139]. As was already mentioned in Section 6.2.B., in [138], Telatar has defined an erasure-decoding scheme. His scheme, when used as the core of an ARQ scheme, achieves a tighter error exponent than the one in [53] and [33], whereas for symmetric channels, the error exponent of Telatar coincides with Forney's one. Let $\varphi_{\text{Telatar}}(y, R, T)$ be the criterion by which the observation space was partitioned in Telatar's work[9], that is, for any $y \in \mathcal{Y}^n$ and any given rate $R = \frac{\log M}{n}$, the decision regions of Telatar's decoding scheme are given by:

$$\mathcal{R}_m^{\text{Tel}} = \left\{ y \in \mathcal{Y}^n : \varphi_{\text{Telatar}}(y, R, T) = m \right\}, \tag{6.2.54}$$

$$\mathcal{R}_M^{\text{Tel}} = \bigcap_{m=0}^{M-1} \left(\mathcal{R}_m^{\text{Tel}} \right)^c \tag{6.2.55}$$

[9]The exact definition of $\varphi_{\text{Telatar}}(y, R, T)$ was left out here since it does not contribute to the understanding of the decoding scheme. This definition can be found in [138, Section 4.4].

where, as in Forney's decision rule, $T \geq 0$ is a tunable parameter, used to adjust the different decision regions and hence to trade off the undetected error probability with the erasure probability. Returning to the discussion on two-phase scheme described above, Draper *et al.*[45] chose the following criterion for partitioning the observation space:

$$\varphi_{\text{Draper}}\left(y,R,T\right) = \begin{cases} i & \text{if} \quad \varphi_{\text{Telatar}}\left(y,R,T\right) = i, \quad \forall j \neq i, j \in \{0,\dots M-1\} \\ & \text{and} \quad \varphi_{\text{Telatar}}\left(y,R-R_{\text{fb}},T\right) = i, \quad \forall j \neq i, j \in \mathcal{B}\left(i\right) \\ \text{erase} & \qquad\qquad \text{otherwise} \end{cases} . \tag{6.2.56}$$

By closely examining this rule, we see that the first condition for $\varphi_{\text{Draper}}\left(y,R,T\right) = i$ is exactly the same as the condition for $\varphi_{\text{Telatar}}\left(y,R,T\right) = i$ to hold. This is an erasure-like decision rule for which the observation space is divided into an erasure region and $M-1$ decoding regions (as depicted in Figure 6.2.1) and is known to be good in the sense of achieving Forney's error exponents for any positive T. In addition, utilizing the second condition, the set of codewords are classified into bins in a way that assures that each bin constitutes a good lower-rate subcode, again, in the sense that the error event illustrated in Figure 6.2.4 has vanishing error probability as long as $T > 0$.

The expected transmission rate, $\bar{R}$, is determined by the number of length-n transmissions, which is, as in Forney's ARQ scheme, a geometrical random variable. Using erasure regions that assure that the erasure probability decays to zero exponentially fast (as in the case where the parameter T in (6.2.56) is strictly positive) and the Chernoff-Stein lemma [29], it can be shown that the probability of retransmission can be made arbitrarily small by taking n large enough. This is not surprising, keeping in mind both Forney's and Yamamoto-Itoh's scheme.

In addition, choosing the decision regions according to (6.2.56) makes both the undetected error probability and the probability of the error event $\{\hat{m} \neq m\} \cap \{m, \hat{m} \in \mathcal{B}_k\}$ exponentially small. The exact analysis in very similar to that of [33] and can be found in [45]. In this work, we illuminate the importance of this scheme in terms of performance and highlight some interesting conclusions.

- The main contribution of this scheme is that it reduces the required feedback rate in comparison to Yamamoto-Itho scheme, by feeding back only the index of the bin in which the estimated codeword lays. By doing so, it provides an achievable error exponent for any feedback rate.

- Both analytical results and simulations carried out in [45] for the BSC, indicate quite surprisingly that for some channels, Burnashev's error exponent can be achieved with $M_{\text{fb}} < M$ and so, for these channels, reduced feedback rates suffice.

- In some sense, this scheme offers a unified look at both Yamamoto and Itho's scheme and Telatar's scheme[10], since it transitions smoothly from one to the other, depending on the feedback rate available. This can easily be seen by taking the erasure probability to be zero in the first phase of the scheme, and therefore constructing a classical codebook (without the erasure option). This scheme performs as Yamamoto and Itoh's scheme, where error-free feedback is considered, achieves Burnashev's error exponent. On the other hand, taking $\gamma = 1$ and $R_{\text{fb}} = 0$ (but still keep one bit of feedback in order too indicate whether to retransmit or not), leads to Telatar's ARQ scheme, using an erasure-decoder as described in the previous section, and hence to $E_{\text{Forney}}(R)$ as a lower bound on the error exponent.

6.3 ARQ with Hard Deadline on the Decoding Time

So far, the total transmission length was considered only in the sense of its expected value, for determining the rate of communication defined in (5.2.10). Nevertheless, in many modern applications with feedback, retransmissions usually have costs beyond their effect on the average transmission time. Such are, for example, in which a communication systems with multiple layers. In these systems, retransmissions often affect the performance of higher layers and hence require care [113]. Systems that are sensitive to overflow in memory or to delay are also examples. In these examples, the cost paid for long transmissions is sometimes very high, and, mathematically, it is translated to harder constraints on the probability distribution of the decoding time.

The most natural way to implement a restriction on the transmission time of an ARQ scheme, such as the one described in the previous chapter, is to impose an upper bound, $L \in \mathbb{N}$, on the maximum number of ARQ rounds. In [62], two such decoding schemes were studied. The first naively carries out Forney's communication scheme, with a modification that takes into account the deadline restriction. The second scheme

[10] And, as mentioned before, this scheme achieves the same error exponent as does Forney's scheme.

differs from Forney's memoryless scheme in that it does not necessarily perform ARQ rounds that are equal in length, and, more importantly, it does not disregard the past after every round. In particular, the first coding algorithm is equivalent to Forney's scheme with one exception: due the restriction on the total number of rounds, the decoder can no longer use the decision regions defined in (6.2.4) and (6.2.4) during the L'th round. Therefore, after the L'th round, the decoder employs the ML decoding rule to decide. The error exponent achieved by this decoding scheme, denoted by $E_{\mathrm{MD}}(R, L)$,[11] is lower and upper bounded as follows [62, Theorem 2]:

$$E_{\mathrm{r}}(R) + (L-1)\left[\max_{0\le s\le\rho\le 1,\mathbf{p}}\left\{\frac{E_0(s,\rho,\mathbf{p}) - \rho R - sE_{\mathrm{r}}(R)}{1+s(L-2)}\right\}\right] \le E_{\mathrm{MD}}(R, L), \quad (6.3.1)$$

$$E_{\mathrm{MD}}(R, L) \le L E_{\mathrm{sp}}(R) \quad (6.3.2)$$

where $E_{\mathrm{r}}(R)$ and $E_{\mathrm{sp}}(R)$ denote the random coding and sphere-packing exponents [143], and $E_0(s,\rho,\mathbf{p})$ is given in (6.2.13).

Using this result, we have the following main conclusions:

- In general, this decoding scheme does not achieve Forney's error exponent $E_f(R)$, when the maximum number of ARQ rounds is restricted to L, at least at high rates where $LE_{\mathrm{SP}}(R) < E_f(R)$.
- As expected, when $L \to \infty$ the lower bound on the error exponent becomes

$$\lim_{L\to\infty} E_{\mathrm{MD}}(R, L) \ge \max_{0\le s\le\rho\le 1,\mathbf{p}}\left\{\frac{E_0(s,\rho,\mathbf{p}) - \rho R}{s}\right\} = E_f(R). \quad (6.3.3)$$

The second scheme in [62] is a well known variant of Forney's ARQ, often referred to as the incremental redundancy automatic repeat request (IR-ARQ) decoding algorithm [20]. In IR-ARQ, the transmitter, upon receiving NACK, transmits n *new* coded symbols (from the same message). The case studied is the one for which these new symbols are obtained as i.i.d. realizations from the capacity-achieving distribution. In the IR-ARQ, decoder does not discard the received observations in the case of erasure, it uses the received sequences of all ARQ rounds jointly in order to decode.

[11] Two notes are in order: (a) A tighter lower bound may be obtained by using the expurgated exponent at low rates, but this result was omitted here since it does not add much to the understanding of the results and (b) The MD stands for memoryless decoding, and indicates that the decoder "forgets" the past after each round terminates.

The following erasure decoding rule is employed by the receiver: after the k'th ARQ round, the decoder rules in favor of the $x_m \in \mathcal{X}^{kn}$ if and only if $y \in \mathcal{R}'_m$ where

$$\mathcal{R}'_m = \left\{ y \in \mathcal{Y}^{kn} : \frac{p\left(y \mid x_m\right)}{\sum_{m' \neq m} p\left(y \mid x'_m\right)} \geq \exp\left\{knT_k\right\} \right\}, \quad m \in \{0, \ldots, M-1\}. \tag{6.3.4}$$

As before, y and $x_m, m \in \{0, \ldots, M-1\}$ are vectors of length kn, which contain the received sequence and transmitted codeword, respectively, corresponding to the k'th ARQ round. Note that (6.3.4) allows a different threshold for each round. Since the error probability is dominated by events occurring at the first rounds, it makes sense to take a non-increasing sequence of thresholds. In particular, in the proof of (6.3.5), the sequence $\left\{T_k = \frac{T}{k}, T > 0\right\}$ was used. If no codeword satisfies the above condition, then an erasure is declared by the decoder, unless $L-1$ erasures have already been declared. In that case, an ML rule is used (on the entire nL-length sequence) in order to decide.

Define $E_{\mathrm{IR}}\left(R, L\right)$ to be the error exponent achieved by IR-ARQ under a deadline constraint L. Then[12]:

$$E_{\mathrm{IR}}\left(R, L\right) \geq \min\left\{E_{\mathrm{f}}\left(R\right), LE_{\mathrm{r}}\left(R/L\right)\right\}, \quad 0 \leq R \leq C. \tag{6.3.5}$$

Altogether, we conclude that:

- When a deadline is given, disregarding the past in every ARQ round, deteriorates the error exponent performance. Moreover, for finite L, it is shown that the IR-ARQ outperforms such a decoding algorithm.

- From (6.3.5), it is clear that if the deadline constraint L is large enough to satisfy $E_{\mathrm{f}}\left(R\right) \leq LE_{\mathrm{r}}\left(R/L\right)$ then IR-ARQ achieves the feedback exponent $E_{\mathrm{f}}\left(R\right)$.

- In [62], for the BSC and VNC models, sufficient conditions on the size of L were given for the IR-ARQ scheme to achieve $E_{\mathrm{f}}\left(R\right)$. In addition, some simulation results that demonstrate the superiority of the IR-ARQ scheme over ARQ in the case of finite L are shown.

[12] Note that, as in the previous result, replacing $E_{\mathrm{r}}\left(R\right)$ by the expurgated exponent may yield a tighter lower bound at low rates.

6.4 Stop-Feedback and Sequential Multiple Hypothesis Testing

In this section, a stronger relation will be established, between the problem of sequentially testing multiple hypotheses and communication with limited feedback. So far, in order to cope with a feedback limitation of one bit per message, some ARQ decoding schemes were introduced. In these schemes, the idea was to use a randomly chosen fixed block-length codebook and an erasure decoder, working at some rate $\tilde{R} > 0$ in the following manner. Let n denote the block length of the code and let $P_X = \{P_X(x), \forall x \in \mathcal{X}\}$ denote the capacity-achieving distribution according to which the code is drawn. In order to send a message, the transmitter transmits the same codeword repeatedly. After receiving each batch of n symbols, the decoder uses an erasure decoding algorithm to make a decision whether to:

(a) Wait n additional time units, in the case of an erasure, or,

(b) Decode the message and signal back to the transmitter to stop repeating the current message and move on to the consecutive one.

In the previous section it was argued that for rates below capacity, one can choose the decision regions of the erasure decoder in a way that assures that the probability of an undetected error and the probability of erasure will decay exponentially in the block-length n. Applying these results to the ARQ scheme described above yields that

$$\mathbb{E}[N] = n + o(1), \tag{6.4.1}$$

$$R = \tilde{R}, \tag{6.4.2}$$

$$E(R) \geq E_{\text{Forney}}(R), \tag{6.4.3}$$

where $\mathbb{E}[N]$, R and $E(R)$ are defined in Section 5.2.E. and $E_{\text{Forney}}(R)$ is defined in (6.2.46). Although the ARQ schemes that are constructed using erasure decoders were proven to be amenable for analysis using classical tools, as well as being able to achieve good performance, no claim regarding their optimality was made. In fact, ARQ schemes of this sort are members in a richer family of codes, sometimes referred to as *stop-feedback codes* [118], [153], where the communication over the feedback channel is restricted to the a single signal, used in order to stop the transmission once the decoder is ready to decode.

Before we proceed to define a different type of a stop-feedback scheme, another important term, which was used vastly throughout the work, is generalized - the *random coding* technique. Thus far in this work, as well as in classical information theory literature, a random codebook of rate $R > 0$ is a codebook that was constructed by drawing $M = \exp(nR)$ codewords, each of length n, at random, using a predetermined probability distribution in an i.i.d. fashion. The underlying assumption is that after the code is constructed, it is revealed to both the transmitter and receiver before the transmission starts, and is kept fixed throughout the decoding process. Using the same concept, an *infinite random code* will be defined in a very similar way, with one difference - instead of assigning each message with a fixed number of i.i.d.-drawn symbols, we assign an infinitely long such sequence. The question of how to construct such a codebook in practice, as well as how should the receiver and the transmitter share it, will not be dealt with here[13].

The infinite random coding method will be used in this section, where a more general decoding scheme for the stop-feedback setup will be considered. The main motivation behind the new scheme is that it will enable to relax the restriction that the stopping time is always an integer multiplication of n. A more general framework is, one of where the decoder can signal back a stopping symbol at *any* time step, i.e.,at each time instant, the observation sequence is re-evaluated. Note that the traditional random coding method was used in all memoryless ARQ schemes that were reviewed in the previous sections. These ARQ schemes can be slightly modified to fall into the infinite random coding framework. This is done by constructing an infinite random codebook, and dividing it into consecutive segments, each of length n. Then, at the k'th ARQ round, the k'th segment of the codebook is used. Since the decoding is memoryless, the performance of the modified scheme will be equal to those of the original schemes.[14]

[13]One elegant way around this problem is the assumption that a common source of randomness is available to both the encoder and the decoder, and hence the two sides can maintain such an infinite codebook. This assumption is sometimes made in the information theory literature on the subject of variable length coding, but it still does not solve the implementation problem in practical systems.

[14]As a matter of fact, Forney's result and results follow-up prove more than merely an achievable error exponent for stop-feedback coding; the power of these scheme is also in the fact that they prove that in order to achieve this error exponent, some observations are not needed and can be disregarded in the decoding process.

The main idea behind the results to be presented, is that the problem of infinite random coding and stop-feedback decoding can be formalized as the sequential multiple hypothesis testing problem, that was discussed in Section 4.3. To that end, let us denote the codebook by $\mathcal{C} \triangleq \left\{\mathbf{x}^{(0)}, \mathbf{x}^{(1)} \ldots, \mathbf{x}^{(M-1)}\right\}$, where $\mathbf{x}^{(i)} = \left\{x_1^{(i)}, x_2^{(i)} \ldots\right\}$ is the infinite codeword assigned to the message $i \in \{0, \ldots, M-1\}$. Furthermore, let $\mathbf{z} = z_1, z_2, \ldots$ be the following sequence:

$$z_k \in \mathcal{X}^M \times \mathcal{Y}, \quad z_k = \left(x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(M-1)}, y_k\right) \tag{6.4.4}$$

where $\mathbf{y} = (y_1, y_2, \ldots)$ is the observation sequence.

For any infinite sequence $\mathbf{v}$, let $[\mathbf{v}]_n$ be the first n samples of the infinite sequence $\mathbf{v}$. In this section we use the same DMC model as was defined in Section 5.2.A., i.e.,

$$P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n\right) = \prod_{k=1}^{n} p(y_k \mid x_k) \tag{6.4.5}$$

Next, define the following M simple hypotheses:

$$H_i \quad : \quad \Pr\left([\mathbf{z}]_n\right) = P_i\left([\mathbf{z}]_n\right), \quad i \in \{0, \ldots, M-1\}, \tag{6.4.6}$$

where we have defined the joint probability distributions

$$P_i\left([\mathbf{z}]_n\right) = P_i\left(\left[\mathbf{x}^{(0)}\right]_n, \left[\mathbf{x}^{(1)}\right]_n \ldots, [\mathbf{x}]^{(M-1)}, [\mathbf{y}]_n\right) \tag{6.4.7}$$

$$\triangleq P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n\right) \prod_{l=0}^{M-1} P_{\mathbf{X}}\left(\left[\mathbf{x}^{(l)}\right]_n\right). \tag{6.4.8}$$

Note that for each $i \in \{0, \ldots, M-1\}$, $P_i\left([\mathbf{z}]_n\right)$ is the distribution of the random process $\mathbf{z} = \left(\mathbf{x}^{(0)}, \mathbf{x}^{(1)} \ldots, \mathbf{x}^{(M-1)}, \mathbf{y}\right)$, where all the $\mathbf{x}$'s are independent, and $\mathbf{y}$ is generated by sending $\mathbf{x}^{(i)}$ through the DMC. In other words, if we assume a uniform prior on the hypotheses, the problem of sequentially testing the hypotheses (6.4.6) is equivalent to the problem of deciding which one of the M equiprobable sequences, $\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(M-1)}$, was sent through the DMC. Once a decision is made, the feedback can be used to indicate that a new message should be sent. Throughout this section, we denote the class of sequential tests that select one of the hypotheses H_i in (6.4.6), by $\Delta = (N, d)$, where, as in Part I, N denotes the stopping time, and d denotes the decision function.

An important feature is that under hypothesis H_i, the random vectors $z_1, z_2, \ldots$ are i.i.d. under P_i. This will be used to simplify the calculation.

For a given $n \in \mathbb{N}$ and $[\mathbf{z}]_n$, denote the log-likelihood ratio sequence between two hypotheses i and j $(i \neq j)$ by

$$\Lambda_{i,j}(n) \triangleq \log \left[\frac{P_i([\mathbf{z}]_n)}{P_i([\mathbf{z}]_n)} \right] = \sum_{k=1}^{n} \Delta\Lambda_{i,j}(k) \tag{6.4.9}$$

where

$$\Delta\Lambda_{i,j}(k) \triangleq \log \left[\frac{p\left(y_k \mid x_k^{(i)}\right)}{p\left(y_k \mid x_k^{(j)}\right)} \right]. \tag{6.4.10}$$

Note that the Kullback-Leibler (KL) divergence between hypotheses i and j is

$$D(i \parallel j) \triangleq \mathbb{E}_i[\Delta\Lambda_{i,j}(k)] \tag{6.4.11}$$

where $\mathbb{E}_i[\cdot]$ denotes the expected value, under H_i.

The setting described above bears a symmetry in the sense that if we change the role of two hypotheses, the problem stays the same (this is evident by the structure of the different hypotheses and the uniform distribution over the messages). Due to this symmetry, it is clear that $D(i \parallel j)$ are all equal, for any $i \neq j$, and hence

$$\min_{j \neq i} D(i \parallel j) = D(0 \parallel 1) \triangleq D \tag{6.4.12}$$

Also,

$$C \leq D \leq C_1 \tag{6.4.13}$$

where C is the capacity and C_1 is defined in (5.3.5). This can be easily proven by first noticing that

$$\begin{aligned}
D &= \sum_{x^{(0)} \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_X\left(x^{(0)}\right) p\left(y \mid x^{(0)}\right) \sum_{x^{(1)} \in \mathcal{X}} P_X\left(x^{(1)}\right) \log \left[\frac{p\left(y \mid x^{(0)}\right)}{p\left(y \mid x^{(1)}\right)} \right] && (6.4.14)\\
&\geq \sum_{x^{(0)} \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_X\left(x^{(0)}\right) p\left(y \mid x^{(0)}\right) \log \left[\frac{p\left(y \mid x^{(0)}\right)}{\sum_{x^{(1)} \in \mathcal{X}} P_X\left(x^{(1)}\right) p\left(y \mid x^{(1)}\right)} \right] && (6.4.15)\\
&= \sum_{x^{(0)} \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_X\left(x^{(0)}\right) p\left(y \mid x^{(0)}\right) \log \left[\frac{p\left(y \mid x^{(0)}\right)}{P_Y(y)} \right] && (6.4.16)\\
&= C && (6.4.17)
\end{aligned}$$

where (6.4.15) follows from the convexity of $t \mapsto \log\left(\frac{1}{t}\right)$ and Jensen's inequality. Secondly, note that

$$D = \mathbb{E}_{X^{(0)},X^{(1)}} \left[\sum_{y \in \mathcal{Y}} p\left(y \mid x^{(0)}\right) \log \left[\frac{p\left(y \mid x^{(0)}\right)}{p\left(y \mid x^{(1)}\right)} \right] \right] = \mathbb{E}\left[D\left(p\left(\cdot \mid i\right) \parallel p\left(\cdot \mid k\right)\right)\right], \tag{6.4.18}$$

whereas

$$C_1 = \max_{i,k} \left\{ D\left(p\left(\cdot \mid i\right) \parallel p\left(\cdot \mid k\right)\right) \right\}, \tag{6.4.19}$$

and so the right inequality of (6.4.13) holds as well.

In the forthcoming sections, we will demonstrate how the equivalence between stop-feedback schemes and sequential multiple-hypothesis testing can be used, and in particular, how can results regarding asymptotically optimal tests, discussed in Section 4.3, can be of aid in the analysis of the communication problem.

A. Exact Error Exponent at Zero Rate

In this section, we concentrate on a single point of the $E(R)$ curve, namely, $R = 0$. Note that in general, there are two notions of zero rate in the information-theoretic literature. The first is the asymptotic regime in which the number of messages, M, goes to infinity such that $R = \frac{\log(M)}{\mathbb{E}[N]} \to 0$ as $\mathbb{E}[N] \to \infty$,[15] and the second is the asymptotic regime where M is fixed and $\mathbb{E}[N] \to \infty$. The notion that will be of interest to us in this section is the latter. The reason is that for this zero-rate regime, the fact that the problem of variable length coding with stop-feedback can be formalized as a sequential multiple-hypothesis testing problem, makes it possible to apply the results from section 4.3 (or, more precisely, from [44]), where the MSPRT was presented, and asymptotic optimality was obtained, for exactly the case where the number of hypotheses was kept fixed. Nevertheless, following closely the derivation of the result obtained in this section, will show that it holds for both of the two zero-rate regimes. This point will be re-emphasized later on.

Let $\Delta = (N, d)$ be a sequential multiple hypothesis test which is optimal in the error exponent sense for the problem in (6.4.6) (i.e., Δ is assumed to achieve the best achievable error exponent among all sequential multiple hypothesis tests). Throughout this section, the following assumptions will be made regarding Δ and the probability model used:

Assumption (i): The hypotheses are equiprobable to occur.

Assumption (ii): As mentioned before, the problem at hand is symmetric in the hypotheses in a sense that without prior knowledge no hypothesis can be preferred over another. For this reason we will assume that the decoding function d bears a symmetry such that non of the elements in the matrix $\left\{P_j\left(d = i\right)\right\}_{i,j}$ depend on i

[15]Another way to write this is $M = e^{o(\mathbb{E}[N])}$ for large enough $\mathbb{E}[N]$.

or j. In other words, it will be assumed that, without loss of generality, for any $i \neq j \in \{0, \ldots, M-1\}$, $P_j (d = i)$'s are all equal.

Under these assumptions, the error probability of Δ, denoted by $p_e (\Delta)$, satisfies:

$$p_e (\Delta) = \sum_{j=0}^{M-1} \pi_j P_j (d \neq j) = \sum_{j=0}^{M-1} \pi_j \sum_{i=0, i\neq j}^{M-1} P_j (d = i) = \sum_{i=0, i\neq j}^{M-1} P_j (d = i) , \qquad (6.4.20)$$

where in the last equality Assumption (ii) was used. By taking the 0-1 loss function for wrong decision, the risk of hypothesis i, defined in (4.3.3), takes the form:

$$R_i (\Delta) = \sum_{j=0, j\neq i}^{M-1} \pi_j P_j (d = i) = \frac{p_e (\Delta)}{M}. \qquad (6.4.21)$$

Moreover, let ϵ be some positive constant. Since under any hypothesis H_i, $\mathbf{z}$ is i.i.d. (under any $i \in \{0, \ldots, M-1\}$), Theorem 9 can be invoked, yielding the following results:

- For small enough ϵ,

$$\inf_{\Delta : p_e(\Delta) \leq \epsilon} \mathbb{E}_i [N] \geq \frac{1}{D} \log \left(\frac{M}{p_e (\Delta)} \right) (1 + o(1)) . \qquad (6.4.22)$$

- Recall that in Section 4.3.A., two hypothesis testing schemes, denoted by $\Delta_a = (N_a, d_a)$ and $\Delta_b = (N_b, d_b)$, were defined. Using Theorem 9, we conclude that applying these tests to the hypothesis testing problem at hand, yields

$$\mathbb{E}_i [N_a] \leq \frac{1}{D} \log \left(\frac{M}{p_e (\Delta)} \right) (1 - o(1)) , \quad \mathbb{E}_i [N_b] \leq \frac{1}{D} \log \left(\frac{M}{p_e (\Delta)} \right) (1 - o(1)) . \qquad (6.4.23)$$

Using the definition for $E(R)$ at $R = 0$ in (5.2.10),

$$E(0) = \max_{P_X} \left\{ \lim_{\mathbb{E}[N] \to \infty, M fixed} \frac{-\log (p_e (\Delta))}{\mathbb{E}[N]} \right\} , \qquad (6.4.24)$$

combined with (6.4.22) and (6.4.23), and using the fact that, from symmetry considerations (Assumption (i)), $\mathbb{E}[N] = \sum_{i=1}^{M-1} \pi_i \mathbb{E}_i [N] = \mathbb{E}_i [N]$ for any $i \in \{0, \ldots, M-1\}$,

$$E(0) = \max_{P_X} D \qquad (6.4.25)$$

where D, defined in (6.4.12), can be written explicitly as:

$$D = D(0 \parallel 1) = \sum_{x^{(0)} \in \mathcal{X}} \sum_{x^{(1)} \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_X \left(x^{(0)}\right) P_X \left(x^{(1)}\right) p \left(y \mid x^{(0)}\right) \log \left[\frac{p \left(y \mid x^{(0)}\right)}{p \left(y \mid x^{(1)}\right)} \right] . \qquad (6.4.26)$$

In [138] the same result was obtained by Telatar[16] using the method of types and an MMI decoder, to show that this error exponent is achievable. Although in [138] Telatar mentions that this result is "curious", by using the interpretation of the problem given in the section, the result (6.4.25) makes perfect sense. Another reason why (6.4.25) is important, is that it shows that the lower bound on the error exponent, using Telatar's scheme, is indeed tight for zero rate, while Forney's error exponent, (6.2.46), is tight at zero rate for a large family of channels, including symmetric channels.

B. Lower Bound on the Performance of Stop-Feedback Schemes

In Section 5.8, we have seen how results and analytical tools, of sequential multiple-hypothesis testing with observation control, can be used in evaluating the performance of variable-length coding with a perfect feedback. The first step, taken in Section 5.8, was to establish the analogy between the problems. It was then demonstrated how to apply results from hypothesis testing theory to prove an upper bound on the error exponent function and, show that it is indeed achievable. By doing so, we have provided an alterative proof of the optimality of Bunrashev's error exponent.

The natural question that arises is whether the same can be done for variable length coding with limited feedback, and, particularly, for the stop-feedback coding setup. Since the mathematical connection between the problems has already been established in Section 6.4 (under random coding), the next step is to harness results of sequential multiple hypothesis tests as bounds for the communication model.

To this end we establish a lower bound on the performance of stop-feedback schemes. We have already seen how the test $\Delta_a = (N_a, d_a)$, defined in Section 4.3.A., can be of use in the analysis of the error exponent at zero rate. In fact, Δ_a was shown to be optimal in that case. In the previous section it was also mentioned that this analysis was straightforward, since zero-rate is the asymptotic regime that is usually assumed in hypothesis testing, that is, M fixed while $\mathbb{E}[N] \to \infty$ (or equivalently $p_e \to 0$). In this section, Δ_a we will re-examined, this time for a non-zero rate. For simplicity, we concentrate on the BSC with crossover probability $0 < \epsilon < \frac{1}{2}$. Define $E_a(R)$ to be the error exponent of Δ_a. Then,

$$E_a(R) \geq E_{\text{Forney}}(R) = E_{\text{sp}}(R) + C - R. \tag{6.4.27}$$

[16]When deriving the result (6.4.25), we were unaware of the previous work done in [138].

The meaning of (6.4.27) is that when applying Δ_a as a decoding algorithm in stop-feedback over a BSC, it performs no worse than Forney's ARQ scheme. Note, however, that Δ_a is quite different from Forney's ARQ scheme. One of the most obvious disadvantages of Forney's ARQ scheme is that in case of erasure, the decoder delays the decision by at least n time units, where n is the block length of the erasure decoder, that is assumed to be very large. It is highly unlikely that, when the confidence is low after n channel outputs, an additional n observations will be needed to achieve the desired confidence. This is also evident from the structure of the memoryless schemes, where past observations are ignored after an erasure. Obviously some useful information is lost this way. Instead, a more reasonable test will use the past observations, and add just a small number of extra observations to "complete the picture". This is especially true in our problem, where performance is measured with respect to the time delay (this is evident by recalling the definition of the error exponent as the limit of $\frac{-\log p_e}{\mathbb{E}[N]}$). The test Δ_a is an example.

Although the main interest in both Forney's ARQ scheme and Δ_a is theoretical, it pays to note that Forney's ARQ scheme is somewhat easier to implement. The reason is that it does not require an infinite random code. In addition, while intuitively Δ_a will take, on average, less time to decode (since it does not require the transmission of an additional n symbols for each erasure event that occurs), it requires a random use of the memory resource, whereas in Forney's ARQ schemes, only a fixed amount of memory is needed.

We next try to understand the relation between the two decoding schemes and the reason for the similar bounds for their performance; in some sense, these schemes are two sides of the same coin.Note that by the proofs of (6.2.46) and (6.4.27), one learns that both are based on the fact that, with high probability, the decoding procedure will end after about $\frac{-\log p_e}{E_{\text{Forney}}(R)}$ time units. In Forney's ARQ scheme, this is the reason to why the decoder can afford to disregard the past when erasure occurs: since this is a rare event, the expected decoding time is about the same as in with Δ_a. The improvement of the two schemes over the fixed block-length coding schemes may be understood by the analogy with lossless source coding. In lossless source coding, more likely (*typical*) sequences are assigned codewords whose description length is roughly equal to the entropy of the source. Such sequences capture most of the probability mass. By contrast, less likely sequences are assigned longer descriptions. While the expected description length is roughly equal to that of a typical sequence, zero-error is attained by (rarely) using longer descriptions. A feedback channel over which one bit per message is transmitted, enables a similar variable-length paradigm for channel coding, such as the memoryless schemes of the previous section, as well as Δ_a.

The proof of the claim in (6.4.27) can be found in Appendix C, along with other properties of Δ_a when applied to the stop-feedback problem.

C. Upper Bound on the Performance of Stop-Feedback Schemes for the BSC Model

In the previous section, a lower bound on the error exponent was obtained for the BSC in the stop-feedback setup. This lower bound agrees with Forney's error exponent, which was derived using the ARQ restrictions. In this section, an upper bound on the error exponent of a general family of stop-feedback decodes will be given, for the same forward channel model under random coding. This upper bound coincides with the lower bound of the previous section, and therefore, it is the exact error exponent.

Let $\Delta = (N, d)$ be an optimal sequential multiple hypothesis test for (6.4.6) in the error exponent sense. Throughout this subsection, all assumptions regarding the random coding setting, as were defined in Section 6.4.A., will be used. In particular, **Assumption (i)** and **Assumption (ii)** will be made, and hence, for any $i, j \in \{0, \ldots, M-1\}$ we will assume that

$$p_e(\Delta) = \sum_{j=0, j\neq i}^{M-1} P_j(d = i). \tag{6.4.28}$$

Define

$$\Lambda_i(N) \triangleq \log\left[\frac{p\left(y \mid x^{(i)}\right)}{\sum_{j=0,j\neq i}^{M-1} p\left(y \mid x^{(j)}\right)}\right] \tag{6.4.29}$$

where $y = \{y_1, \ldots, y_N\}$ and $x^{(k)} = \left\{x_1^{(k)}, \ldots, x_N^{(k)}\right\}$ for any $k \in \{0, \ldots, M-1\}$. In addition, let a be a positive constant and, for any $\bar{n} \in \mathbb{N}$, define the event $\Omega_{i,\bar{n}}$ as

$$\Omega_{i,\bar{n}} \triangleq \{d = i, N \leq \bar{n}\}. \tag{6.4.30}$$

The values of a and $\bar{n}$ will be determined later. Note that for any a and $\bar{n}$ the following chain of equalities and inequalities hold:

$$\sum_{j=0,j\neq i}^{M-1} P_j\left(d=i\right) = \sum_{j=0,j\neq i}^{M-1} \sum_{\mathbf{z}} \mathbb{I}\left\{\mathbf{z}: d=i\right\} P_j\left(\mathbf{z}\right) \tag{6.4.31}$$

$$= \sum_{\mathbf{z}} \sum_{j=0,j\neq i}^{M-1} \mathbb{I}\left\{\mathbf{z}: d=i\right\} \frac{P_j\left(\mathbf{z}\right)}{P_i\left(\mathbf{z}\right)} P_i\left(\mathbf{z}\right) \tag{6.4.32}$$

$$= \mathbb{E}_i\left[\mathbb{I}\left\{\mathbf{z}: d=i\right\} \frac{\sum_{j=0,j\neq i}^{M-1} P_j\left(\mathbf{z}\right)}{P_i\left(\mathbf{z}\right)}\right] \tag{6.4.33}$$

$$= \mathbb{E}_i\left[\mathbb{I}\left\{d=i\right\} \exp\left\{-\log\left[\frac{P_i\left(\mathbf{z}\right)}{\sum_{j=0,j\neq i}^{M-1} P_j\left(\mathbf{z}\right)}\right]\right\}\right] \tag{6.4.34}$$

$$= \mathbb{E}_i\left[\mathbb{I}\left\{d=i\right\} \exp\left\{-\Lambda_i\left(N\right)\right\}\right] \tag{6.4.35}$$

$$\geq \mathbb{E}_i\left[\mathbb{I}\left\{\Omega_{i,\bar{n}}, \Lambda_i\left(N\right) < a\right\} \exp\left\{-\Lambda_i\left(N\right)\right\}\right] \tag{6.4.36}$$

$$\geq e^{-a}\mathbb{E}_i\left[\mathbb{I}\left\{\Omega_{i,\bar{n}}, \Lambda_i\left(N\right) < a\right\}\right] \tag{6.4.37}$$

$$\geq e^{-a} P_i\left(\Omega_{i,\bar{n}}, \sup_{n\leq \bar{n}}\left\{\Lambda_i\left(n\right) < a\right\}\right). \tag{6.4.38}$$

where $P_i\left(\mathbf{z}\right)$ is defined in (6.4.8) for any $i \in \{0, \ldots, M-1\}$. Using the union bound and the symmetry implied by Assumption 2 it can be shown that:

1. $P_i\left(\Omega_{i,\bar{n}}, \sup_{n\leq\bar{n}}\left\{\Lambda_i\left(n\right) < a\right\}\right) \geq P_i\left(\Omega_{i,\bar{n}}\right)$ - $P_i\left(\sup_{n\leq\bar{n}}\left\{\Lambda_i\left(n\right) > a\right\}\right)$.

2. $P_i\left(\Omega_{i,\bar{n}}\right) \geq P_i\left(d=i\right) - P_i\left(N \geq \bar{n}\right) = 1 - p_e\left(\Delta\right) - P_i\left(N \geq \bar{n}\right)$.

By combining these observations with (6.4.28), we get

$$p_e\left(\Delta\right) e^a \geq 1 - p_e\left(\Delta\right) - P_i\left(N \geq \bar{n}\right) - P_i\left(\sup_{n\leq\bar{n}}\left\{\Lambda_i\left(n\right) > a\right\}\right). \tag{6.4.39}$$

Applying the Markov inequality on the third term on the right-hand side of (6.4.39) yields:[17]

$$\frac{\mathbb{E}\left[N\right]}{\bar{n}} \geq 1 - p_e\left(\Delta\right)\left(e^a - 1\right) - P_i\left(\sup_{n\leq\bar{n}}\left\{\Lambda_i\left(n\right) > a\right\}\right). \tag{6.4.40}$$

We now choose the parameters a and $\bar{n}$. To that end, let ϵ_1 and ϵ_2 be arbitrarily small positive numbers, and let $a \triangleq -\left(1-\epsilon_1\right)\log p_e\left(\Delta\right)$ and $\bar{n} \triangleq \left(1+\epsilon_2\right)\mathbb{E}\left[N\right]$. Using

[17] Here we used the symmetry of the problem again in order to conclude that for any $i \in \{0, \ldots, M-1\}$ it holds that $\mathbb{E}_i\left[N\right] = \mathbb{E}\left[N\right]$. In other words, following Assumptions (i) and (ii), the expected value of the stopping time does not depend on the specific underlying hypothesis.

these definitions, we get:

$$p_e(\Delta)(e^a-1)=p_e(\Delta)\left(\frac{1}{[p_e(\Delta)]^{1-\epsilon_1}}-1\right)=[p_e(\Delta)]^{\epsilon_1}-p_e(\Delta). \tag{6.4.41}$$

Considering the asymptotic regime $p_e(\Delta)\to 0$ and $\mathbb{E}[N]\to\infty$, we take the limits over both sides of (6.4.40) and obtain::

$$\frac{1}{1+\epsilon_2}\geq 1-\lim P_i\left(\sup_{n\leq\bar{n}}\{\Lambda_i(n)>a\}\right)\geq 1-\lim\sum_{n\leq\bar{n}}P_i(\Lambda_i(n)>a). \tag{6.4.42}$$

Define $\bar{R}\triangleq\frac{R}{1+\epsilon_2}$, $\bar{T}\triangleq\left[\frac{1-\epsilon_1}{1+\epsilon_2}\right]E(R)$ and

$$\kappa_0\triangleq\max_{\kappa>0}\left\{\epsilon-\kappa>0 \text{ and } \epsilon+\kappa<\delta_{\mathrm{GV}}\left(\bar{R}\right)\right\}. \tag{6.4.43}$$

Assume conversely that there exist positive number ϵ_1,ϵ_2 and $\kappa\in(0,\kappa_0)$ such that

$$\bar{T}>\beta\left[\delta_{\mathrm{GV}}\left(\bar{R}\right)-(\epsilon-\kappa)\right]. \tag{6.4.44}$$

Then, by Claim E.3 and (E.0.37) in Appendix E, $\lim\sum_{n\leq\bar{n}}P_i(\Lambda_i(n)>a)=0$ as $\mathbb{E}[N]\to\infty$ and $p_e(\Delta)\to 0$. Since the left hand side of (6.4.42) is strictly smaller than 1 for any $\mathbb{E}[N]$, this leads to a contradiction. Therefore, we conclude that for any such ϵ_1,ϵ_2 and κ,

$$E(R)\leq\left[\frac{1+\epsilon_2}{1-\epsilon_1}\right]\beta\left[\delta_{\mathrm{GV}}\left(\bar{R}\right)-\epsilon\right]+\kappa\beta\left[\frac{1+\epsilon_2}{1-\epsilon_1}\right]. \tag{6.4.45}$$

Specifically, ϵ_1,ϵ_2 and κ can be made arbitrarily small. By taking the limit as $\epsilon_1,\epsilon_2,\kappa\to 0$ (but with $\epsilon_1,\epsilon_2,\kappa>0$) we further conclude that

$$E(R)\leq\beta\left[\delta_{\mathrm{GV}}(R)-\epsilon\right]\equiv E_{\mathrm{Forney}}(R). \tag{6.4.46}$$

Recall that in the previous section, we have seen a decoding scheme that achieves this upper bound under random coding. Therefore, by combining (6.4.27) and (6.4.46), we see that, $E_{\mathrm{Forney}}(R)$ is the best achievable error exponent function for the BSC under random coding at rate $R<C$.

D. Upper Bound on Performance of Stop-Feedback Schemes for a General DMC

In this section, a novel upper bound on the error exponent of stop-feedback schemes will be given, and its proof will be briefly outlined. The full derivation can be found in Appendix H.

Theorem 35 *For the DMC of Section 5.2.A., the error exponent of any stop-feedback coding scheme, using infinite random coding, is upper bounded as follows:*

$$E(R) \leq D\left(1 - \frac{R}{C}\right) \tag{6.4.47}$$

and R is as in (5.2.9).

Note that in Section 6.2.C., we saw that Burnasev's error exponent, $E_B(R)$, can sometimes be attained using random coding and a limited feedback. On the other hand, it follows from (6.4.13) that for non-trivial channels, $D < C_1$, and so, Theorem 35 asserts that for the forward channel in Section 5.2.A., a single bit of feedback is not enough to attain $E_B(R)$.

Outline of proof: Let $\Delta = (N, d)$ be a general decoding algorithm for the stop-feedback problem, using infinite random coding, i.e., N is the (random) time at which the decoder sends the stop symbol through the feedback channel, and $d = d(y_1, \ldots, y_N)$ is the decision function associated with Δ. The main idea is to artificially divide the decoding process of any such stop-feedback decoding procedure into two phases, and separately bound the time it takes for the two phases to terminate. This idea was first used in [9] to provide a simple proof of the optimality of Burnashev's error exponent in the variable-length coding problem over a DMC. Later, a similar idea was used in [28] to extend this result to finite-state ergodic Markov channels.

Define the following stopping times:

$$N_1 \triangleq \inf_{n>0} \left\{p_e^{MAP}(y^n) < \delta\right\} \tag{6.4.48}$$

$$\tau \triangleq \min\{N_1, N\} = \inf_{n>0} \left\{p_e^{MAP}(y^n) < \delta \text{ or } N = n\right\} \tag{6.4.49}$$

where δ is some positive constant, and $p_e^{MAP}(y^n)$ is the error probability of a MAP decoder at time n, given $\{Y_1, \ldots, Y_n\} = \{y_1, \ldots, y_n\}$. Since, by definition, $\tau \leq N$, one can look at the decoding time N, as having two segments; the first is of length N_1 and the second is the remainder. It can be proven (as done in Appendix F), that δ can be chosen such that, with high probability, $N_1 < N$, and hence both parts are non-empty.

Recall that $\pi_i(n) \triangleq \Pr(\theta = i \mid y^n)$ is the a posteriori probability of the i'th message to be sent given all the information available for the decoder at time n.

The next lemma summarizes some of the properties of τ.

Lemma 36 *Let τ be as defined in* (6.4.49). *Then the following hold:*

1. *There exists at least one hypothesis, $i^\star \in \{0, \ldots, M-1\}$, satisfying*

$$\pi_{i^\star}(\tau) = \Pr(\theta = i^\star \mid y^\tau) > 1 - \delta. \tag{6.4.50}$$

2. *For the same $i^\star$*

$$\sum_{j=0, j\neq i^\star}^{M-1} \pi_j(\tau) \geq \lambda\delta \tag{6.4.51}$$

where $\lambda \triangleq \min_{x,y\in\mathcal{X}\times\mathcal{Y}} p(x \mid y)$.

3. *For any $0 < \delta \leq 1/2$,*

$$\mathbb{E}[\tau] \geq \left(1 - \delta - \frac{p_e}{\delta}\right)\frac{\log M}{C} - \frac{h_2(\delta)}{C} \tag{6.4.52}$$

where $p_e = p_e(\Delta)$ is the error probability of Δ.

Note that for the DMC model defined in Section 5.2.A., $\lambda > 0$. The proof is in Appendix F. The second key point in the proof, is that after $\{y_1, \ldots, y_\tau\}$ were observed, a new phase starts in which the test infers among M distinct hypotheses, where at the "starting point" of this test, both (6.4.50) and (6.4.51) hold. The second mission is, therefore, to lower bound the stopping time of any multiple-hypothesis test satisfying these conditions. To that end, we have the following auxiliary lemma:

Lemma 37 *Let δ, λ, c and ρ be some constants satisfying $\delta > 0, \lambda < \frac{1}{2}$, $c > 1$, $0 < \rho < \frac{1}{c}$, and $\tilde{\Delta} = \left(\tilde{N}, \tilde{d}\right)$ be a sequential multiple hypothesis test among the hypotheses* (6.4.6)*, where the prior probabilities satisfy:*

1. *The exists at least one hypothesis, satisfying*

$$\pi_{i^\star} > 1 - \delta. \tag{6.4.53}$$

2. *For the same $i^\star$*

$$\sum_{j=0, j\neq i^\star}^{M-1} \pi_j \geq \lambda\delta. \tag{6.4.54}$$

Then the following holds:

$$\mathbb{E}\left[\tilde{N}\right] \geq (1-\delta)\frac{\rho}{D}\log\left[\frac{\lambda\delta}{p_e\left(\tilde{\Delta}\right)}\right] \times \\ \left(1 - \frac{p_e\left(\tilde{\Delta}\right)}{1-\delta} - \left[\frac{p_e\left(\tilde{\Delta}\right)}{\lambda\delta}\right]^{1-c\rho} - \frac{KD^2}{\rho^2\log^2\left[\lambda\delta/p_e\left(\tilde{\Delta}\right)\right]}\right), \tag{6.4.55}$$

where $p_e\left(\tilde{\Delta}\right)$ *is the error probability of the scheme* $\tilde{\Delta}$*, and* $K>0$ *is a constant that depends solely on the channel.*

The proof is in Appendix G. Theorem 35 is proven by using the fact that any stopping time N of the sequential hypothesis problem proposed in the beginning of this section, can be divided into two phases: the duration of the first phase is τ, which is either equal to N, or, it is the time by which a parallel sequential MAP test with threshold $\delta>0$ as its test statistics, would have stopped. The second's phase duration is the time it takes the original test to stop from that point and on. Since, by the definition of τ, we know that properties 1 and 2 of Lemma 36 hold, the second stopping time can be bounded using Lemma 37. Using item 3 of Lemma 36 and Lemma 37, we obtain:

$$\mathbb{E}\left[N\right] \simeq \mathbb{E}\left[\tau+\tilde{N}\right] \tag{6.4.56}$$

$$\geq \left(1-\delta-\frac{p_e}{\delta}\right)\frac{\log M}{C}-\frac{h_2\left(\delta\right)}{C} \tag{6.4.57}$$

$$+\left(1-\delta\right)\frac{\rho}{D}\log\left[\frac{\lambda\delta}{p_e\left(\tilde{\Delta}\right)}\right]\left(1-\frac{p_e}{1-\delta}-\left(\frac{p_e}{\lambda\delta}\right)^{1-c\rho}-\frac{KD^2}{\rho^2\log^2\left[\lambda\delta/p_e\left(\tilde{\Delta}\right)\right]}\right). \tag{6.4.58}$$

Taking the limit $p_e\to 0$ and using the definition of R and $E\left(R\right)$ in (5.2.9) and (5.2.10), respectively, yields Theorem 35. The proof is in Appendix H. For the BSC with crossover probability 0.1, the upper bound of Theorem 35 is illustrated in Figure 6.4.1, where it is plotted along with Forney's error exponent $E_{\text{Forney}}\left(R\right)$ and the sphere-packing bound, which is known to be an upper bound on the error exponent function for fixed block-length codes without any feedback [143].

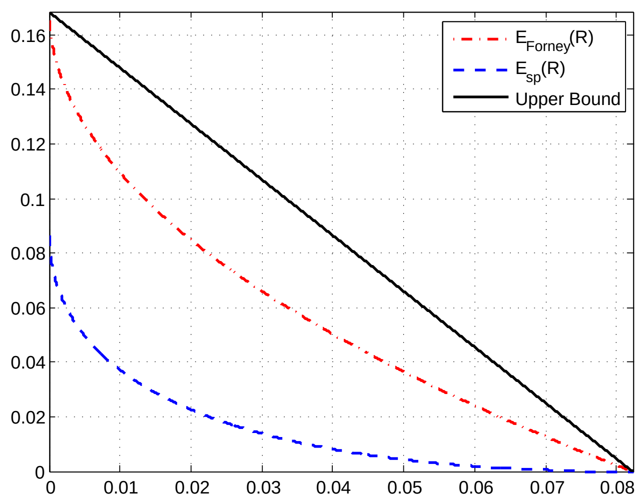

Figure 6.4.1: The upper bound of Theorem 35 plotted for a BSC with a crossover probability 0.1, plotted with $E_{\text{Forney}}(R)$ and $E_{sp}(R)$.

Chapter 7

Conclusions and Future Work

We have reviewed aspects of the interplay between sequential hypothesis testing and variable length channel coding with feedback. Specifically, we focused on the connection between the following two families of problems: the first is sequential hypotheses testing. The second is an information-theoretic problem of communication over a noisy channel in the presence of feedback. The goal is to come up with an encoder-decoder pair that assures reliable communication. Although feedback cannot increase the capacity of a memoryless channel, it can improve the reliability, i.e., error exponent.

Various examples emerge from the above description of the two problems. In the first part, we reviewed results about testing of two simple hypotheses, multiple simple hypotheses, testing composite hypotheses and hypothesis testing setups with control. The second part was dedicated to communication, and results on the error exponent of several communication schemes were given, including perfect and non restricted feedback along with results on some of the effects of limiting feedback. For several models, a connection to the sequential hypothesis tests was established. In general, we have concentrated on harnessing analysis tools of sequential hypothesis testing theory, to obtain results on the reliability of communication systems, as well as to gain better intuition.

The main contribution of this work is in that it clusters together some important examples where hypothesis testing theory can be used in information-theoretic problems of channel coding with variable length block length. In the course of writing of this paper, an effort was made in order for it to be self-contained as well as a concise and thorough source of reference to the subject of sequential hypothesis testing. In addition, some novel ideas and results were obtained. For example, a new scheme that achieves Burnashev's exponent was introduced in Section B., and new bounds on the error exponent function of the when a single bit of feedback is allowed were proven in Section 6.4. Specifically, we have demonstrated how a family of ARQ communication setups, denoted in the sequel by *stop-feedback schemes*, can be defined in a way that fits the i.i.d. observation model of simple and multiple hypothesis testing framework. This connection was then further stretched in order to gain both intuition and new results regarding the reliability of such communication schemes. Using these results we concluded that Forney's exponent is tight for the BSC model, as conjectured by Forney himself in 1969. Another importance of these results is in that they tighten the already established connection between the two problems discussed.

Nevertheless, the theory of stop-feedback schemes is far from being complete. Following are some direction for future research:

- **Generalizing the channel model:** Note that the tight bound that appears in Sections 6.4.B. and 6.4.C. was obtained under the assumption that the forward channel is binary and symmetric. Under this assumption, the analysis boiled down to that of evaluating the "time n enumerator" random variables denoted by $\left\{N_{[\mathbf{y}]_n}\right\}_{n>0}$. Since the asymptotic behaviour of $N_{[\mathbf{y}]_n}$ is fully characterized in [134], this assumption made the entire analysis simpler. It is worth noting that in [134] the authors have expanded the results obtained for the BSC to other, quite general, DMCs. Since both the derivation in [134] and the one carried out in Sections 6.4.B. and 6.4.C. concern the random sequence defined in Section 6.4 as $\{\Lambda_i(n)\}$, it may be possible to utilize of the techniques used in [134] to achieve more general results than those of Section 6.4.

  Another approach that may lead to a generalized theory regarding the forward channel in ARQ schemes is the following: note that the derivation of (6.4.31) - (6.4.38) holds for any DMC and the loss of generality was made in the next step to simplify the analysis of the expression

$$P_i\left(\sup_{n\le\bar{n}}\Lambda_i(n)>a\right). \tag{7.0.1}$$

  Specifically, techniques similar to those of [134] were used for the asymptotic evaluation of the *probability* of $\left\{\sup_{n\le\bar{n}}\Lambda_i(n)>a\right\}$. As was mentioned in Section 6.1, in many cases it is possible to apply the Markov inequality on $P_i(\Lambda_i(n)>a)$ and then analyse the *moments* of $\Lambda_i(n)$. For example, in [103] such an analysis was carried out and was shown to yield useful bounds for a large family of channels. We propose the use of the same technique to simplify the analysis of (7.0.1). A crucial obstacle is, however, that the Markov inequality does not hold for (7.0.1) due to the supremum operator. Nevertheless, it is still possible to bound (7.0.1)by moments of $\Lambda_i(n)$ using Doob's inequality [46], provided that $\{\Lambda_i(n)\}$ is a submartingale. In Appendix J, this property was proven to hold for the BSC (were the process $\{\Lambda_i(n)\}$ was considered with respect to its natural filtration).We conjecture that the fact that $\Lambda_i(n)$ is a submartingale holds also in more general cases, and hence Doob's inequality can be harnessed to obtain similar bounds for general DMCs.

- **Generalizing the coding scheme:** Another important fact regarding the bound in Section 6.4 is that it was derived under random coding. Although this assumption is often made in information-theoretic literature, there is no evidence that random coding is optimal in the error exponent sense. Extending the bounds to *general* coding schemes is of interest. One idea how to obtain such results is to the technique in Section 5.8. Specifically, in Section 5.8 we have seen how to pose the communication setup in which a DMC with perfect feedback is used, as a DP problem. The next step was to use the well known theory of DP and closed-loop feedback schemes to obtain upper and lower bounds on the error exponent, in addition to gaining some intuition regarding the optimal coding scheme. If one could define the problem of stop-feedback as a different DP problem, then perhaps open-loop results can be used to reach some conclusions regarding the optimal control policy, which will reveal some intuition regarding the optimal coding scheme for this case as well.

- **Practical considerations:** As explained in the text, in many modern applications with feedback retransmissions and average transmission time are only part of the parameters to be taken into account. Other important parameters, that were partially discussed, are upper bounds on the actual transmission time and higher moments of the stopping time. In Section 6.3, we have concentrated on ARQ systems where the number of retransmission requests is restricted to L. The same idea can be applied to the stop-feedback scenario, where the number of samples is bounded. Note that for stop-feedback schemes, this restriction takes an even more important role since L will determine, not only an upper bound on the transmission time, but also to restrict the length of the codewords, where in the original setup this length is thought of as infinite. We propose two such schemes: the first carries out the same stopping and decision rules as in Section 6.4 at time instances that are less than L. If by the time L, a decision is not reached, then the encoder uses an ML decision, based on the L observations. A second, and more reasonable scheme is to adjust both the stopping and decision rules to be time-dependent. Such rules may be derived using ideas from the theory of stopped random walks (and, specifically, using *backward induction*) [63]. As for higher moments of the stopping time, we propose to use the tail formula for $\mathbb{E}\left[N^2\right]$ instead of that for $\mathbb{E}\left[N\right]$ as in (C.0.5), and Chebyshev's inequality instead of Markov's inequality in (6.4.39). Of course, to do so, we may have to redefine $\Omega_{i,n}$ and analyse higher moment of $\{\Lambda_i\left(n\right)\}$, but we conjecture that such analysis is feasible using the same tools of Section 6.4.

To conclude, we hope that this review will resolve in better understanding of the two research fields that were discussed and will help to attract interest to the interplay between them.

Appendices

Appendix A

Proof of the Achievability of Burnashev's Exponent

In this appendix, a proof of the direct part of Burnashev's reliability function will be given, using the modified Yamamoto-Itoh scheme, described in Section 5.6.B..

By its definition, the coding algorithm bears a regenerative property, i.e., after each communication cycle, the encoding and decoding procedures "forget" about their past and start over. For this reason, the random variables $\{N_{I,1}, N_{I,2}, \ldots\}$ are i.i.d. given the message sent, and the same holds for $\left\{N_{I,1}^i, N_{I,2}^i, \ldots\right\}$ (for $i \in \{0, \ldots, M-1\}$) and $\{N_{II,1}, N_{II,2}, \ldots\}$.

<u>*Phase I' analysis:*</u>

Recall that in phase I', M one sided SPRTs are performed, each with a stopping time

$$N_{I,k}^i = \min_{n \geq 0} \left\{ \sum_{j=1}^{n} \xi_j^i \geq (1+\epsilon) \log M \right\}, \quad i \in \{0, \ldots M-1\}. \tag{A.0.1}$$

where $\xi_j^i \triangleq \log \left[\frac{p\left(y_j | x_j^i\right)}{\Pr(y_j)} \right]$. The stopping time and decision function of the first operation phase are defined to be:

$$N_{I,k} \triangleq \min_{i \in \{0,\ldots,M-1\}} N_{I,k}^i, \quad d_{I,k}\left(N_{I,k}\right) = \operatorname*{argmin}_{i \in \{0,\ldots,M-1\}} N_{I,k}^i. \tag{A.0.2}$$

Assume, without loss of generality, that the 0'th message was sent, and define the error event at the k'th cycle by:

$$\mathcal{E}_{k,0} \triangleq \left\{ d_{I,k}\left(N_{I,k}\right) \neq 0 \right\}. \tag{A.0.3}$$

Since $\{N_{I,1}, N_{I,2}\dots\}$ are identically distributed,

$$
\begin{aligned}
\Pr\left(\mathcal{E}_{k,0}\right) &= \Pr\left(\mathcal{E}_{1,0}\right) && \text{(A.0.4)}\\
&= \Pr \bigcup_{i=1}^{M-1} \left\{ \sum_{j=1}^{N_{I,1}} \xi_j^i \geq \sum_{j=1}^{N_{I,1}} \xi_j^0 \right\} && \text{(A.0.5)}\\
&\leq \Pr \bigcup_{i=1}^{M-1} \left\{ \sum_{j=1}^{N_{I,1}} \xi_j^i \geq (1+\epsilon)\log(M) \right\} && \text{(A.0.6)}\\
&\leq (M-1) \Pr \left\{ \sum_{j=1}^{N_{I,1}} \xi_j^1 \geq (1+\epsilon)\log(M) \right\} && \text{(A.0.7)}\\
&\leq (M-1) \Pr\left(N_{I,1}^1 < \infty\right), && \text{(A.0.8)}
\end{aligned}
$$

where (A.0.6) follows from (A.0.1).

Lemma 38 *Under the assumption that the 0'th message was sent,*

$$
\Pr\left(N_{I,1}^1 < \infty\right) \leq \frac{1}{M^{1+\epsilon}}. \tag{A.0.9}
$$

Proof. Let H_0 and H_1 be as in (5.6.8) and (5.6.9). Then, $P_0\left(N_{I,1}^1 < \infty\right) = P_1\left(N_{I,1}^0 < \infty\right)$ and

$$
\begin{aligned}
P_1\left(N_{I,1}^0 < \infty\right) &= \mathbb{E}_0\left[\exp\left\{-\sum_{j=1}^{N_{I,1}^0} \xi_j^0\right\} \mathbb{I}\left(N_{I,1}^0 < \infty\right)\right] && \text{(A.0.10)}\\
&\leq \exp\left\{-(1+\epsilon)\log M\right\} && \text{(A.0.11)}\\
&= \frac{1}{M^{1+\epsilon}}, && \text{(A.0.12)}
\end{aligned}
$$

where in the first equality Wald's LR identity (2.2.6) was used, and the second equality follows directly from the definition of $N_{I,1}^0$ in (A.0.1). $\square$

Setting the result from Lemma 38 in (A.0.8) yields $\Pr\left(\mathcal{E}_{k,0}\right) \leq \frac{1}{M^\epsilon}$, and so, for large enough M, $\Pr\left(\mathcal{E}_{k,0}\right) \leq P_{e,1}$ for any $\epsilon > 0$.

Since the messages are uniform in $\{0,\dots,M-1\}$, i.e., $\pi_i = \frac{1}{M}$ for all $i \in \{0,\dots,M-1\}$, and following the definition of $N_{I,k}$,

$$
\mathbb{E}\left[N_{I,k}\right] = \mathbb{E}\left[N_{I,k} \mid 0 \text{ sent}\right] \leq \mathbb{E}\left[N_{I,k}^0 \mid 0 \text{ sent}\right]. \tag{A.0.13}
$$

Using Wald's first equation (2.3.3) and the definition of $N_{I,k}^0$ yields

$$\mathbb{E}\left[N_{I,k}^0 \mid 0 \text{ sent}\right] C = \mathbb{E}\left[\sum_{j=0}^{N_{I,k}^0} \xi_j^i \mid 0 \text{ sent}\right] \leq (1+\epsilon)\log M + \tilde{\Delta} \tag{A.0.14}$$

where $\tilde{\Delta} = \max \xi_j^i$, which, by the assumption on the channel, is finite. Using (A.0.14)

$$\mathbb{E}\left[N_{I,k}\right] \leq (1+\epsilon)\frac{\log M}{C} + \Delta \tag{A.0.15}$$

where $\Delta \triangleq \frac{\tilde{\Delta}}{C}$ is a constant that depends solely on the channel.

A few notes are in order:

- The multiple hypotheses test defined above lends itself to the asymptotic regime for which the number of hypotheses goes to infinity while the rate is kept fixed. This is not the case for most of the tests described in Part I of this work, where we were only interested in the asymptotic regime where the expected number of samples taken is large, while the number of hypotheses is kept fixed.

- Equation (A.0.15) implies that

$$C \leq (1+\epsilon)\frac{\log M}{\mathbb{E}\left[N_{I,k}\right]} + \frac{C\Delta}{\mathbb{E}\left[N_{I,k}\right]} \triangleq (1+\epsilon) R' + \frac{C\Delta}{\mathbb{E}\left[N_{I,k}\right]} \tag{A.0.16}$$

  where R' is the rate of the code used in the first communication phase. Keeping in mind that for large enough M, the error probability can be made as small as desired, this is yet another proof that for $R < C$ reliable communication is possible (and, indeed, the code rate used in the first phase is very close to the channel's capacity). In addition, this is also analogous to the fixed block length code of rate $C(1-\epsilon)$ that was used in the communication operation mode in Yamamoto and Itoh's original scheme.

Phase II' analysis:

Assume that M is large enough such that the probability of error in the first phase is bounded by $P_{e,1} \in \left(0, \frac{1}{3}\right)$. By the definition of the coding algorithm, this means that the prior probability, at the beginning of the second phase, of the "NACK" hypothesis is upper bounded by $P_{e,1}$, and so, for any k,

$$\begin{aligned} \mathbb{E}\left[N_{II,k}\right] &= \pi_A \mathbb{E}_{P_A}\left[N_{II,k}\right] + \pi_N \mathbb{E}_{P_N}\left[N_{II,k}\right] && \text{(A.0.17)} \\ &\leq \mathbb{E}_{P_A}\left[N_{II,k}\right] + P_{e,1}\mathbb{E}_{P_N}\left[N_{II,k}\right]. && \text{(A.0.18)} \end{aligned}$$

Let $\zeta_j = \log\left[\frac{P_A(y_j)}{P_N(y_j)}\right]$. Note that

$$N_{II,k} = \min_{n\geq 0}\left\{\sum_{j=1}^{n}\zeta_j \geq \log A \text{ or } \sum_{j=1}^{n}\zeta_j \leq \log B\right\} \leq \inf_{n\geq 0}\left\{\sum_{j=1}^{n}\zeta_j \geq \log A\right\}. \quad (A.0.19)$$

Since, by the SPRT properties, $\alpha \leq \alpha_0 = \frac{1}{A}$ and $\beta \leq \beta_0 = B$ (where α and β are as defined in (5.6.2) and (5.6.3) respectively), following the same exact reasoning as is (A.0.14) with the aid of (A.0.19) yields

$$\begin{aligned} \mathbb{E}_{P_A}\left[N_{II,k}\right] &\leq \frac{\log A}{C_1} + \Delta_A \leq \frac{-\log\alpha}{C_1} + \Delta_A, & (A.0.20)\\ \mathbb{E}_{P_N}\left[N_{II,k}\right] &\leq \frac{|\log B|}{\bar{C}_1} + \Delta_N & (A.0.21)\end{aligned}$$

where Δ_A and Δ_N are defined as

$$\frac{\max\log\left[\frac{P_A(Y_n)}{P_N(Y_n)}\right]}{C_1} \quad (A.0.22)$$

and

$$\frac{\min\left|\log\left[\frac{P_A(Y_n)}{P_N(Y_n)}\right]\right|}{\bar{C}_1}, \quad (A.0.23)$$

respectively, and $C_1, \bar{C}_1$ are as defined in (5.5.1) and (5.5.2). By substituting these results in (A.0.18) we get

$$\mathbb{E}\left[N_{II,k}\right] \leq \frac{-\log\alpha}{C_1} + P_{e,1}\frac{|\log B|}{\bar{C}_1} + \Delta_C \quad (A.0.24)$$

where $\Delta_C = \Delta_A + \Delta_N$ is a constant that solely depends on the channel.

To analyze the performance, assume again, that the 0'th message was sent, and define the following events:

$$\begin{aligned} \mathcal{C}_{II,k} &= \left\{d_{I,k} = 0, d_{II,k} = ACK\right\}, & (A.0.25)\\ \mathcal{R}_{II,k} &= \left\{d_{I,k} = 0, d_{II,k} = NACK\right\}\bigcup\left\{d_{I,k} \neq 0, d_{II,k} = NACK\right\}, & (A.0.26)\\ \mathcal{E}_{II,k} &= \left\{d_{I,k} \neq 0, d_{II,k} = ACK\right\}, & (A.0.27)\end{aligned}$$

that is, $\mathcal{C}_{II,k}$ is the event of stopping and correctly decoding at cycle k, the probability of which is $\Pr\left(\mathcal{C}_{II,k}\right) = \left(1 - P_{e,1}\right)\left(1 - \beta\right)$, $\mathcal{R}_{II,k}$ is the event of retransmission at cycle k, the probability of which is $\Pr\left(\mathcal{R}_{II,k}\right) = \left(1 - P_{e,1}\right)\beta + P_{e,1}\left(1 - \alpha\right)$ and $\mathcal{E}_{II,k}$ is the event of stopping and incorrectly decoding at cycle k, the probability of which is $\Pr(\mathcal{C}_{II,k}) = P_{e,1}\alpha$. Setting $B = P_{e,1}$ yields the following results:

$$\begin{aligned}
\Pr(\mathcal{R}_{II,k}) &\leq 2P_{e,1} && \text{(A.0.28)}\\
P_e &= P_{e_1}\alpha\sum_{j=0}^{\infty}\left[\Pr(\mathcal{R}_{II,k})\right]^k && \text{(A.0.29)}\\
&= \frac{Pe_1\alpha}{1-\Pr(\mathcal{R}_{II,k})} && \text{(A.0.30)}\\
&\leq \frac{P_{e,1}}{1-2P_{e,1}}\alpha && \text{(A.0.31)}\\
\mathbb{E}[K] &= \frac{1}{1-\Pr(\mathcal{R}_{II,k})} \leq \frac{1}{1-P_{e,1}}. && \text{(A.0.32)}
\end{aligned}$$

By (5.6.19) we can then bound N as follows:

$$\begin{aligned}
\mathbb{E}[N] &= \mathbb{E}\left[\sum_{k=0}^{K} N_{I,k}\right] + \mathbb{E}\left[\sum_{k=0}^{K} N_{II,k}\right] && \text{(A.0.33)}\\
&= \mathbb{E}[K]\left(\mathbb{E}[N_{I,0}] + \mathbb{E}[N_{II,0}]\right) && \text{(A.0.34)}\\
&\leq \frac{1}{1-P_{e,1}}\left[(1+\epsilon)\frac{\log M}{C} + \frac{-\log\alpha}{C_1} + P_{e,1}\frac{\log P_{e,1}}{\bar{C}_1} + \bar{\Delta}\right] && \text{(A.0.35)}
\end{aligned}$$

where (A.0.34) is Wald's first equation (2.3.3) applied to the stopping time K and the i.i.d. random sequences $N_{I,k}$ and $N_{II,k}$, and $\bar{\Delta} = \Delta + \Delta_A$ is a (finite positive) constant that depends only on the channels. By (A.0.29) - (A.0.31) we have:

$$-\log\alpha \leq -\log P_e + \log\left(\frac{P_{e,1}}{1-2P_{e,1}}\right) \leq -\log P_e. \qquad \text{(A.0.36)}$$

where the second inequality holds since $P_{e,1} \in (0, 1/3)$. Since $P_{e,1} > 0$ can be chosen to be arbitrarily small, and since the bound in $\mathbb{E}[N]$ holds for any ϵ, it then follows that

$$\mathbb{E}[N] \leq (1+\epsilon'')\left(\frac{\log M}{C} - \frac{\log P_e}{C_1}\right) + \bar{\Delta} \qquad \text{(A.0.37)}$$

for arbitrarily small $\epsilon'' > 0$, which is exactly the leading term in Burnashev's bound.

Appendix B

Alternative Derivation of Forney's Exponent for the BSC

Consider the BSC with crossover probability $0 < \epsilon < \frac{1}{2}$, and assume the uniform random coding distribution For this model, a slightly different derivation of Forney's error exponent for ARQ schemes, defined in (6.2.46), will be given.

In general, we revisit Forney's ARQ scheme, i.e., embedding Forney's erasure-decoder in an ARQ scheme in the same way that was done in Section 6.2.B.. The difference is that this time, the exact error exponent for both the erasure and the undetected error events are used. The idea is that as long as the probability of an erasure event decays exponentially as the block-length grows (i.e., as long as $e_1(R,T)$, defined in (6.2.9), is strictly positive), the expected decoding time, denoted by $\mathbb{E}[N]$, is close to the (fixed) block-length of the erasure-decoder codebook, while the error exponent is lower bounded by $e_2(R,T)$, defined in (6.2.9). These properties of Froney's ARQ scheme were proven in Section 6.2.B..

Following [53], we define the following achievable error exponent function for Forney's ARQ scheme:

$$\hat{E}_{\mathrm{f}}(R) \triangleq \lim_{e_1(R,T)\to 0} e_2(R,T). \tag{B.0.1}$$

For the case at hand, we know, using Theorem 34, that if $R \geq \log 2 - h_2\left(\epsilon + \frac{T}{\alpha}\right)$, $e_1(R,T) = 0$ and otherwise

$$e_1(R,T) = \min_{\nu\in\left[\epsilon,\delta_{\mathrm{GV}}(R)-\frac{T}{\alpha}\right]} \left\{ D(\nu \,\|\, \epsilon) - h_2\left(\nu + \frac{T}{\alpha}\right) + \log 2 - R \right\} \tag{B.0.2}$$

and $e_2(R,T) = e_1(R,T) + T$. Therefore, it holds that

$$\hat{E}_{\mathrm{f}}(R) = \lim_{e_1(R,T)\to 0} T. \tag{B.0.3}$$

Note that according to [54]:

$$E_{\mathrm{sp}}(R) = D(\delta_{\mathrm{GV}}(R) \,\|\, \epsilon), \quad \delta_{\mathrm{GV}}(C) = \epsilon, \quad C = \log 2 - h_2(\epsilon), \tag{B.0.4}$$

where $\delta_{\mathrm{GV}}(R)$ is the G-V distance, defined in Section 6.2.A.. Moreover, since $T \mapsto e_1(R,T)$ is continuous around zero, it will be enough to concentrate on the limiting threshold T_0, for which $e_1(R,T) = 0$. By Theorem 34, this happens when $R = \log 2 - h_2\left(\epsilon + \frac{T_0}{\alpha}\right)$, or, equivalently, when

$$T_0 = \alpha\left[h_2^{-1}(\log 2 - R) + \epsilon\right] = \alpha\left[\delta_{\mathrm{GV}}(R) - \epsilon\right] = \alpha\left[\delta_{\mathrm{GV}}(R) - \delta_{\mathrm{GV}}(C)\right] \tag{B.0.5}$$

where the definition of $\delta_{\mathrm{GV}}(R)$ and (B.0.4) were used in the first and second equality signs respectively. It then follows that:

$$\begin{aligned}
E_{\mathrm{Forney}}(R) &= E_{\mathrm{sp}}(R) + C - R && \text{(B.0.6)}\\
&= D(\delta_{\mathrm{GV}}(R) \,\|\, \epsilon) + \log 2 - h_2(\epsilon) - R && \text{(B.0.7)}\\
&= -h_2(\delta_{\mathrm{GV}}(R)) - \delta_{\mathrm{GV}}(R)\log\epsilon - [1 - \delta_{\mathrm{GV}}(R)]\log(1-\epsilon) && \text{(B.0.8)}\\
&+ \epsilon\log(\epsilon) + (1-\epsilon)\log(1-\epsilon) + \log 2 - R && \text{(B.0.9)}\\
&= \log\left(\frac{1-\epsilon}{\epsilon}\right)\left[\delta_{\mathrm{GV}}(R) - \delta_{\mathrm{GV}}(C)\right] && \text{(B.0.10)}
\end{aligned}$$

where in the last equality we have used the fact that $-h_2(\delta_{\mathrm{GV}}(R)) + \log 2 - R = 0$, which follows directly from the definition of $\delta_{\mathrm{GV}}(R)$. We therefore showed that indeed:

$$\hat{E}_{\mathrm{f}}(R) = E_{\mathrm{Forney}}(R) = \alpha\left[\delta_{\mathrm{GV}}(R) - \delta_{\mathrm{GV}}(C)\right]. \tag{B.0.11}$$

To conclude, we have seen how results from [134] can be harnessed to simplify calculations regarding Forney's ARQ scheme. This simplification is also apparent through the fact that the closed form expression for $e_1(R,T)$ and $e_2(R,T)$, which are calculated in [134], were not needed in our analysis. For a fixed R, the only interesting point T on the curves $e_1(R,T)$ and $e_2(R,T)$, was the one at which $e_1(R,T)$ becomes strictly positive.

Although this was only demonstrated for the BSC, the analysis tools used above can also be generalized to any DMC. In addition, for the BSC, this analysis gave rise to a simple alternative expression for $E_{\mathrm{Forney}}(R)$ in terms of the difference between the G-V distance at the capacity and the G-V distance at the communication rate R.

Appendix C

Lower Bounding the Error Exponent Function of Δ_a

Recall that the stop-feedback communication problem under random coding can be viewed as a sequential multiple hypothesis testing problem, where the hypotheses are defined by (6.4.6), and the uniform prior on $\{H_0, \ldots, H_{M-1}\}$. This structure bears a symmetry between the different hypotheses[1]. By using the symmetry between the different hypotheses, taking the $0-1$ loss function, defined in Section 4.3, and plugging in the definition of the hypotheses (6.4.6) in the definition of the sequential multiple hypothesis test defined as $\Delta_a = (N_a, d_a)$ in Section 4.3.A., the test Δ_a takes on the following form:

Test Δ_a applied to the stop-feedback problem: For all $i \in \{0, \ldots, M-1\}$ and any positive threshold value a, the stopping times N_i, defined in (4.3.5), take on the following form:

$$N_i = \min_{n \geq 0} \left\{ L_i(n) \geq a + \log \left(\sum_{j \neq i} \exp \{ L_j(n) \} \right) \right\} \tag{C.0.1}$$

$$= \min_{n \geq 0} \left\{ \log \left[\frac{P_i([\mathbf{z}]_n)}{\sum_{j \neq i} P_j([\mathbf{z}]_n)} \right] \geq a \right\} \tag{C.0.2}$$

$$= \min_{n \geq 0} \left\{ \log \left[\frac{P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(i)}\right)}{\sum_{j \neq i} P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)}\right)} \right] \geq a \right\} \tag{C.0.3}$$

[1]for obvious reasons, these hypotheses will sometimes be referred to as 'messages'.

where for an infinite sequence $\mathbf{w}$, $[\mathbf{w}]_n$ is defined the restriction of $\mathbf{w}$ to its first n instances, $L_i(n), i \in \{0, \ldots, M-1\}$ are the log-likelihood ratios at time n, defined in (4.3.1) for some dominating measure Q, and $P_i([\mathbf{z}]_n)$ and $P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(i)}\right)$ are defined in (6.4.6) and (6.4.5), respectively, for any $i \in \{0, \ldots, M-1\}$. The test procedure $\Delta_a = (N_a, d_a)$ is then defined as follows:

$$N_a = \min_{0 \leq k \leq M-1} N_k, \quad d_a = i \text{ if } N_a = N_i. \tag{C.0.4}$$

Analysis: Fix $M > 1$ and $a > 0$, where M is the number of messages, and a is the threshold for Δ_a. Note that in the asymptotic regime of interest, $M = M(\mathbb{E}[N_a])$ goes to infinity as $\mathbb{E}[N_a]$ does (so that $R = \frac{\log(M(\mathbb{E}[N_a]))}{\mathbb{E}[N_a]}$ is fixed), although this dependence of M on $\mathbb{E}[N]$ will be omitted for readability reasons. Later in this Appendix it will also be made evident that in order for Δ_a to satisfy $\mathbb{E}[N_a] \to \infty$, the threshold, a, should tend to infinity as M grows. In turn, this will imply that $p_e(\Delta_a) \to 0$ as M grows.

Throughout the analysis it will be assumed, without loss of generality, that the 0'th message was sent. In addition, note that by the definitions of N_a and $N_i, i \in \{0, \ldots, M-1\}$, $N_a \leq N_0 \quad$ a.s., and so

$$\mathbb{E}_0[N_a] \leq \mathbb{E}_0[N_0] = \sum_{n=1}^{\infty} P_0(N_0 \geq n) \tag{C.0.5}$$

where $\mathbb{E}_0[\cdot]$ denotes the expectation with respect to the P_0.

Let $\bar{n} \in \mathbb{N} \cup \{\infty\}$ be a positive integer. The value on $\bar{n}$ will be determined later. For any such $\bar{n}$, one can trivially bound $P_0(N_0 \geq n)$ for all $n \leq \bar{n}$, and get the following bound in $\mathbb{E}_0[N_a]$:

$$\mathbb{E}_0[N_a] \leq \bar{n} + \sum_{n=\bar{n}+1}^{\infty} P_0(N_0 \geq n). \tag{C.0.6}$$

Note that the derivation thus far was done for a general DMC. For simplicity, the BSC with crossover probability $0 < \epsilon < \frac{1}{2}$ will be taken to be the forward channel from this point and on. For this channel, results from [134] will be used to bound the error exponents of Δ_a, when applied to the stop-feedback coding problem. Specifically, define $\bar{n} \in \mathbb{N} \cup \{\infty\}$ to be:

$$\bar{n} \triangleq \max_{n \in \mathbb{N}} \left\{ \hat{R} \geq \log 2 - h_2\left(\epsilon + \frac{a}{n\beta}\right) - \delta_1 \right\} \tag{C.0.7}$$

where $\delta_1 > 0$ is an, arbitrarily small constant, a is the threshold of Δ_a, and $\hat{R} = \frac{\log(M)}{n}$.

Two important properties of $\bar{n}$, following (C.0.7), are the following:

1. The following equivalences hold:

$$\hat{R} \geq \log 2 - h_2\left(\epsilon + \frac{a}{n\beta}\right) - \delta_1 \Leftrightarrow \tag{C.0.8}$$

$$h_2\left(\epsilon + \frac{a}{n\beta}\right) \geq \log 2 - \delta_1 - \hat{R} \Leftrightarrow \tag{C.0.9}$$

$$n \leq \frac{a}{\beta\left(\delta_{\text{GV}}\left(\hat{R} + \delta_1\right) - \epsilon\right)} \Leftrightarrow \tag{C.0.10}$$

$$n \leq \frac{a}{E_{\text{Forney}}\left(\hat{R} + \delta_1\right)}, \tag{C.0.11}$$

where (C.0.10) is due to the definition of $\delta_{\text{GV}}(\cdot)$ in Section 6.2.A., and (C.0.11) was shown to hold for the BSC in Appendix B. Hence,

$$\bar{n} \leq \frac{a}{E_{\text{Forney}}\left(\bar{R}\right)} \tag{C.0.12}$$

where $\bar{R} \triangleq \frac{\log(M)}{\bar{n}}$.

2. From the definition of $\bar{n}$ in (C.0.7), we see that $\bar{n} = \bar{n}(M)$ is a function of the M. Moreover, $\bar{n}(M)$ increases with M and $\bar{n}(M) \to \infty$ as $M \to \infty$.

Next, using properties 1 and 2 above, $\mathbb{E}_0[N_a]$ will be further bounded. This will be done by closely examining the terms on the right-hand-side of (C.0.6). To that end, fix $n_0 > \bar{n}$ and denote the fraction $\frac{a}{n_0}$ by T_0. By the definition of $\bar{n}$, the following hold for each such n_0:

- Since $\bar{n} \to \infty$ as $M \to \infty$, and $n_0 > \bar{n}$, the same holds for n_0 as well.

- Using the definitions of T_0 and of the definition of the stopping time N_0 of the test procedure Δ_a, it holds that:

$$P_0\left(N_0 \geq n_0\right) \leq P_0\left(\log\left[\frac{P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_{n_0} \mid [\mathbf{x}]_{n_0}^{(i)}\right)}{\sum_{j\neq i} P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_{n_0} \mid [\mathbf{x}]_{n_0}^{(j)}\right)}\right] < a\right) \tag{C.0.13}$$

$$= P_0\left(\frac{P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_{n_0} \mid [\mathbf{x}]_{n_0}^{(i)}\right)}{\sum_{j\neq i} P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_{n_0} \mid [\mathbf{x}]_{n_0}^{(j)}\right)} < e^{n_0 T_0}\right). \tag{C.0.14}$$

- Since $n_0 > \bar{n}$,

$$R_0 < \log 2 - h_2\left(\epsilon + \frac{a}{n_0\alpha}\right) - \delta_1 \tag{C.0.15}$$
$$= \log 2 - h_2\left(\epsilon + \frac{T_0}{\alpha}\right) - \delta_1 \tag{C.0.16}$$
$$< \log 2 - h_2\left(\epsilon + \frac{T_0}{\beta}\right) \tag{C.0.17}$$

where $R_0 \triangleq \frac{\log(M)}{n_0}$.

Interestingly, the event

$$\mathcal{E}_1\left(n_0, T_0\right) \triangleq \left\{\frac{P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_{n_0} \mid [\mathbf{x}]_{n_0}^{(i)}\right)}{\sum_{j\neq i} P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_{n_0} \mid [\mathbf{x}]_{n_0}^{(j)}\right)} < e^{n_0 T_0}\right\} \tag{C.0.18}$$

is exactly the event that Forney's erasure decoder, will not make the right decision when a random code of block length n_0 is used, and the threshold is $T_0 > 0$. As mentioned in Section 6.2.A., the exact asymptotic exponential behavior of $P_0\left(\mathcal{E}_1\left(n_0, T_0\right)\right)$ was found in [134], for any DMC. Following [134], define

$$e_1\left(n_0, T_0\right) = \limsup_{n_0\to\infty} -\frac{1}{n_0}\log P_0\left[\mathcal{E}_1\left(n_0, T_0\right)\right]. \tag{C.0.19}$$

Recall that the discussion was restricted to the BSC. For this case, we know, using Theorem 34, that if $R_0 \geq \log 2 - h_2\left(\epsilon + \frac{T_0}{\alpha}\right)$, $e_1\left(R_0, T_0\right) = 0$, and otherwise[2]

$$e_1\left(R_0, T_0\right) \triangleq \min_{\nu\in\left[\epsilon, \delta_{\mathrm{GV}}(R_0) - \frac{T_0}{\alpha}\right]} \left\{D\left(\nu \parallel \epsilon\right) - h_2\left(\nu + \frac{T_0}{\alpha}\right) + \log 2 - R_0\right\} > \delta_1. \tag{C.0.20}$$

Using this result, and keeping in mind that for any $n_0 > \bar{n}$, (C.0.14) and (C.0.20) hold, the following is implied:

1. For an arbitrarily small δ_1, there exists a large enough M such that

$$P_0\left(N_0 \geq n_0\right) \leq \exp\left\{-\delta_1 n_0 + O\left(\log n_0\right)\right\}. \tag{C.0.21}$$

[2]Note that for the BSC case, the closed form expression for $c\left(n_0\right)$ was found in [134], but in order to prove (6.4.27), this expression will not be needed.

2. Since (C.0.21) holds for any $n_0 > \bar{n}$, $\sum_n P_0(N_0 \geq n)$ converges. In addition, $\bar{n} \to \infty$ as $M \to \infty$, and so for an arbitrarily small δ_2, there exists a large enough M such that

$$\sum_{n=\bar{n}+1}^{\infty} P_0(N_0 \geq n) < \delta_2, \tag{C.0.22}$$

as a tail of a converging sum.

3. Plugging in (C.0.22) into (C.0.6), we get $\mathbb{E}_0[N_a] \leq \bar{n} + \delta_2$ for an arbitrarily small δ_2, and so for any $\delta_3 > 0$, there exists a large enough M such that

$$\bar{R} - \delta_3 = \frac{\log M}{\bar{n} + \delta_2} \leq \frac{\log M}{\mathbb{E}_0[N_a]} = R. \tag{C.0.23}$$

Since $E_{\text{Forney}}(\cdot)$ is monotonically decreasing for $\cdot \in [0, C]$, and $a > 0$,

$$\frac{a}{E_{\text{Forney}}\left(\bar{R} - \delta_3 + \delta_1\right)} \leq \frac{a}{E_{\text{Forney}}(R + \delta_1)}, \tag{C.0.24}$$

and since $E_{\text{Forney}}(\cdot)$ is also differentiable (and so is $x \mapsto \frac{1}{x}$ for $x > 0$), it follows that for any arbitrarily small constant $\delta_4 > 0$, there exists large enough M such that

$$\frac{a}{E_{\text{Forney}}\left(\bar{R} - \delta_3 + \delta_1\right)} = \frac{a}{E_{\text{Forney}}\left(\bar{R} + \delta_1\right)} - \delta_4. \tag{C.0.25}$$

Combining (C.0.6), (C.0.22) and (C.0.25) and using the differentiability of $E_{\text{Forney}}(\cdot)$ again yields the following bound on $\mathbb{E}_0[N_0]$

$$\mathbb{E}_0[N_a] \leq \frac{a}{E_{\text{Forney}}(R)} + \delta \tag{C.0.26}$$

where $\delta > 0$ can be made arbitrarily small for large enough M.

In order to evaluate a, note that the following chain of inequalities holds:

$$\begin{aligned}
p_e(\Delta_a) &= \sum_{i=0}^{M-1} \sum_{j=0, j\neq i}^{M-1} \pi_j P_j(N_a = N_i, N_i < \infty) && \text{(C.0.27)}\\
&= \sum_{j=1}^{M-1} P_j(N_a = N_0, N_0 < \infty) && \text{(C.0.28)}\\
&= \mathbb{E}_0\left[\mathbb{I}\{N_a = N_0, N_0 < \infty\} \frac{\sum_{j=1}^{M-1} P_j(\mathbf{z})}{P_0(\mathbf{z})}\right] && \text{(C.0.29)}\\
&\leq e^{-a} && \text{(C.0.30)}
\end{aligned}$$

where (C.0.28) is due to symmetry, and (C.0.29) follows from the definition of Δ_a. This result is also evident using Lemma 6, and keeping in mind the same type of symmetry. Using (C.0.30) we finally get

$$\mathbb{E}_0\left[N_a\right] \leq \frac{-\log p_e\left(\Delta_a\right)}{E_{\text{Forney}}\left(R\right)} + \delta. \tag{C.0.31}$$

Since $\delta > 0$ is arbitrarily small in the asymptotic regime of interest, it follows, using the error exponent definition in Section 5.2.E., that

$$E\left(R\right) \geq E_{\text{Forney}}\left(R\right), \tag{C.0.32}$$

which proves (6.4.27).

From the derivations done above, one can also learn the following:

- Since, by (C.0.6),
$$\mathbb{E}\left[N_a\right] \leq \frac{a}{E_{\text{Forney}}\left(R\right)} + \delta_2, \tag{C.0.33}$$
it follows that for $\mathbb{E}\left[N_a\right] \to \infty$ to hold (which is required in the asymptotic regime of interest), one must take $a \to \infty$. Keeping this observation in mind, the definition $T_0 = \frac{a}{n_0}$ for every $n_0 > \bar{n}$, makes sense. In addition, by (C.0.30), we see that the requirement that the error probability goes to zero, implies, again, that $a \to \infty$. We conclude that for the sequential test Δ_a, the threshold a is tunable and it determines the tradeoff between $\mathbb{E}\left[N_a\right]$ and the error probability $p_e\left(\Delta_a\right)$, and the asymptotic regime of interest is that where a is taken to be large.

- Note that both (C.0.6) and (C.0.14) hold for a general DMC as well. Moreover, as in Appendix B, the exact error exponents of the error events, defined for the erasure-decoder, where not used. Instead, the focus was on a single point on the $R \mapsto e_1\left(R, T\right)$ curve - the point at which $e_1\left(R, T\right)$ first becomes non-zero. This implies that the same techniquehere, can also be invoked for more complicated channel models, where closed form results for the error exponents of the erasure-decoder were not yet found.

Appendix D

Gallager-Type Lower Bound on the Error Exponent of Δ_a

In this appendix we give an alternative proof that the best achievable error exponent in the stop-feedback setup satisfies $E(R) \geq E_{\text{Forney}}(R)$. As in Appendix C, this will done by utilizing Δ_a . This time, the proof is a natural extension of Forney's proof of the achievability of this error exponent in [53], except that here we will not restrict ourselves to block codes. A similar idea was also presented in [69].where Hashimoto showed that to obtain $E_{\text{Forney}}(R)$ in an ARQ setup (using block codes), it is enough to use a decoding algorithm similar to that of Forney, with the exception that instead of using the test statistics

$$\left\{ \log \left[\frac{P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(i)} \right)}{\sum_{j \neq i} P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)} \right)} \right] \right\}, \quad i = 0, \ldots, M-1, \tag{D.0.1}$$

it is enough to use:

$$\left\{ \log \left[\frac{P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(i)} \right)}{\left[\sum_{j \neq i} P_{\mathbf{Y}|\mathbf{X}}^{1/(\rho+1)}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)} \right) \right]^{\rho+1}} \right] \right\}, \quad i = 0, \ldots, M-1, \tag{D.0.2}$$

where $\rho > 0$ is a parameter of the algorithm, that can be chosen either arbitrarily or by an optimization procedure. Akin to Forney (and to Gallager's classic derivation of error exponents), we use the inequality $\sum_i a_i \leq \left(\sum_i a_i^{\lambda} \right)^{1/\lambda}$, that holds for any

$0 < \lambda < 1$, to show that for any DMC, any $\rho > 0$, and any $i \in \{0, \ldots, M-1\}$,

$$\log \left[\frac{P_{\mathbf{Y}|\mathbf{X}} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(i)} \right)}{\sum_{j \neq i} P_{\mathbf{Y}|\mathbf{X}} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)} \right)} \right] \geq \log \left[\frac{P_{\mathbf{Y}|\mathbf{X}} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(i)} \right)}{\left[\sum_{j \neq i} P_{\mathbf{Y}|\mathbf{X}}^{1/(\rho+1)} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)} \right) \right]^{\rho+1}} \right] \quad a.s., \tag{D.0.3}$$

and therefore,

$$P_0 \left(\frac{P_{\mathbf{Y}|\mathbf{X}} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)} \right)}{\sum_{j \neq 0} P_{\mathbf{Y}|\mathbf{X}} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)} \right)} < e^a \right) \leq P_0 \left(\frac{P_{\mathbf{Y}|\mathbf{X}} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)} \right)}{\left[\sum_{j \neq 0} P_{\mathbf{Y}|\mathbf{X}}^{1/(\rho+1)} \left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)} \right) \right]^{\rho+1}} \leq e^a \right). \tag{D.0.4}$$

To calculate an upper bound on the error exponent, we continue the derivation in C from (C.0.13) using (D.0.4) and get, for any $n \in \mathbb{N}$

$$P_0\left(N_0 \geq n\right) \leq P_0\left(\frac{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)}{\sum_{j\neq 0} P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)} < e^a\right) \tag{D.0.5}$$

$$\leq P_0\left(\frac{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)}{\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right]^{\rho+1}} \leq e^a\right) \tag{D.0.6}$$

$$= \sum_{\left[\mathbf{z}\right]_n \in \mathcal{Z}^n} P_0\left(\left[\mathbf{z}\right]_n\right) \mathbb{I}\left\{\frac{e^a\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right]^{\rho+1}}{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)} \geq 1\right\} \tag{D.0.7}$$

$$\leq \sum_{\left[\mathbf{z}\right]_n \in \mathcal{Z}^n} P_0\left(\left[\mathbf{z}\right]_n\right) \left(\frac{e^a\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right]^{\rho+1}}{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)}\right)^s \tag{D.0.8}$$

$$= e^{sa} \sum_{\left[\mathbf{z}\right]_n \in \mathcal{Z}^n} P_0\left(\left[\mathbf{z}\right]_n\right) \left(\frac{\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right]^{\rho+1}}{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)}\right)^s \tag{D.0.9}$$

$$= e^{sa} \sum_{\left[\mathbf{y}\right]_n, \left[\mathbf{x}\right]_n^{(0)}} P_X\left(\left[\mathbf{x}\right]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right) \times \tag{D.0.10}$$

$$\left[\sum_{\left[\mathbf{x}\right]_n^{(l)}, l=1\ldots, M-1} \prod_{l=1}^{M-1} P_X\left(\left[\mathbf{x}\right]_n^{(l)}\right) \left(\frac{\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right]^{\rho+1}}{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)}\right)^s\right] \tag{D.0.11}$$

$$= e^{sa} \sum_{\left[\mathbf{y}\right]_n, \left[\mathbf{x}\right]_n^{(0)}} P_X\left(\left[\mathbf{x}\right]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right) \times \tag{D.0.12}$$

$$\mathbb{E}_{\left[\mathbf{x}\right]_n^{(l)}, l=1\ldots M-1}\left[\left(\frac{\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right]^{\rho+1}}{P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)}\right)^s\right] \tag{D.0.13}$$

$$= e^{sa} \sum_{\left[\mathbf{y}\right]_n, \left[\mathbf{x}\right]_n^{(0)}} P_X\left(\left[\mathbf{x}\right]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right) \frac{1}{P^s_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(0)}\right)} \times \tag{D.0.14}$$

$$\mathbb{E}_{\left[\mathbf{x}\right]_n^{(l)}, l=1\ldots M-1}\left[\left(\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left(\left[\mathbf{y}\right]_n \mid \left[\mathbf{x}\right]_n^{(j)}\right)\right)^{s(\rho+1)}\right] \tag{D.0.15}$$

$$\leq e^{sa} \sum_{[\mathbf{y}]_n, [\mathbf{x}]_n^{(0)}} P_X\left([\mathbf{x}]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right) \frac{1}{P^s_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right)} \times \tag{D.0.16}$$

$$\left(\mathbb{E}_{[\mathbf{x}]_n^{(l)}, l=1\ldots M-1}\left[\sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)}\right)\right]\right)^{s(\rho+1)} \tag{D.0.17}$$

$$= e^{sa} \sum_{[\mathbf{y}]_n, [\mathbf{x}]_n^{(0)}} P_X\left([\mathbf{x}]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right) \frac{1}{P^s_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right)} \times \tag{D.0.18}$$

$$\left(\sum_{[\mathbf{x}]_n^{(l)}, l=1\ldots, M-1} \prod_{l=1}^{M-1} P_X\left([\mathbf{x}]_n^{(l)}\right) \sum_{j\neq 0} P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)}\right)\right)^{s(\rho+1)} \tag{D.0.19}$$

$$= e^{sa} \sum_{[\mathbf{y}]_n, [\mathbf{x}]_n^{(0)}} P_X\left([\mathbf{x}]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right) \frac{1}{P^s_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right)} \times \tag{D.0.20}$$

$$\left(\sum_{j\neq 0}\left[\sum_{[\mathbf{x}]_n^{(l)}, l=1\ldots, M-1} P_X\left([\mathbf{x}]_n^{(l)}\right) P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)}\right)\right]\right)^{s(\rho+1)} \tag{D.0.21}$$

$$= e^{sa}\left(M-1\right)^{s(1+\rho)} \sum_{[\mathbf{y}]_n, [\mathbf{x}]_n^{(0)}} P_X\left([\mathbf{x}]_n^{(0)}\right) P_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right) \frac{1}{P^s_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(0)}\right)} \times \tag{D.0.22}$$

$$\left(\sum_{[\mathbf{x}]_n^{(l)}, l=1\ldots, M-1} P_X\left([\mathbf{x}]_n^{(l)}\right) P^{1/(\rho+1)}_{\mathbf{Y}|\mathbf{X}}\left([\mathbf{y}]_n \mid [\mathbf{x}]_n^{(j)}\right)\right)^{s(\rho+1)} \tag{D.0.23}$$

$$= e^{sa}\left(M-1\right)^{s(1+\rho)} \times \tag{D.0.24}$$

$$\left[\sum_{x,y\in\mathcal{X}\times\mathcal{Y}} P_X\left(x\right) p\left(y \mid x\right) \frac{1}{p^s\left(y \mid x\right)} \left(\sum_{x\in\mathcal{X}} P_X\left(x\right) p^{1/(\rho+1)}\left(y \mid x\right)\right)^{s(\rho+1)}\right]^n \tag{D.0.25}$$

$$= e^{sa}\left(M-1\right)^{s(1+\rho)} \mathbb{E}_{X,Y}\left[\frac{1}{p^s\left(Y \mid X\right)} \left(\sum_{x\in\mathcal{X}} P_X\left(x\right) p^{1/(\rho+1)}\left(Y \mid x\right)\right)^{s(\rho+1)}\right]^n \tag{D.0.26}$$

where (D.0.8) holds for any $s > 0$ and in (D.0.11) the definition of P_0 was used. The operator $\mathbb{E}_{[\mathbf{x}]_n^{(l)}, l=1\ldots, M-1}\left[\cdot\right]$ in (D.0.13) is the expectation taken with respect to $\prod_{l=1}^{M-1} P_X\left(\mathbf{x}^{(\mathbf{l})}\right)$. In (D.0.19) Jensen's inequality was used, (D.0.23) is due the symmetry of the coding procedure and the channel, (D.0.25) is since the channel is memoryless and since random coding is used, and so both $P_{\mathbf{Y}|\mathbf{X}}\left(\cdot \mid \cdot\right)$ and $P_X\left(\cdot\right)$ have a product form, where all the products have the same statistics, and in (D.0.26) $\mathbb{E}_{X,Y}[\cdot]$ is the expectation over the joint measure of X and Y. We now use the fact that we assume a BSC with crossover probability $0 < \epsilon < \frac{1}{2}$. Since this channel is symmetric, we have chosen the singleton probability to be $P_X(x) = \frac{1}{2}$ for $x \in \mathcal{X} = \{0,1\}$. In addition, by the definition of the BSC, the random variable $p(Y \mid X)$ is equal to the following:

$$p(Y \mid X) = (1-\epsilon) e^{-\alpha X \oplus Y}. \tag{D.0.27}$$

Moreover, note that for any realization of Y

$$\sum_{x \in \mathcal{X}} P_X(x) p^{1/(\rho+1)}(Y \mid x) = \frac{1}{2}\left[p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right]. \tag{D.0.28}$$

Since the right hand side of (D.0.28) does not depend on Y, we can simplify (D.0.26) and get:

$$P_0(N_0 \geq n) \tag{D.0.29}$$

$$\leq e^{sa}(M-1)^{s(1+\rho)}\left[\frac{1}{2}\left(p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right)\right]^{ns(\rho+1)} \mathbb{E}_{X,Y}\left[\frac{1}{p^s(Y \mid X)}\right]^n \tag{D.0.30}$$

$$= \exp\left\{s\left[a + (1+\rho)\log(M-1) + n(1+\rho)\left(\log\left(p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right) - \log 2\right)\right]\right\} \mathbb{E}_{X,Y}\left[\frac{1}{p^s(Y \mid X)}\right]^n. \tag{D.0.31}$$

Using (D.0.27) we can carry out the expectation in (D.0.31) as follows:

$$\mathbb{E}_{X,Y}\left[\frac{1}{p^s(Y \mid X)}\right] = (1-\epsilon)^{1-s} + (1-\epsilon)^{-s} e^{\alpha s}\epsilon \tag{D.0.32}$$

$$= (1-\epsilon)^{1-s} + (1-\epsilon)^{-s}\left(\frac{1-\epsilon}{\epsilon}\right)^s \epsilon \tag{D.0.33}$$

$$= (1-\epsilon)^{1-s} + \epsilon^{1-s}. \tag{D.0.34}$$

Define the function $g(s,n)$ to be:

$$g(s,n) \triangleq a + (1+\rho)\log(M-1) + n(1+\rho)\log\left(p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right) \tag{D.0.35}$$

$$- n(1+\rho)\log(2) + n\frac{\log\left((1-\epsilon)^{1-s} + \epsilon^{1-s}\right)}{s}. \tag{D.0.36}$$

Using this definition we can rewrite the above bound as $P_0\left(N_0 \geq n\right) \leq e^{sg(s,n)}$.

Combining this with (C.0.6) (that holds for any $\bar{n} \in \mathbb{R}$) we get

$$\mathbb{E}_0\left[N_a\right] \leq \bar{n} + \sum_{n=\bar{n}+1}^{\infty} P_0\left(N_0 > n\right) \leq \bar{n} + \sum_{n=\bar{n}+1}^{\infty} e^{sg(s,n)}. \tag{D.0.37}$$

Analogous to the proof in Appendix C, let $s > 0$ be a parameter to be defined later, and define $\bar{n}$ to be[1]

$$\bar{n} \triangleq \max_{n\in\mathbb{R}} \left\{g\left(s,n\right) > 0\right\}. \tag{D.0.38}$$

Carrying out the sum in (D.0.37) yields:

$$\sum_{n=\bar{n}+1}^{\infty} e^{sg(s,n)} = \exp\left\{s\left[a + (1+\rho)\log\left(M-1\right)\right]\right\} \sum_{n=\bar{n}+1}^{\infty} \exp\Bigg\{-ns\Bigg[(1+\rho)\log 2 - \frac{\log\left((1-\epsilon)^{1-s} + \epsilon^{1-s}\right)}{s} - (1+\rho)\log\left(p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right)\Bigg]\Bigg\}. \tag{D.0.39}$$

The following lemma will help simplify the above expression:

Lemma 39 *The function* $s \mapsto \frac{\log\left[(1-\epsilon)^{1-s}+\epsilon^{1-s}\right]}{s}$ *satisfies:*

$$\frac{\log\left[(1-\epsilon)^{1-s} + \epsilon^{1-s}\right]}{s} = -h_2\left(\epsilon\right) + O\left(s\right). \tag{D.0.40}$$

Proof. The proof follows the Taylor series expansion of the numerator of the function, i.e., $s \mapsto \log\left((1-\epsilon)^{1-s} + \epsilon^{1-s}\right)$, around $s = 0$. Specifically, note that

$$\frac{d}{ds}\left[(1-\epsilon)^{1-s} + \epsilon^{1-s}\right]\Bigg|_{s=0} = -h_2\left(\epsilon\right). \tag{D.0.41}$$

□

Using this lemma, the coefficient of $-n$ in the exponent's argument can be written as:

$$s\left[\log 2 - h_2\left(\epsilon\right)\right] + s\left[\rho\log 2 - (1+\rho)\log\left(p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right)\right] + O\left(s\right). \tag{D.0.42}$$

As in [144], define the function

$$E_0\left(\rho\right) = \left[\rho\log 2 - (1+\rho)\log\left(p^{1/(\rho+1)} + (1-p)^{1/(\rho+1)}\right)\right]. \tag{D.0.43}$$

Using these results, we can make the following simplifications:

[1]Since $g\left(s,n\right)$ is linear in n, $\bar{n}$ exists and is finite.

1. The function $g(s,n)$ can be written as

$$g(s,n) = a + (1+\rho)\log(M-1) - n\left[C + E_0(\rho) + O(1)\right], \tag{D.0.44}$$

and so, for as given s,

$$\bar{n} = \max_{n\in\mathbb{R}}\left\{n \le \frac{a + (1+\rho)\log(M-1)}{C + E_0(\rho) + O(1)}\right\} \le \frac{a + (1+\rho)\log(M-1)}{C + E_0(\rho) + O(1)} \tag{D.0.45}$$

2. The expression in (D.0.39) can be written as:

$$\sum_{n=\bar{n}+1}^{\infty} e^{sg(s,n)} = \exp\left\{s\left[a + (1+\rho)\log(M-1)\right]\right\} \times$$
$$\sum_{n=\bar{n}+1}^{\infty} \exp\left\{-ns\left[C + E_0(\rho) + O(1)\right]\right\}. \tag{D.0.46}$$

3. Choosing s to be small enough, the expression $C + E_0(\rho) + O(1)$ can be made positive. Carrying out the sum in (D.0.46) then yields:

$$\sum_{n=\bar{n}+1}^{\infty} e^{sg(s,n)} = \exp\left\{s\left[a + (1+\rho)\log(M-1)\right]\right\} \times$$
$$\frac{\exp\left\{-(\bar{n}+1)s\left[C + E_0(\rho) + O(1)\right]\right\}}{1 - \exp\left\{-s\left[C + E_0(\rho) + O(1)\right]\right\}} \tag{D.0.47}$$
$$= \frac{\exp\left\{sg(s,\bar{n}+1)\right\}}{1 - \exp\left\{-s\left[C + E_0(\rho) + O(1)\right]\right\}} \tag{D.0.48}$$
$$\le \frac{1}{1 - \exp\left\{-s\left[C + E_0(\rho) + O(1)\right]\right\}} \tag{D.0.49}$$

where the last inequality follows from the definition of $\bar{n}$.

In addition, recall that in (C.0.30) we have shown that for the decoding scheme at hand, $a \le -\log p_e$ where p_e is the average error probability. Combining these results with (D.0.37) yields:

$$\mathbb{E}_0\left[N_a\right] \le \bar{n} + \sum_{n=\bar{n}+1}^{\infty} e^{sg(s)} \tag{D.0.50}$$
$$\le \frac{-\log(p_e) + (1+\rho)\log(M-1)}{C + E_0(\rho) + O(1)} \tag{D.0.51}$$
$$+ \frac{1}{1 - \exp\left\{-s\left[C + E_0(\rho) + O(1)\right]\right\}}, \tag{D.0.52}$$

or

$$\frac{-\log p_e}{\mathbb{E}_0\left[N_a\right]} \geq C + E_0\left(\rho\right) + O\left(1\right) - \left(1+\rho\right)\frac{\log\left(M-1\right)}{\mathbb{E}_0\left[N_a\right]} \tag{D.0.53}$$

$$- \frac{1}{\mathbb{E}_0\left[N_a\right]\left[1-\exp\left\{-s\left[C+E_0\left(\rho\right)+O\left(1\right)\right]\right\}\right]} \tag{D.0.54}$$

$$\geq C + E_0\left(\rho\right) + O\left(1\right) - \left(1+\rho\right)\frac{\log\left(M\right)}{\mathbb{E}_0\left[N_a\right]} \tag{D.0.55}$$

$$- \frac{1}{\mathbb{E}_0\left[N_a\right]\left[1-\exp\left\{-s\left[C+E_0\left(\rho\right)+O\left(1\right)\right]\right\}\right]}. \tag{D.0.56}$$

Choosing $s = \frac{1}{\mathbb{E}_0[N_a]}$, taking $\mathbb{E}_0\left[N_a\right]$ to infinity[2] and using the definitions of R $E\left(R\right)$, the last inequality can be written as:

$$E\left(R\right) \geq C + E_0\left(\rho\right) + \left(1+\rho\right)R = E_0\left(\rho\right) - \rho R + C - R. \tag{D.0.57}$$

Optimizing over ρ yields:

$$E\left(R\right) \geq \sup_{\rho\geq 0}\left[E_0\left(\rho\right) - \rho R\right] + C - R = E_{\text{sp}}\left(R\right) + C - R \equiv E_{\text{Forney}}\left(R\right) \tag{D.0.58}$$

where in the first equality we have used the definition of $E_{\text{sp}}\left(R\right)$ [144].

[2]Note that taking $\mathbb{E}_0\left[N_a\right]$ to infinity and the error probability to zero is the asymptotic regime of interest. In addition, under this asymptotic regime $s \to 0$ as was assumed throughout this section

Appendix E

Proofs For Section 6.4.C.

Consider the hypothesis testing problem (6.4.6) for the BSC with crossover probability of ϵ, and let $\bar{n}$ and a be as defined in Section 6.4.C. Assume W.L.O.G that the message zero was sent through the channel, and let $\left[\mathbf{x}^{(i)}\right]_n$ and $[\mathbf{y}]_n$ be the first n symbols of the i'th codeword and the received sequence respectively. Define $\delta_i(n) \triangleq d_H\left(\left[\mathbf{x}^{(\mathbf{i})}\right]_n, [\mathbf{y}]_n\right)/n$ where $d_H(x, y)$ is the Hamming distance between the sequences x and y.

Using the result of Section 6.4.B., we know that there exists at least one sequential test Δ_a, that achieves a non negative error exponent for rates below capacity. Therefore, defining $\Delta = (N, d)$ to be the *optimal* sequential test in the the error exponent sense, we can conclude that there exists a non-negative function $E \mapsto (0, C)$ such that for large enough $\mathbb{E}[N]$,

$$-\log p_e(\Delta) = \mathbb{E}[N]\, E(R) + o(\mathbb{E}[N]). \tag{E.0.1}$$

For brevity, reasons we omit the $o(\mathbb{E}[N])$ term of $-\log p_e(\Delta)$. This factor will play no role in the final results as $\mathbb{E}[N]$ will be taken to infinity. In this section, three claims will be stated and proven. These claims, together with (6.4.42), will then be used in Section 6.4.C. to upper bound the error exponent.

Claim E.1 *Let a and $\Lambda_0(\cdot)$ be as defined as in Section 6.C. Then, for any fixed $\mathbb{E}[N]$ there exists an $n_0 = n_0(\mathbb{E}[N])$ such that for any $n \le n_0$,*

$$P_0(\Lambda_0(n) \ge a) = 0. \tag{E.0.2}$$

Proof. For a memoryless BSC

$$\Lambda_0(n) = \log\left[\frac{p\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(0)}\right]_n\right)}{\sum_{j>0} p\left([\mathbf{y}]_n \mid [\mathbf{x}^{(j)}]_n\right)}\right] \leq \log\left(\frac{1}{e^{\mathbb{E}[N]R}\epsilon^n}\right). \tag{E.0.3}$$

Therefore, if

$$a > \log\left(\frac{1}{e^{\mathbb{E}[N]R}\epsilon^n}\right) = -\mathbb{E}[N]R - n\log\epsilon, \tag{E.0.4}$$

then (E.0.2) holds. Combining this condition with (E.0.1) and the definition of a, we get

$$a = (1-\epsilon_1)\left[-\log\left(p_e(\Delta)\right)\right] = (1-\epsilon_1)\,\mathbb{E}[N]\,E(R) > -\mathbb{E}[N]R - n\log\epsilon, \tag{E.0.5}$$

and so the claim holds for

$$n_0 \triangleq \mathbb{E}[N]\frac{\left[(1-\epsilon_1)E(R)+R\right]}{-\log(\epsilon)}. \tag{E.0.6}$$

□

Corollary 40 *For the optimal sequential test Δ, it holds that*

$$\frac{((1-\epsilon_1)E(R)+R)}{-\log(\epsilon)(1+\epsilon_2)} < 1 \tag{E.0.7}$$

Proof. Assume Assume the converse is true, then

$$\sum_{n\leq\bar{n}} P_0\left(\Lambda_0(n) > a\right) = \sum_{n=n_0}^{\mathbb{E}[N](1+\epsilon_2)} P_i\left(\Lambda_i(n) > a\right) \equiv 0, \tag{E.0.8}$$

and so, by (6.4.42) $\frac{1}{1+\epsilon_2} \geq 1$, which is a contradiction since the left hand side is strictly less than 1 for any $\epsilon_2 > 0$. □

Note that for any $\epsilon_1, \epsilon_2 > 0$ the former corollary also implies the following upper bound:

$$E(R) < \frac{\left[\log(1/\epsilon) - R\right](1+\epsilon_2)}{1-\epsilon_1}. \tag{E.0.9}$$

However, in Section 6.4.B. this bound turns out to be loose, and a tighter bound will be given using the results in this appendix. Towards that end, define $T \triangleq (1-\epsilon_1)E(R)$

and note that

$$P_0\left(\Lambda_0\left(n\right)\geq a\right)=P_0\left(\log\left[\frac{p\left(\left[\mathbf{y}\right]_n\mid\left[\mathbf{x}^{(0)}\right]_n\right)}{\sum_{j>0}p\left(\left[\mathbf{y}\right]_n\mid\left[\mathbf{x}^{(j)}\right]_n\right)}\right]\geq a\right)\tag{E.0.10}$$

$$=P_0\left(\frac{p\left(\left[\mathbf{y}\right]_n\mid\left[\mathbf{x}^{(0)}\right]_n\right)}{\sum_{j>0}p\left(\left[\mathbf{y}\right]_n\mid\left[\mathbf{x}^{(j)}\right]_n\right)}\geq e^{\mathbb{E}[N]T}\right)\tag{E.0.11}$$

$$=P_0\left(\sum_{j>0}p\left(\left[\mathbf{y}\right]_n\mid\left[\mathbf{x}^{(0)}\right]_n\right)\leq e^{-\mathbb{E}[N]T}p\left(\left[\mathbf{y}\right]_n\mid\left[\mathbf{x}^{(0)}\right]_n\right)\right)\tag{E.0.12}$$

$$=P_0\left(\sum_{d=0}^{n}N_{[\mathbf{y}]_n}\left(d\right)e^{-\beta d}\leq e^{-\mathbb{E}[N]T}e^{-n\beta\delta_0(n)}\right)\tag{E.0.13}$$

$$=P_0\left(\sum_{\delta}N_{[\mathbf{y}]_n}\left(n\delta\right)e^{-n\beta\delta}\leq e^{-\mathbb{E}[N]T}e^{-n\beta\delta_0(n)}\right)\tag{E.0.14}$$

$$\leq P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right)e^{-n\beta\bar{\delta}}\leq e^{-\mathbb{E}[N]T}e^{-n\beta\delta_0(n)}\right)\tag{E.0.15}$$

$$=P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right)\leq e^{-\mathbb{E}[N]T}e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)}\right)\tag{E.0.16}$$

where $N_{[\mathbf{y}]_n}\left(d\right)$ is defined as the number of codewords among the set $\left\{\left[\mathbf{x}^{(j)}\right]_n, j\neq i\right\}$ at Hamming distance d from $\left[\mathbf{y}\right]_n$, and in (E.0.14) we have defined the set $\{\delta\}\triangleq\left\{\frac{d}{n}, n=0\ldots d\right\}$. In (E.0.15) we took $\bar{\delta}$ to be some member of $\{\delta\}$ and the inequality holds since all the $\left\{N_{[\mathbf{y}]_n}\left(n\delta\right)e^{-n\beta\delta}\right\}$'s in the sum are non-negative random variables. Specifically, let $\bar{\delta}$ be defined as:

$$\bar{\delta}\triangleq\underset{\delta\in\mathcal{G}_{\bar{R}}}{\operatorname{argmax}}\ \left\{h_2\left(\delta\right)-\beta\delta\right\}\tag{E.0.17}$$

where $\bar{R}=\frac{R}{1+\epsilon_2}$ and

$$\mathcal{G}_{\bar{R}}\triangleq\left\{\delta:\delta_{\mathrm{GV}}\left(\bar{R}\right)\leq\delta\leq 1-\delta_{\mathrm{GV}}\left(\bar{R}\right)\right\}\tag{E.0.18}$$

In order to further bound $P_0\left(\Lambda_0\left(n\right)\geq a\right)$, define, for any positive number κ, the set $\mathcal{B}_\kappa$ as:

$$\mathcal{B}_\kappa\triangleq\left\{\delta:\left|\delta-\epsilon\right|<\kappa\right\}.\tag{E.0.19}$$

Conditioning on the event $\left\{\left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n : \delta_0(n) \in \mathcal{B}_\kappa\right\}$,

$$
\begin{aligned}
P_0\left(\Lambda_0(n) \geq a\right) &\leq P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)}\right) && \text{(E.0.20)}\\
&= P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n : \delta_0(n) \in \mathcal{B}_\kappa\right)\\
&\qquad \times P_0\left(\delta_0(n) \in \mathcal{B}_\kappa\right) && \text{(E.0.21)}\\
&+ P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n : \delta_0(n) \notin \mathcal{B}_\kappa\right)\\
&\qquad \times P_0\left(\delta_0(n) \notin \mathcal{B}_\kappa\right) && \text{(E.0.22)}\\
&\leq P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n : \delta_0(n) \in \mathcal{B}_\kappa\right)\\
& && \text{(E.0.23)}\\
&+ P_0\left(\delta_0(n) \notin \mathcal{B}_\kappa\right). && \text{(E.0.24)}
\end{aligned}
$$

We will continue by bounding (E.0.23) and (E.0.24) separately, staring with (E.0.23), and making use of the following claim:

Claim E.2 *For any $\epsilon_2 > 0$ and $\delta \in \mathcal{G}_{\bar{R}}$ $\epsilon < \delta$, and, there exists a number $\kappa_0 > 0$ such that for any $\kappa \in (0, \kappa_0)$, $\epsilon + \kappa < \delta$.*

The proof will be given at the end of this appendix. By the definition of $\bar{\delta}$, $\bar{\delta} \in \mathcal{G}_{\bar{R}}$ and hence, by Claim E.2, there exists a $\kappa \in (0, \kappa_0)$ such that $\bar{\delta} - \epsilon - \kappa > 0$. Moreover, if we restrict n only to the case where $n_0 \leq n \leq \mathbb{E}[N](1+\epsilon_2)$ (which is the only interesting values corresponding to (6.4.42)) and pairs $\left\{\left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n\right\}$ such that $\delta_0(n) \in \mathcal{B}_\kappa$ (as in (E.0.23)), then

$$
\begin{aligned}
&P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n\right) \leq && \text{(E.0.25)}\\
&P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-\mathbb{E}[N](1+\epsilon_2)\beta\left(\epsilon-\kappa-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n\right) \leq && \text{(E.0.26)}\\
&P_0\left(N_{[\mathbf{y}]_{(1+\epsilon_2)\mathbb{E}[N]\bar{\delta}}}\left((1+\epsilon_2)\,\mathbb{E}[N]\,\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-\mathbb{E}[N](1+\epsilon_2)\beta\left(\epsilon-\kappa-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n\right)\\
& && \text{(E.0.27)}
\end{aligned}
$$

where the last inequality holds since, by the definition of $N_{[\mathbf{y}]_n}(n\delta)$, it can be written as:

$$
N_{[\mathbf{y}]_n}(n\delta) = \sum_{0<j\leq M-1} \mathbb{I}\left\{d_H\left(\left[\mathbf{x}^{(j)}\right]_n, [\mathbf{y}]_n\right) = n\delta\right\}. \tag{E.0.28}
$$

Since, for a given $[\mathbf{y}]_n$, the events $\left\{d_H\left(\left[\mathbf{x}^{(j)}\right]_n, [\mathbf{y}]_n\right) = n\delta\right\}_{j>0}$ are i.i.d., $N_{[\mathbf{y}]_n}(n\delta)$ is a binomial random variable with $M-1$ "trials", and a probability of a "success"

$\Pr\left\{d_H\left(\left[\mathbf{x}^{(j)}\right]_n, [\mathbf{y}]_n\right) = n\delta\right\}$ for $j \neq 0$. According to the method of types,

$$\Pr\left\{d_H\left(\left[\mathbf{x}^{(0)}\right]_n, [\mathbf{y}]_n\right) = n\delta\right\} \doteq \exp\left\{-n\left[\log 2 - h_2(\delta)\right]\right\}. \tag{E.0.29}$$

As this probability is a descending function of n, for a sufficiently large $\mathbb{E}[N]$ the inequality in (E.0.27) holds.

In addition, similar to the proof in [104], it can be shown that for any $\delta \in \mathcal{G}_{\bar{R}}$, $\left\{N_{[\mathbf{y}]_{(1+\epsilon_2)\mathbb{E}[N]\bar{\delta}}}\left((1+\epsilon_2)\,\mathbb{E}[N]\,\bar{\delta}\right)\right\}$ are random variables that concentrate double-exponentially rapidly around their expectations $e^{\mathbb{E}[N](1+\epsilon_2)\left[\bar{R}+h_2\left(\bar{\delta}\right)-\log(2)\right]}$ and so if

$$e^{\mathbb{E}[N](1+\epsilon_2)\left[\bar{R}+h_2\left(\bar{\delta}\right)-\log(2)\right]} > e^{-\mathbb{E}[N]T} e^{-\mathbb{E}[N](1+\epsilon_2)\beta\left(\epsilon-\kappa-\bar{\delta}\right)}, \tag{E.0.30}$$

the expression in (E.0.23) will tend to zero in a double exponential rate as $\mathbb{E}[N] \to \infty$. The former condition is equivalent to

$$\mathbb{E}[N](1+\epsilon_2)\left[\bar{R}+h_2(\delta)-\log 2\right] > \mathbb{E}[N](1+\epsilon_2)\left[\beta\left(\bar{\delta}-(\epsilon-\kappa)\right)-\bar{T}\right] \tag{E.0.31}$$

or simply

$$h_2\left(\bar{\delta}\right) - \beta\bar{\delta} + \bar{R} - \log 2 > -\beta(\epsilon+\kappa) - \bar{T} \tag{E.0.32}$$

where $\bar{T} \triangleq \frac{T}{1+\epsilon_2}$. Note that

$$\max_{\delta\in\mathcal{G}_{\bar{R}}}\left[h_2(\delta)-\beta\delta\right] + \bar{R} - \log 2 = -\beta\delta_{\text{GV}}\left(\bar{R}\right) \tag{E.0.33}$$

and since $\bar{\delta} = \underset{\delta\in\mathcal{G}_{\bar{R}}}{\operatorname{argmax}}\ \{h_2(\delta)-\beta\delta\}$ the condition in (E.0.31) boils down to

$$\bar{T} > \beta\left[\delta_{\text{GV}}\left(\bar{R}\right) - (\epsilon-\kappa)\right]. \tag{E.0.34}$$

Therefore, the following claim holds:

Claim E.3 *If $\bar{T} > \beta\left[\delta_{GV}\left(\bar{R}\right) - (\epsilon-\kappa)\right]$ for some $\kappa \in (0, \kappa_0)$, then*

$$P_0\left(N_{[\mathbf{y}]_n}\left(n\bar{\delta}\right) \leq e^{-\mathbb{E}[N]T} e^{-n\beta\left(\delta_0(n)-\bar{\delta}\right)} \mid \left[\mathbf{x}^{(0)}\right]_n [\mathbf{y}]_n : \delta_0(n) \in \mathcal{B}_\kappa\right) \to 0 \tag{E.0.35}$$

double exponentially in $\mathbb{E}[N]$ as $\mathbb{E}[N] \to 0$.

As for the term in (E.0.24), note that as an application of Sanov's theorem, $\sum_{n=1}^{\infty} P_0\left(\delta_0(n) \notin \mathcal{B}_\kappa\right)$ converges, and hence for

$$n_0 = \mathbb{E}[N]\,\frac{\left((1-\epsilon_1)\,E(R)+R\right)}{-\log(\epsilon)}, \tag{E.0.36}$$

$$\lim_{\mathbb{E}[N]\to\infty} \sum_{n=n_0}^{\mathbb{E}[N](1+\epsilon_2)} P_0\left(\delta_0\left(n\right) \notin \mathcal{B}_\kappa\right) \to 0. \tag{E.0.37}$$

We conclude this appendix with the proof of Claim E.2.

Proof. Recall that $\bar{R} \triangleq \frac{R}{1+\epsilon_2}$ for some $\epsilon_2 > 0$, and so $\bar{R} < R$. In addition, we have defined for any fixed $r \leq C$ the following set:

$$\mathcal{G}_r \triangleq \left\{\delta : \delta_{\mathrm{GV}}\left(r\right) \leq \delta \leq 1 - \delta_{\mathrm{GV}}\left(r\right)\right\}, \tag{E.0.38}$$

and hence $\mathcal{G}_{\bar{R}} \subset \mathcal{G}_R$.

Next, assume conversely that there exists a $\delta \in \mathcal{G}_R$ that satisfies $\delta < \epsilon$. Since $\epsilon < \frac{1}{2}$, this implies

$$\delta_{\mathrm{GV}}\left(R\right) \leq \epsilon \Leftrightarrow \tag{E.0.39}$$

$$\log 2 - R \leq h_2\left(\epsilon\right) \Leftrightarrow \tag{E.0.40}$$

$$C = h_2\left(\epsilon\right) - \log 2 \leq R. \tag{E.0.41}$$

This is in contradiction with the basic assumption that $R < C$. Therefore we conclude that $\epsilon < \delta$ for all $\delta \in \mathcal{G}_R$, or, in other words, $\epsilon \in \left[0, \frac{1}{2}\right) \cap \mathcal{G}_R^c$. Since $\mathcal{G}_{\bar{R}} \subset \mathcal{G}_R$ (where the inclusion is *strict*), Claim E.2 holds. $\square$

Appendix F

Proof of Lemma 36

The proof of Lemma 36 can be found in [9], using results from [18]. It is brought here for the sake of completeness. We start the proof by analyzing τ, defined in (6.4.49). Since the MAP decoder will choose, at time n, the message $i \in \{0, \ldots, M-1\}$ that has the largest a posteriori probability (or chooses at random between all messages that attain this maximum together), the probability of error can be written as $p_e^{MAP}(y^n) = 1 - p_{max}$ where $p_{max} = \max_{i \in \{0,\ldots,M-1\}} \Pr(\theta = i \mid y^n)$. Let us denote by $p_e\left(y^N\right)$ the probability of error given the observation $\{y_1, \ldots, y_N\}$. The unconditioned probability of error is then given by $p_e = \mathbb{E}\left[p_e\left(Y^N\right)\right]$.

Since, by the definition of τ, $\left\{p_e^{MAP}(y^\tau) \geq \delta\right\} \subseteq \left\{p_e\left(y^N\right) \geq \delta\right\}$ it holds that

$$\Pr\left(p_e^{MAP}(y^\tau) \geq \delta\right) \leq \Pr\left(p_e\left(y^N\right) \geq \delta\right) \leq \frac{p_e}{\delta} \tag{F.0.1}$$

and

$$\Pr\left(p_e^{MAP}(y^\tau) < \delta\right) \geq 1 - \frac{p_e}{\delta} \tag{F.0.2}$$

where the second inequality in (F.0.1) is an application of Chebyshev's inequality.

Define $H(i \mid y^n)$ to be the entropy of the a posteriori distribution $\Pr(\theta = \cdot \mid y^n)$ (and so $\mathbb{E}[H(i \mid y^n)] = H(i \mid Y^n)$). The random variable we will next consider is $H(i \mid y^\tau)$ which is defined in the same manner as $H(i \mid y^n)$ only that the available observations are now given up to the (random) time τ. The following holds:

$$\begin{aligned}
\mathbb{E}[H(i \mid y^\tau)] &= \mathbb{E}\left[H(i \mid y^\tau) \mid p_e^{MAP}(y^\tau) < \delta\right] \Pr\left(p_e^{MAP}(y^\tau) < \delta\right) && \text{(F.0.3)}\\
&+ \mathbb{E}\left[H(i \mid y^\tau) \mid p_e^{MAP}(y^\tau) \geq \delta\right] \Pr\left(p_e^{MAP}(y^\tau) \geq \delta\right) && \text{(F.0.4)}\\
&\leq h_2(\delta) + \delta \log(M) + \log(M) \frac{p_e}{\delta} && \text{(F.0.5)}\\
&= h_2(\delta) + \left(\delta + \frac{p_e}{\delta}\right) \log(M) && \text{(F.0.6)}
\end{aligned}$$

where in (F.0.5), (F.0.1) was used, in addition to Fano's inequality, that was harnessed in order to bound $\mathbb{E}\left[H\left(i \mid y^{\tau}\right) \mid p_e^{MAP}\left(y^{\tau}\right) < \delta\right]$. The proof of the third item in Lemma 36 is now strait-forward, keeping in mind (F.0.6) and the fact that the random process $\{H\left(i \mid y^{n}\right) + nC, \mathcal{F}_n\}$ is a bounded submartingale (a fact proved in [18]). Using Doob's optional sampling theorem yields

$$\mathbb{E}\left[H\left(i \mid y^{\tau}\right) + \tau C\right] \geq H\left(i \mid y^{0}\right) = \log\left(M\right). \tag{F.0.7}$$

The result (6.4.52) then follows by combining (F.0.6) and (F.0.7).

In order to prove the first two item of Lemma 36, we need to use the following lemma, which is a direct consequence of the Bayes rule. The proof of this lemma can be found in [9, Lemma 2].

Lemma 41 *Define* $\lambda = \min_{\mathcal{X},\mathcal{Y}} p\left(\cdot \mid \cdot\right)$. *For any DMC channel such that* $0 < \lambda \leq 1/2$ *it holds that*

$$\lambda \Pr\left(i \mid y^{n-1}\right) \leq \Pr\left(i \mid y^{n}\right) \leq \frac{\Pr\left(i \mid y^{n-1}\right)}{\lambda} \tag{F.0.8}$$

for all $i \in \{0, \ldots, M-1\}$

To see why (6.4.50) and (6.4.51) hold, note that:

1. The strict inequality $\tau < N$ implies $p_e^{MAP}\left(y^{\tau}\right) < \delta$, and therefore

$$1 - \max_{i \in \{0, \ldots M-1\}} \Pr\left(i \mid y^{n}\right) < \delta. \tag{F.0.9}$$

The last relation then implies that there exists an $i^{\star} \in \{0, \ldots, M-1\}$ such that

$$\pi_{i^{\star}}\left(\tau\right) = \Pr\left(\theta = i^{\star} \mid y^{\tau}\right) > 1 - \delta. \tag{F.0.10}$$

2. Notice that, by the definition of τ, it holds that $p_e^{MAP}\left(y^{\tau-1}\right) > \delta$. Just as in the previous item, this implies the following relations

$$\max_{i \in \{0, \ldots M-1\}} \Pr\left(\theta = i \mid y^{\tau-1}\right) < 1 - \delta \quad \Rightarrow \tag{F.0.11}$$

$$\forall i \in \{0, \ldots M-1\} \;:\; \Pr\left(\theta = i \mid y^{\tau-1}\right) < 1 - \delta, \tag{F.0.12}$$

and so for any $i \in \{0, \ldots M-1\}$:

$$\begin{aligned}
\delta &< 1 - \Pr\left(\theta = i \mid y^{\tau-1}\right) && \text{(F.0.13)}\\
&= \Pr\left(\theta \neq i \mid y^{\tau-1}\right) && \text{(F.0.14)}\\
&\leq \sum_{j=0, j\neq i}^{M-1} \Pr\left(\theta = j \mid y^{\tau-1}\right) && \text{(F.0.15)}\\
&= \sum_{j=0, j\neq i}^{M-1} \pi_j\left(\tau - 1\right) && \text{(F.0.16)}\\
&\leq \sum_{j=0, j\neq i}^{M-1} \frac{\pi_j\left(\tau\right)}{\lambda}. && \text{(F.0.17)}
\end{aligned}$$

So, for τ defined as in (6.4.49), for $i \in \{0, \ldots M-1\}$, $\sum_{j=0, j\neq i}^{M-1} \pi_j\left(\tau\right) \geq \lambda\delta$. In particular, if we choose $i = i^\star \in \{0, \ldots, M-1\}$, defined in the previous item, we get

$$\sum_{j=0, j\neq i^\star}^{M-1} \pi_j\left(\tau\right) \geq \lambda\delta \qquad \text{(F.0.18)}$$

and so Lemma 36 is proven.

Appendix G

Proof of Lemma 37

Let $\tilde{\Delta} = \left(\tilde{N}, \tilde{d}\right)$ be the sequential test defined in (6.4.6), and assume that both (6.4.53) and (6.4.54) hold. For any $i \in \{0, 1, \ldots, M-1\}$ and any positive constant L, define the following event

$$\Omega_{i,L} = \left\{\tilde{d} = i\right\} \bigcap \left\{\tilde{N} \leq L\right\}. \tag{G.0.1}$$

The following chain of equalities and inequalities hold for any i and j such that $i \neq j \in \{0, 1, \ldots, M-1\}$:

$$\begin{aligned}
P_j\left(d = i\right) &= \mathbb{E}_i\left[\mathbb{I}\left\{\tilde{d} = i\right\}\exp\left\{\Lambda_{j,i}\left(\tilde{N}\right)\right\}\right] && \text{(G.0.2)}\\
&= \mathbb{E}_i\left[\mathbb{I}\left\{\tilde{d} = i\right\}\exp\left\{-\Lambda_{i,j}\left(\tilde{N}\right)\right\}\right] && \text{(G.0.3)}\\
&\geq \mathbb{E}_i\left[\mathbb{I}\left\{\Omega_{i,L}, \Lambda_{i,j}\left(\tilde{N}\right) < B\right\}\exp\left\{-\Lambda_{i,j}\left(\tilde{N}\right)\right\}\right] && \text{(G.0.4)}\\
&\geq \exp\left\{-B\right\} P_i\left(\Omega_{i,L}, \sup_{n \leq L}\left\{\Lambda_{i,j}\left(n\right)\right\} < B\right) && \text{(G.0.5)}\\
&\geq \exp\left\{-B\right\}\left[P_i\left(\Omega_{i,L}\right) - P_i\left(\sup_{n \leq L}\left\{\Lambda_{i,j}\left(n\right)\right\} \geq B\right)\right] && \text{(G.0.6)}
\end{aligned}$$

where B is a positive constant and $\Lambda_{j,i}\left(n\right)$ is defined in (6.4.9). Using the fact that

$$P_i\left(\Omega_{i,L}\right) \geq P_i\left(\tilde{d} = i\right) - P_i\left(\tilde{N} > L\right), \tag{G.0.7}$$

(G.0.6) implies that

$$\begin{aligned}
P_i\left(\tilde{N} > L\right) \geq 1 - P_i\left(\tilde{d} \neq i\right) - \exp\left\{B\right\} P_j\left(\tilde{d} = i\right) && \text{(G.0.8)}\\
- P_i\left(\sup_{n \leq L}\left\{\Lambda_{i,j}\left(n\right)\right\} \geq B\right). && \text{(G.0.9)}
\end{aligned}$$

Given (6.4.53) and (6.4.54), one can bound $p_e\left(\tilde{\Delta}\right)$ (defined in Lemma 37) by

$$p_e\left(\tilde{\Delta}\right) = \sum_{j=0}^{M-1} \pi_j P_j\left(\tilde{d} \neq j\right) \geq \pi_{i^\star} P_{i^\star}\left(\tilde{d} \neq i^\star\right) \geq (1-\delta)\, P_{i^\star}\left(\tilde{d} \neq i^\star\right), \qquad \text{(G.0.10)}$$

and so

$$P_{i^\star}\left(\tilde{d} \neq i^\star\right) \leq \frac{p_e\left(\tilde{\Delta}\right)}{1-\delta}. \qquad \text{(G.0.11)}$$

In addition, if we define $j_0 = \text{argmin}_{j \neq i^\star} P_j\left(\tilde{d} = i^\star\right)$ then

$$p_e\left(\tilde{\Delta}\right) = \sum_{j=0}^{M-1} \pi_j \sum_{i=0, i\neq j}^{M-1} P_j\left(\tilde{d} = i\right) \geq \sum_{j=0, j\neq i^\star}^{M-1} \pi_j P_j\left(\tilde{d} = i^\star\right) \geq P_{j_0}\left(\tilde{d} = i^\star\right) \lambda\delta, \qquad \text{(G.0.12)}$$

and hence it is possible to bound $P_{j_0}\left(\tilde{d} = i^\star\right)$ by

$$P_{j_0}\left(\tilde{d} = i^\star\right) \leq \frac{p_e\left(\tilde{\Delta}\right)}{\lambda\delta}. \qquad \text{(G.0.13)}$$

Taking $i = i^\star$ and $j = j_0$ in (G.0.9) and using (G.0.11) and (G.0.13) then yields

$$P_{i^\star}\left(\tilde{N} > L\right) \geq 1 - P_{i^\star}\left(\tilde{d} \neq i^\star\right) - e^B P_{j_0}\left(\tilde{d} = i^\star\right) - P_{i^\star}\left(\sup_{n \leq L} \left\{\Lambda_{i^\star, j_0}(n)\right\} \geq B\right) \qquad \text{(G.0.14)}$$

$$\geq 1 - \frac{p_e\left(\tilde{\Delta}\right)}{1-\delta} - e^B \frac{p_e\left(\tilde{\Delta}\right)}{\lambda\delta} - P_{i^\star}\left(\sup_{n \leq L} \left\{\Lambda_{i^\star, j_0}(n)\right\} \geq B\right). \qquad \text{(G.0.15)}$$

In order to further bound $P_{i^\star}\left(\tilde{N} > L\right)$, notice that, provided that $B - LD > 0$, and for any $i \neq j \in \{0, 1, \ldots, M-1\}$,

$$P_i\left(\sup_{n \le L}\left\{\Lambda_{i,j}(n)\right\} \ge B\right) = P_i\left(\sup_{n \le L}\left\{\sum_{k=1}^{n} \Delta\Lambda_{i,j}(k) - LD\right\} \ge B - LD\right) \tag{G.0.16}$$

$$\le P_i\left(\sup_{n \le L}\left\{\sum_{k=1}^{n} \Delta\Lambda_{i,j}(k) - nD\right\} \ge B - LD\right) \tag{G.0.17}$$

$$\le P_i\left(\sup_{n \le L}\left|\sum_{k=1}^{n} \Delta\Lambda_{i,j}(k) - nD\right| \ge B - LD\right) \tag{G.0.18}$$

$$= P_i\left(\sup_{n \le L}\left(\sum_{k=1}^{n} \Delta\Lambda_{i,j}(k) - nD\right)^4 \ge \{B - LD\}^4\right) \tag{G.0.19}$$

$$\le \frac{\mathbb{E}_i\left[\left(\sum_{k=1}^{L}\left[\Delta\Lambda_{i,j}(k) - D\right]\right)^4\right]}{(B - LD)^4} \tag{G.0.20}$$

where in (G.0.20) we have used the Doob's inequality [46, Page 247] applied to the P_i-submartingale

$$\left\{\left(\sum_{k=1}^{n} \Delta\Lambda_{i,j}(k) - nD\right)^4, \mathcal{F}_n\right\}. \tag{G.0.21}$$

Setting $B = cDL$ where $c > 1$, then yields:

$$P_i\left(\sup_{n \le L}\left\{\Lambda_{i,j}(n)\right\} \ge B\right) \le \frac{\mathbb{E}_i\left[\left(\sum_{k=1}^{L}\left[\Delta\Lambda_{i,j}(k) - D\right]\right)^4\right]}{(c-1)^4 L^4 D^4}. \tag{G.0.22}$$

Since the LLRs are i.i.d. at different time steps (with mean D), it follows that $\{\Delta\Lambda_{i,j}(k) - D\}$ are i.i.d. zero-mean random variables (with finite fourth moment), so there exist $L_0, K \in \mathbb{R}_+$ such that for any $L_0 < L$,

$$P_i\left(\sup_{n \le L}\left\{\Lambda_{i,j}(n)\right\} \ge B\right) \le \frac{K}{L^2}. \tag{G.0.23}$$

Plugging (G.0.23) into (G.0.15) with $i = i^\star, j = j_0$, and applying the Chebyshev inequality leads to

$$\mathbb{E}_{i^\star}\left[\tilde{N}\right] \ge L - L\frac{p_e\left(\tilde{\Delta}\right)}{1-\delta} - Le^B\frac{p_e\left(\tilde{\Delta}\right)}{\lambda\delta} - \frac{K}{L}. \tag{G.0.24}$$

We have yet to determine L. We will take L to be

$$L = \frac{\rho}{D} \log \left(\frac{\lambda \rho}{p_e \left(\tilde{\Delta} \right)} \right) \tag{G.0.25}$$

where $0 < \rho < 1/c$ is some constant. For this choice of L, and using the assumption (6.4.53), we conclude that

$$\mathbb{E} \left[\tilde{N} \right] \geq (1 - \delta) \, \mathbb{E}_{i^\star} \left[\tilde{N} \right] \tag{G.0.26}$$

$$\geq (1 - \delta) \frac{\rho}{D} \log \left(\frac{\lambda \rho}{p_e \left(\tilde{\Delta} \right)} \right) \left(1 - \frac{p_e \left(\tilde{\Delta} \right)}{1 - \delta} - \left(\frac{p_e \left(\tilde{\Delta} \right)}{\lambda \delta} \right)^{1 - c\rho} - \frac{K}{\left[\frac{\rho}{D} \log \left(\frac{\lambda \rho}{p_e \left(\tilde{\Delta} \right)} \right) \right]^2} \right), \tag{G.0.27}$$

and Lemma 37 is proven.

Appendix H

Proof of Theorem 35

Let $\Delta = (N, d)$ be any coding scheme for the stop-feedback problem, when infinite random coding is assumed to be used in the construction of the codebook. As described in Section 6.4.D., we can represent the stopping time N of any such test as

$$N = \tau + \hat{N} \tag{H.0.1}$$

where τ is defined in (6.4.49) and $\hat{N}$ is defined as

$$\hat{N} \triangleq \begin{cases} 0 & \text{if } p_e^{MAP}\left(y^\tau\right) \geq \delta \\ \tilde{N} & \text{otherwise} \end{cases} \tag{H.0.2}$$

and $\tilde{N}$ is defined as the time it takes the test to terminate, after observing $\{y_1, \ldots, y_\tau\}$, given that $p_e^{MAP}\left(y^\tau\right) < \delta$. Using Lemma 36, we know that at time τ, conditions 1 and 2 of Lemma 37 hold, and so, $\tilde{N}$ can be described as the time it takes Δ to infer among the M hypotheses (6.4.6), given that these conditioned are satisfied. Therefore, $\hat{N}$ can be bounded, using Lemma 37, as follows:

$$\mathbb{E}\left[\hat{N}\right] = \mathbb{E}\left[\hat{N} \mid p_e^{MAP}\left(y^\tau\right) < \delta\right] \Pr\left(p_e^{MAP}\left(y^\tau\right) < \delta\right) \tag{H.0.3}$$

$$+ \mathbb{E}\left[\hat{N} \mid p_e^{MAP}\left(y^\tau\right) \geq \delta\right] \Pr\left(p_e^{MAP}\left(y^\tau\right) \geq \delta\right) \tag{H.0.4}$$

$$\geq \mathbb{E}\left[\tilde{N}\right]\left(1 - \frac{p_e}{\delta}\right) \tag{H.0.5}$$

$$\geq (1-\delta)\frac{\rho}{D}\log\left(\frac{\lambda\rho}{p_e\left(\tilde{\Delta}\right)}\right) \times \tag{H.0.6}$$

$$\left(1 - \frac{\tilde{p}_e}{1-\delta} - \left(\frac{\tilde{p}_e}{\lambda\delta}\right)^{1-c\rho} - \frac{K}{\left[\frac{\rho}{D}\log\left(\frac{\lambda\rho}{p_e\left(\tilde{\Delta}\right)}\right)\right]^2}\right)\left(1 - \frac{p_e}{\delta}\right) \tag{H.0.7}$$

where $\tilde{p}_e$ is defined to be the conditioned error probability of the decoding scheme Δ given that $\left\{p_e^{MAP}\left(y^\tau\right) < \delta\right\}$, and p_e is defined to be the unconditioned error probability associated with Δ. Notice that p_e can be bounded using (F.0.2) as follows:

$$p_e \geq \tilde{p}_e \Pr\left(p_e^{MAP}\left(y^\tau\right) < \delta\right) \geq \tilde{p}_e\left(1 - \frac{p_e}{\delta}\right) \geq 0. \tag{H.0.8}$$

From (H.0.8) we can deduce that $\tilde{p}_e \to 0$ as $p_e \to 0$, and that $\hat{p}_e \leq \frac{p_e}{1-p_e/\delta}$. The later relation thus implies that

$$-\log\left(\frac{\tilde{p}_e}{\lambda\delta}\right) \geq -\log\left(\frac{p_e}{\lambda\left(\delta - p_e\right)}\right). \tag{H.0.9}$$

Combining (6.4.52), (H.0.7) and (H.0.9) then yields:

$$\mathbb{E}\left[N\right] \geq (1-\delta)\left(\frac{\rho\log\left(1/p_e\right)}{D}\right) \times \tag{H.0.10}$$

$$\left(1 - \frac{\tilde{p}_e}{1-\delta} - \left(\frac{\tilde{p}_e}{\lambda\delta}\right)^{1-c\rho} - \frac{K}{\left(\frac{\rho}{D}\log\left(\frac{\tilde{p}_e}{\lambda\rho}\right)\right)^2}\right)\left(1 - \frac{p_e}{\delta}\right) \tag{H.0.11}$$

$$+ (1-\delta)\frac{\rho}{D}\log\left(\lambda\left(\delta - p_e\right)\right) \times \tag{H.0.12}$$

$$\left(1 - \frac{\tilde{p}_e}{1-\delta} - \left(\frac{\tilde{p}_e}{\lambda\delta}\right)^{1-c\rho} - \frac{K}{\left(\frac{\rho}{D}\log\left(\frac{\tilde{p}_e}{\lambda\rho}\right)\right)^2}\right)\left(1 - \frac{p_e}{\delta}\right) \tag{H.0.13}$$

$$+ \left(1 - \delta - \frac{p_e}{\delta}\right)\frac{\log M}{C} - \frac{h_2\left(\delta\right)}{C}. \tag{H.0.14}$$

Recall that the asymptotic regime of interest is $p_e \to 0$ (and $\mathbb{E}[N] \to \infty$) while $\lim \frac{\log M}{\mathbb{E}[N]} = R$ (for $0 \leq R \leq C$). Using the error exponent in (5.2.10) and some algebra, the relation above implies that

$$E(R) \leq \frac{1}{\rho}\left(\frac{D}{1-\delta} - D\frac{R}{C}\right). \tag{H.0.15}$$

Since δ can be chosen as arbitrarily close to 0 and ρ arbitrarily close to 1, we see that for any $\epsilon > 0$ the error exponent in upper bounded by $E(R) \leq D\left(1 - \frac{R}{C}\right) + \epsilon$.

Appendix I

The Weak Converse of VL Coding

In this appendix, the notation and a few results obtained in Appendix F will be used to prove a weak converse for Chapter 5. Specifically, for any decoding scheme $\Delta = (N, d)$ and for R and C defined in Section 5.2, it will be shown that if $R > C$, then the probability of error is bounded away from zero for sufficiently large $\mathbb{E}[N]$ (and hence for all $\mathbb{E}[N]$, since if $P_e(\Delta) \to 0$ for small $\mathbb{E}[N]$, we can construct codes for large $\mathbb{E}[N]$ with $P_e(\Delta) \to 0$ by concatenating these codes).

Proof. Recall that in Appendix F the random process $H(i \mid y^\tau)$ was defined for any stopping time τ. By Fano's inequality applied to Δ,

$$\begin{aligned} \mathbb{E}\left[H\left(i \mid Y^N\right)\right] &\le \mathbb{E}\left[h_2\left(P_e\left(Y^N\right)\right)\right] + \mathbb{E}\left[P_e\left(Y^N\right)\right]\log M && \text{(I.0.1)} \\ &\le h_2\left(P_e(\Delta)\right) + P_e(\Delta)\log M && \text{(I.0.2)} \end{aligned}$$

where $P_e(y^\tau)$ is as defined in Appendix F, and the second inequality holds due to the Jensen inequality. Combining (I.0.2) and (F.0.7) then yields:

$$\log M \le h_2\left(P_e(\Delta)\right) + P_e(\Delta)\log M + C\mathbb{E}[N]. \tag{I.0.3}$$

After some algebra, we obtain that (I.0.3) implies

$$P_e(\Delta) \ge 1 - \frac{C}{R} - \frac{h_2\left(P_e(\Delta)\right)}{RC} \tag{I.0.4}$$

which proves the claim. □

Appendix J

The Submartingale Property of $\Lambda_i(n)$

Let $\mathcal{C} = \left\{\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(M-1)}\right\}$ be an infinite, randomly drawn codebook as defined in Section 6.4, where $\mathbf{x}^{(i)} = \left\{x_1^{(i)} x_2^{(i)}, \ldots : x_j^{(i)} \in \{0,1\}\right\}$ is the codeword assigned to $i \in \{0, \ldots, M-1\}$. For any $i \in \{0, \ldots, M-1\}$ define the random process $(\Lambda_i(n), \mathcal{F}_n)$ where:

- $\Lambda_i(n) = \log\left[\frac{p\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n\right)}{\sum_{j=0, j\neq i}^{M-1} p\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(j)}\right]_n\right)}\right]$,
- $\mathcal{F}_n$ is the filtration generated by $\left\{\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(M-1)}, \mathbf{y}\right\}$,

Throughout this section it will be assumed that the i'th message was sent.

Claim J.1 *The process $(\Lambda_i(n), \mathcal{F}_n)$ is a submartingale.*

Proof. By definition $\Lambda_n \in \mathcal{F}_n$ and since $p(y \mid x)$ is positive and finite for any $x, y \in \{0,1\}$, it holds that $\mathbb{E}_i[|\Lambda_i(n)|] < \infty$. In order to prove the claim it is left to show that

$$\log\left[\frac{p\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n\right)}{\sum_{j=0, j\neq i}^{M-1} p\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(j)}\right]_n\right)}\right] \leq \mathbb{E}_i\left\{\log\left[\frac{p\left([\mathbf{y}]_{n+1} \mid \left[\mathbf{x}^{(i)}\right]_{n+1}\right)}{\sum_{j=0, j\neq i}^{M-1} p\left([\mathbf{y}]_{n+1} \mid \left[\mathbf{x}^{(j)}\right]_{n+1}\right)}\right] \middle| \mathcal{F}_n\right\}. \tag{J.0.1}$$

Towards that end, note that

$$\mathbb{E}_i \left\{ \log \left[p \left([\mathbf{y}]_{n+1} \mid \left[\mathbf{x}^{(i)}\right]_{n+1} \right) \right] \mid \mathcal{F}_n \right\} =$$

$$\log \left[p \left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n \right) \right] + \mathbb{E}_i \left\{ \log \left[p \left(y_{n+1} \mid x^{(i)}_{n+1} \right) \right] \right\} \tag{J.0.2}$$

$$= \log \left[p \left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n \right) \right] - h_2 (\epsilon), \tag{J.0.3}$$

where the first equality holds due to the memorylessness along with the fact that $\log \left[p \left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n \right) \right]$ is measurable with respect to $\mathcal{F}_n$ and the $n+1$ instant of the random process is independent of $\mathcal{F}_n$. The second equality follows by the definition of the binary entropy function. Therefore, the right hand side of (J.0.1) is equal to:

$$\log \left[p \left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(i)}\right]_n \right) \right] - h_2 (\epsilon) - \tag{J.0.4}$$

$$\mathbb{E}_i \left\{ \log \left[\sum_{j=0, j \neq i}^{M-1} p \left([\mathbf{y}]_{n+1} \middle| \left[\mathbf{x}^{(j)}\right]_{n+1} \right) \right] \middle| \mathcal{F}_n \right\}. \tag{J.0.5}$$

Note that the first term cancels out on both sides of (J.0.1) and so we have left to shown that

$$\mathbb{E}_i \left\{ \log \left[\sum_{j=0, j \neq i}^{M-1} p \left([\mathbf{y}]_{n+1} \mid \left[\mathbf{x}^{(j)}\right]_{n+1} \right) \right] \middle| \mathcal{F}_n \right\} \leq \log \left[\sum_{j=0, j \neq i}^{M-1} p \left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(j)}\right]_n \right) \right] - h_2 (\epsilon). \tag{J.0.6}$$

It is easy to see that indeed the former inequality holds since:

$$\mathbb{E}_i \left\{ \log \left[\sum_{j=0, j\neq i}^{M-1} p\left([\mathbf{y}]_{n+1} \middle| \left[\mathbf{x}^{(j)}\right]_{n+1} \right) \right] \mid \mathcal{F}_n \right\}$$

$$\leq \log \left[\sum_{j=0, j\neq i}^{M-1} \mathbb{E}_i \left\{ p\left([\mathbf{y}]_{n+1} \mid \left[\mathbf{x}^{(j)}\right]_{n+1} \right) \middle| \mathcal{F}_n \right\} \right] \tag{J.0.7}$$

$$= \log \left[\sum_{j=0, j\neq i}^{M-1} \mathbb{E}_i \left\{ p\left([\mathbf{y}]_n \mid \left[\mathbf{x}^{(j)}\right]_n \right) \right\} p\left(y_{n+1} \mid x_{n+1}^{(j)} \right) \right] \tag{J.0.8}$$

$$= \log \left[\frac{1}{2} \sum_{j=0, j\neq i}^{M-1} p\left(y_{n+1} \mid x_{n+1}^{(j)} \right) \right] \tag{J.0.9}$$

$$= \log \left[\sum_{j=0, j\neq i}^{M-1} p\left(y_{n+1} \mid x_{n+1}^{(j)} \right) \right] + \log\left(\frac{1}{2}\right) \tag{J.0.10}$$

$$\leq \log \left[\sum_{j=0, j\neq i}^{M-1} p\left(y_{n+1} \mid x_{n+1}^{(j)} \right) \right] - h_2(\epsilon), \tag{J.0.11}$$

where in (J.0.7) Jensen's inequality was used. □

Recall that the submartingale property was needed to enable the use of Doob's inequality to upper bound the probability of the event $\left\{\sup_{n\leq \bar{n}} \Lambda_i(n) > a\right\}$ as follows:

$$P_i \left(\sup_{n\leq \bar{n}} \Lambda_i(n) > a \right) \leq \frac{\mathbb{E}_i \left\{ [\Lambda_i(\bar{n})]^+ \right\}}{a}, \tag{J.0.12}$$

where for any $z \in \mathbb{R}$,

$$[z]^+ = \begin{cases} 0 & : x < 0 \\ z & : x \geq 0. \end{cases}$$

Carrying out the $[\cdot]^+$ operator may lead to cumbersome expressions. Nevertheless, note that we can avoid it if we are willing to bound the random variable $e^{\Lambda_i(\bar{n})}$ instead of $\Lambda_i(\bar{n})$. This simplification can be achieved by using the fact that, since $\exp\{\cdot\}$ is convex, the process $\left(e^{\Lambda_i(n)}, \mathcal{F}_n\right)$ is a submartingale too, and furthermore, it is positive.

Therefore we may write:

$$P_i\left(\sup_{n\le\bar{n}}\Lambda_i(n)>a\right)=P_i\left(\sup_{n\le\bar{n}}e^{\Lambda_i(n)}>e^a\right) \tag{J.0.13}$$

$$\le\frac{\mathbb{E}_i\left[e^{\Lambda_i(\bar{n})}\right]}{e^a} \tag{J.0.14}$$

$$=e^{-a}\mathbb{E}_i\left\{\frac{p\left([\mathbf{y}]_{\bar{n}}\mid\left[\mathbf{x}^{(i)}\right]_{\bar{n}}\right)}{\sum_{j=0,j\neq i}^{M-1}p\left([\mathbf{y}]_{\bar{n}}\mid\left[\mathbf{x}^{(j)}\right]_{\bar{n}}\right)}\right\}. \tag{J.0.15}$$

This last expression may be amenable to further bounding using the techniques of [103].

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בחינת השערות סדרתית וקידוד ערוץ עם משוב

שי גנזך

המחקר נעשה בהנחיית פרופ' יגאל ששון ופרופ' נרי מרחב
בפקולטה להנדסת חשמל

הכרת תודה

אני מודה מקרב לב למנחים שלי, פרופ' יגאל ששון ופרופ' נרי מרחב, על ההנחייה, הסבלנות והזמן הרב שהוקדשו למחקר שלי ולעזרה הרבה בכתיבת עבודה זו. היו לי הזכות והכבוד לעבוד תחת שני חוקרים מקצועיים ומורים מסורים אלו.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי

תקציר

עבודה זו עוסקת בקשר שבין שני תחומי מחקר שונים, אשר כל אחד מהם זכה להתעניינות מרובה בשנים האחרונות: בחינת השערות סדרתית וקידוד ערוץ באורך בלוק משתנה. מסתבר כי על אף היותם של נושאים אלו מושאים למחקרים רבים, מספר העבודות העוסקות בקשר שביניהם הוא דל יחסית על אף הקשר גורדי ביניהם. עם זאת, קיימים בספרות כמה אזכורים לקשר הנ"ל, ומספר עבודות שבהן נוצלו תוצאות מעולם בחינת ההשערות הסדרתית על מנת לקבל חסמים של ביצועי קוד ערוץ בנוכחות משוב תסקרנה בחלקה השני של עבודה זו. הן תוצאות אלו והן הקשר בין בעיית בחינת ההשערות לבעיית התקשורת מהוות יחד את עיקר המוטיבציה לכתיבת עבודה זו.

לעבודה זו שתי מטרות עיקריות: הראשונה היא מתן סקירה מקיפה וממצה של תוצאות הלקוחות מעולם בחינת ההשערות הסדרתית שיכולות להיות שימושיות בניתוח מערכות תקשורת עם משוב. המטרה השניה היא הדגמה של כמה דרכים לקשור תוצאות אלו לבעיית התקשרות. עם זאת, קצרה היריעה מכדי למצות את כלל ההיקף של שני מטרות אלו. עקב כך, בחרנו להתמקד בעיקר רק בקשר שבין בחינת השערות פשוטות באופן סדרתי ומעריך השגיאה של סכמות קידוד ערוץ באורך משתנה. לאורך העבודה, הכוונה ב"בעיית בחירת השערות פשוטות באופן סדרתי" תהיה קבלת החלטה לגבי אחת מ־ M ההיפותזות:

$$H_i: \quad \Pr\left(\mathbf{x}\right) = P_i\left(\mathbf{x}\right), \quad i \in \{0, \ldots, M-1\}, \ \mathbf{x} = x_0, x_1 \ldots,$$

כאשר $P_i\left(\cdot\right)$ הם פילוגים ידועים, ו־ $\mathbf{x}$ הינה סדרת האובזרבציה. המאפיין הבולט של בעיה זו הוא שאורך סדרת האובזרבציה הינו אינסופי, קרי, מספר המדידות איננו מוגבל[1]. כזכור, בבעיות בחינת ההשערות הקלאסית מספר המדידות שעל בסיסן מתקבלת ההחלטה נקבע מבעוד מועד והינו פרמטר של המבחן. להבדיל מכך, בבעיה הסדרתית על המפענח לקבוע, בנוסף להחלטה באיזו השערה לתמוך, גם את מספר המדידות שעליו לקחת על מנת שהחלטה זו תתקבל ברמת סמך גבוהה. בפרט, בבעיות הנדונות בעבודה מספר המדידות יהיה פונקציה של סדרת האובזרבציה (ולפיכך מספר זה הינו משתנה אקראי). מספר בעיות בחינת השערות שונות יידונו בעבודה זו,

[1]למעשה, בבעיה זו מוגדרים M פילוגים לכל זמן n והמפענח מחליט האם ניתן לקבל החלטה שסדרת האובזרבציה שנצפתה עד כה, x^n, $n \in \mathbb{N}$ יוצרה ע"י אחד המקורות P_i, או שמע על מנת לקבל החלטה כזו בצורה אמינה יש לקחת עוד מדידות

ובהן בחינת זוג השערות פשוטות, בחינת מספר רב (גדול משתיים) של השערות פשוטות, ובחינת השערות עם אפשרות של בקרה על אופיים הסטטיסטי של המדידות. בעיה נוספת שתיסקר בקצרה הינה בחינת השערות מורכבות, אשר בה כל היפותיזה מגדירה אוסף פילוגים ועל המפענח לבחור באיזה מבין האוספים השונים לתמוך. בכל המקרים הנ"ל, המטרה של מקבל ההחלטה היא להחליט בצורה מיטיבית אילו מבין ההיפותזות היא הנכונה, כאשר מדד הטיב הוא שקלול של הסתברות השגיאה הממוצעת ושל הזמן הממוצע הנדרש להגיע להחלטה זו. בפרט, מדד הטיב אחד אשר בו נתרכז הינו מעריך השגיאה, המוגדר כ־

$$\limsup_{P_{\mathrm{e}}\to 0}\frac{-\log\left(P_{\mathrm{e}}\right)}{\mathbb{E}\left[N\right]}$$

כאשר P_{e} הינה הסתברות השגיאה הממוצעת ו־ $\mathbb{E}\left[N\right]$ הינה תוחלת זמן העצירה. חשוב לציין כי הגדרה זו גם מכתיבה משטר אסימפטוטי מסוים שיהיה בעל עניין לאורך העבודה: המשטר בו זמן העצירה שואף לאינסוף והסתברות השגיאה שואפת לאפס. מדד טיב נוסף שיהיה בעל עניין הינו מינימום של תוחלת זמן העצירה עבור רמת סמך נתונה.

החלק השני של העבודה דן בבעיית התקשורת שבה למקודד ישנה האפשרות לקודד כל הודעה ע"י סדרה אינסופית של סימבולי ערוץ. לאחר בחירת הודעה לשידור, המקודד שולח את מילת הקוד המתאימה, סימבול אחרי סימבול, דרך ערוץ התקשורת. המפענח, מצדו, מקבל את רצף הסימבולים לאחר המעבר בערוץ. על המפענח מוטלות המשימות הבאות: (1) להודיע לצד השולח מתי עליו לחדול משליחת ההודעה הנוכחית ולעבור להודעה הבאה ו־ (2) לפענח את ההודעה. באופן זה המפענח יכול להשהות את קבלת ההודעה במידה והמידע שצבר אינו מובהק דיו, ובכך לשפר את אמינותה של התקשורת על חשבון תשלום אפשרי בהשהיה ארוכה. מתח זה שבין הסתברות השגיאה וההשהיה בא לידי ביטוי במעריך השגיאה, שהינו מדד הטיב בו נעסוק. ההבדל בין הגדרת מעריך השגיאה עבור בעיית בחינת ההשערות ובעיית התקשורת הינו שבבעיית התקשורת נדרש כי קצב הקידוד יהיה חיובי. בדומה לבעיית התקשורת הקלאסית, גם עבור סכימות שידור אלו עבור כל קצב (הקטן מקיבול הערוץ) יתקבל מעריך שגיאה אחר.

לאורך עבודה נניח את קיומו של ערוץ משוב מעליו המשדר מודיע למקלט על קבלת ההחלטה. בפרט, נדון בשני סוגי אילוצים על מערכת התקשורת בנוכחות המשוב: הראשון, בו המשוב הוא נקי וחסר השהייה, וכן מותרת הגישה אליו בכל זמן שהוא. מעריך השגיאה האופטימאלי תחת הנחות אלו ידוע והוכחות שונות שלו תובאנה בעבודה. ההוכחות שתסקרנה הינן שונות זו מזו וכל אחת מספקת אינטואיציה שונה לגבי תוצאה זו. בנוסף למספר תוצאות שמופיעות בספרות, תוצג גם סכמה חדשה הכוללת אך ורק אלמנטים סדרתיים המשיגה ביצועים אופטימאליים. בכך גם יודגם הקשר שבין בעיית בחינת ההשערות לבעיית הקידוד שלפנינו. בסוג השני של אילוצים על המשוב, עדיין מניחים שערוץ המשוב חסר השהייה, אך כעת השימוש בו מותר רק פעם אחת לכל הודעה – להורות למשדר לחדול מהמשך שידור ההודעה הנוכחית ולאותת לו על מעבר להודעה הבאה. בעבודה תוצגנה תוצאות חדשות לגבי הביצועים של תקשורת מן הסוג הזה במובן מעריך

השגיאה. בפרט, מעריך השגיאה האופטימאלי עבור ערוץ בינארי סימטרי יחושב תחת ההנחה כי נעשה שימוש בקידוד אקראי וחסמים על מעריך זה עבור ערוצים כלליים יותר יוצגו אף הם.

מבנה העבודה הוא כדלקמן: חלקה הראשון מוקדש לבעיית בחינת ההשערות הסדרתית. בפרק 2 נדון בחינת השערות בינארית והפתרון של הבעיה במובן מבחן סדרתי אופטימאלי. בפרק 3 נדון בבעיית בה ההשערות הינן מורכבות ונציג כמה אבני דרך מרכזיות בהתפתחות התיאוריה שלהן. בפרק 4 נחזור להשערות פשוטות, אך כעת יילקחו בחשבון מספר גדול מ־2 של השערות וכן נדון במקרים בהם קיימת אפשרות לשליטה מסוימת על המדידות (המושג באמצעותו של משוב בין הקלט למשדר). במהלך החלק הראשון של העבודה יושם דגש על הסוגים השונים של המשטרים האסימפטוטיים וכן על קריטריוני הטיב הנהוגים בספרות של בחינת ההשערות הסדרתית.

בחלקה השני של העבודה, נדון בחסמים על ביצועי מערכת תקשורת הפועלת בנוכחות משוב. יתרונו של ערוץ המשוב הינו שהוא מאפשר עבודה עם קודים בעלי אורך בלוק משתנה כתלות בהודעה הנקלטת במקלט. בפרק 5 יוצג הפורמליזם המתמטי של הבעיה וכן יוגדרו פרמטרי התקשורת ומדד הטיב בו נתרכז לאורך העבודה, קרי, מעריך השגיאה. בנוסף ייערך דיון מקיף בנוגע למעריך השגיאה האופטימאלי של מערכת תקשורת עם משוב מושלם. בפרק 6 נדון בתרחיש מעט יותר מעשי שבו, במקום להניח ערוץ משוב מושלם הזמין בכל עת, נניח כי למפענח מותרת גישה בודדת למשוב לכל בלוק אינפורמציה וכן שערוץ המשוב הוא בינארי. עבור תרחיש זה תוצגנה בעבודה, בנוסף לתוצאות קלאסיות, גם מספר תוצאות מקוריות. בפרט, ניסקור סכימת הקידוד שהוצעה ע"י פורני בשנת 1968 ונוכיח כי מעריך השגיאה המתקבל הינו אופטימאלי במקרה של ערוץ בינארי סימטרי. פרק 7 נסכם את העבודה ונציע כמה כיווני מחקר עתידיים.

בחינת השערות סדרתית וקידוד ערוץ בנוכחות משוב

חיבור על עבודת גמר

לשם מילוי חלקי של הדרישות לקבלת התואר מגיסטר למדעים בהנדסת חשמל

שי גנזך

הוגש לסנט של הטכניון - מכון טכנולוגי לישראל.

שבט תשע"ה   חיפה   פברואר 2015