On H-Intersecting Graph Families

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1. Abstract

This poster applies a version of Shearer's lemma to derive a new upper bound on the maximum cardinality of a family of graphs on a fixed number of vertices, in which the intersection of every two graphs in that family contains a subgraph that is isomorphic to a specified graph H. Such families are referred to as H-intersecting graph families. The derived bound is expressed in terms of the chromatic number of H, extending the bound by Chung, Graham, Frankl, and Shearer (1986) with H specialized to a triangle.

2. H-Intersecting Graphs

- An H-intersecting family of graphs is a set of finite, undirected, and simple graphs (i.e., graphs with no self-loops or parallel edges), whose vertices are labelled, and the intersection of every two graphs in the family contains a subgraph isomorphic to H. Let $\mathcal{G}$ be a family of graphs on a common vertex set.
- These graph families play a central role in extremal graph theory. Determining their maximum possible size, for a fixed number of vertices, is a longstanding challenge.

3. Triangle-Intersecting Graphs

- Let $\mathcal{G}$ be a family of graphs on the vertex set $[n] \triangleq \{1, \dots, n\}$, with the property that for every $G_1, G_2 \in \mathcal{G}$, the intersection $G_1 \cap G_2$ contains a triangle (i.e, there are three vertices $i, j, k \in [n]$ such that each of $\{i, j\}$, $\{i, k\}$, $\{j, k\}$ is in the edge sets of both G_1 and G_2). The family $\mathcal{G}$ is referred to as a *triangle-intersecting* family of graphs on n vertices.
- A question, posed by Simonovits and Sós (1978), was how large can $\mathcal{G}$ be?

4. Simonovits & Sós Conjecture, '78

- The family $\mathcal{G}$ can be as large as $2^{\binom{n}{2}-3}$. with the family of all graphs on n vertices that have a particular triangle.
- On the other hand, $|\mathcal{G}|$ cannot exceed $2^{\binom{n}{2}-1}$. The latter upper bound holds since, in general, a family of distinct subsets of a set of size m, where any two of these subsets have a non-empty intersection, can have a cardinality of at most 2^{m-1} ($\mathcal{A}$ and $\mathcal{A}^{\mathrm{c}}$ cannot be members of this family). The edge sets of the graphs in $\mathcal{G}$ satisfy this property, with $m = \binom{n}{2}$.

5. Ellis, Filmus, and Friedgut, '12

Theorem. The size of a family $\mathcal{G}$ of triangle-intersecting graphs on n vertices satisfies $|\mathcal{G}| \le 2^{\binom{n}{2}-3}$, and it is attained by the family of all graphs with a common vertex set of n vertices, and with a fixed common triangle.

This result was proved by using discrete Fourier analysis to obtain the sharp bound, as conjectured by Simonovits and Sós

6. Prior Work (1986)

- The first significant progress towards proving the Simonovits–Sós conjecture came from an information-theoretic approach (1986). Using the combinatorial Shearer lemma, a simple and elegant upper bound on the size of $\mathcal{G}$ was derived in their work.
- That bound is equal to $2^{\binom{n}{2}-2}$, falling short of the Simonovits–Sós conjecture by a factor of 2.

7. Extended Conjecture and Progress

- It was conjectured by Ellis, Filmus, and Friedgut (2012) that, for $t \ge 4$, every K_t-intersecting family of graphs on a common vertex set $[n]$ has size at most $2^{\binom{n}{2}-\binom{t}{2}}$, with equality for the family of all graphs containing a fixed clique on t vertices.
- This conjecture was proved by Berger and Zhao (2023) for $t = 4$, while its validity is left open for $t \ge 5$.

8. Main Result

Theorem. Let H be a non-empty graph, and let $\mathcal{G}$ be a family of H-intersecting graphs on a common vertex set $[n]$. Then,

$$|\mathcal{G}| \le 2^{\binom{n}{2} - (\chi(\mathbf{H})-1)}, \tag{1}$$

where $\chi(H)$ denotes the chromatic number of the graph H.

Applied to K_t-intersecting families of graphs, for an integer $t \ge 3$, $|\mathcal{G}| \le 2^{\binom{n}{2}-(t-1)}$, extending the result by Shearer's lemma.

9. Combinatorial Shearer's lemma

The proof of the main result relies on the following tool.

Theorem.

- Let $\mathscr{F}$ be a finite multiset of subsets of $[n]$ (allowing repetitions of some subsets), where each element $i \in [n]$ is included in at least $k \ge 1$ sets of $\mathscr{F}$, and let $\mathscr{M}$ be a set of subsets of $[n]$.
- For every set $\mathcal{S} \in \mathscr{F}$, let the trace of $\mathscr{M}$ on $\mathcal{S}$, denoted by $\mathrm{trace}_{\mathcal{S}}(\mathscr{M})$, be the set of all possible intersections of elements of $\mathscr{M}$ with $\mathcal{S}$, i.e.,

$$\mathrm{trace}_{\mathcal{S}}(\mathscr{M}) \triangleq \{\mathcal{A} \cap \mathcal{S} : \mathcal{A} \in \mathscr{M}\}, \quad \forall \mathcal{S} \in \mathscr{F}. \tag{2}$$

Then,

$$|\mathscr{M}| \le \prod_{\mathcal{S} \in \mathscr{F}} \left|\mathrm{trace}_{\mathcal{S}}(\mathscr{M})\right|^{\frac{1}{k}}. \tag{3}$$

10. Main Result: Proof Outline

- Identify every graph $G \in \mathcal{G}$ with its edge set $E(G)$, and let $\mathscr{M} = \{E(G) : G \in \mathcal{G}\}$ (all these graphs have the common vertex set $[n]$).
- Let $\mathcal{U} = E(K_n)$. For every $G \in \mathcal{G}$, we have $E(G) \subseteq \mathcal{U}$, and $|\mathcal{U}| = \binom{n}{2}$.
- Let $t \triangleq \chi(H)$. For every unordered equipartition of $[n]$ into $t-1$ disjoint subsets, i.e., $\bigcup_{j=1}^{t-1} \mathcal{A}_j = [n]$, which satisfies $\big||\mathcal{A}_i| - |\mathcal{A}_j|\big| \le 1$ for all $1 \le i < j \le t-1$, let $\mathcal{U}(\{\mathcal{A}_j\}_{j=1}^{t-1})$ be the subset of $\mathcal{U}$ consisting of all those edges that lie entirely inside one of the subsets $\{\mathcal{A}_j\}_{j=1}^{t-1}$.
- We apply the combinatorial version of Shearer's lemma with $\mathscr{F} = \{\mathcal{U}(\{\mathcal{A}_j\}_{j=1}^{t-1})\}$, taken over all unordered equipartitions of $[n]$, $\{\mathcal{A}_j\}_{j=1}^{t-1}$, as described above.

11. Proof Outline (cont.)

- Let $m = |\mathcal{U}(\{\mathcal{A}_j\}_{j=1}^{t-1})|$, which is independent of the equipartition. It can be verified that $m \le \frac{1}{\chi(H)-1}\binom{n}{2}$.
- By a simple double-counting argument in regard to the edges of the complete graph K_n (the set $\mathcal{U}$), if k is the number of elements of $\mathscr{F}$ in which each element of $\mathcal{U}$ occurs, then $m\,|\mathscr{F}| = \binom{n}{2} k$.
- Let $\mathcal{S} \in \mathscr{F}$. It can be verified that $\mathrm{trace}_{\mathcal{S}}(\mathscr{M})$ forms an intersecting family of subsets of $\mathcal{S}$.
- Consequently, $|\mathcal{S}| = m$ yields $|\mathrm{trace}_{\mathcal{S}}(\mathscr{M})| \le 2^{m-1}$.
- The proof of the proposed main result is then completed by an application of the Combinatorial version of Shearer's lemma (and the one-to-one correspondence between $\mathcal{G}$ and $\mathscr{M}$).

Relaxed Bound (Lovász ϑ-Function)

- The computational complexity of the chromatic number of a graph is in general NP-hard. This poses a problem in calculating the upper bound on the cardinality of H-intersecting families of graphs on a fixed number of vertices. This bound is loosened, expressing it in terms of the Lovász ϑ-function of the complement graph $\overline{H}$.
- **Corollary.** Let H be a graph, and let $\mathcal{G}$ be a family of H-intersecting graphs on a common vertex set $[n]$. Then,

$$|\mathcal{G}| \le 2^{\binom{n}{2} - (\lceil \vartheta(\overline{H}) \rceil - 1)}. \tag{4}$$

- The Lovász ϑ-function of $\overline{H}$ can be efficiently computed with a precision of r decimal digits, having a computational complexity that is polynomial in $|V(H)|$ and r by solving a semidefinite programming problem.