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On H-Intersecting Graph Families

(Extended abstract)

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Abstract

This paper applies the combinatorial version of Shearer's inequalities to derive a new upper bound on the maximum cardinality of a family of graphs on a fixed number of vertices, in which the intersection of every two graphs in that family contains a subgraph that is isomorphic to a specified graph H. Such families are referred to as H-intersecting graph families. The derived bound is expressed in terms of the chromatic number of H, extending the bound by Chung, Graham, Frankl, and Shearer (1986) with H specialized to a triangle.

1 Introduction

An H-intersecting family of graphs is a collection of finite, undirected, and simple graphs (i.e., graphs with no self-loops or parallel edges), whose vertices are labelled, and the intersection of every two graphs in the family contains a subgraph isomorphic to H. For instance, if H is an edge or a triangle, then every pair of graphs in the family shares at least one edge or triangle, respectively. These intersecting families of graphs play a central role in extremal combinatorics and graph theory, where determining their maximum possible size remains a longstanding challenge. Different choices of H lead to distinct combinatorial problems and structural constraints.

A pivotal conjecture, proposed in 1976 by Simonovits and Sós, concerned the maximum size of triangle-intersecting graph families—those in which the intersection of any two graphs contains a triangle. Their foundational work, initially presented in [1], along with other results on intersection theorems for families of graphs where the shared subgraphs are cycles or paths, was surveyed in [2]. The first major progress on this conjecture was made by Chung, Graham, Frankl, and Shearer [3], who utilized Shearer's inequality to establish a non-trivial bound on the largest possible cardinality of a family of triangle-intersecting graphs with a fixed number of vertices. This bound lay between the trivial bound and the conjectured bound.

The conjecture by Simonovits and Sós was ultimately resolved by Ellis, Filmus, and Friedgut [4], who proved that the largest triangle-intersecting family comprises all graphs

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containing a fixed triangle. Building on the spectral approach in [4] (see also [5]), a recent work by Berger and Zhao [6] extended the investigation to K_4-intersecting graph families, addressing analogous questions for graph families where every pair of graphs intersects in a complete subgraph of size four. Additionally, Keller and Lifshitz [7] provided high-probability results for constructing, for every graph H, families of large random graphs with a common vertex set such that every pair of graphs contains a subgraph isomorphic to H. These are referred to as families of H-intersecting graphs.

The paper employs the combinatorial version of Shearer's lemma for upper bounding the size of H-intersecting families of graphs. An extended version of this work is available in [8].

2 Preliminaries

Definition 2.1 (Triangle-Intersecting Families of Graphs)**.** Let $\mathcal{G}$ be a family of graphs on the vertex set $[n] \triangleq \{1, \ldots, n\}$, with the property that for every $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}$, the intersection $\mathsf{G}_1 \cap \mathsf{G}_2$ contains a triangle (i.e, there are three vertices $i, j, k \in [n]$ such that each of $\{i, j\}$, $\{i, k\}$, $\{j, k\}$ is in the edge sets of both G_1 and G_2). The family $\mathcal{G}$ is referred to as a *triangle-intersecting* family of graphs on n vertices.

The question that was posed by Simonovits and Sós [1] was how large can $\mathcal{G}$, a family of triangle-intersecting graphs, be?

The family $\mathcal{G}$ can be as large as $2^{\binom{n}{2}-3}$. To that end, consider the family $\mathcal{G}$ of all graphs on n vertices that include a particular triangle. On the other hand, $|\mathcal{G}|$ cannot exceed $2^{\binom{n}{2}-1}$. The latter upper bound holds since, in general, a family of distinct subsets of a set of size m, where any two of these subsets have a non-empty intersection, can have a cardinality of at most 2^{m-1} ($\mathcal{A}$ and $\mathcal{A}^{\mathrm{c}}$ cannot be members of this family). The edge sets of the graphs in $\mathcal{G}$ satisfy this property, with $m = \binom{n}{2}$.

Theorem 2.1 (Ellis, Filmus, and Friedgut, [4])**.** The size of a family $\mathcal{G}$ of triangle-intersecting graphs on n vertices satisfies $|\mathcal{G}| \leq 2^{\binom{n}{2}-3}$, and it is attained by the family of all graphs with a common vertex set of n vertices, and with a fixed common triangle.

This result was proved by using discrete Fourier analysis to obtain the sharp bound in Theorem 2.1, as conjectured by Simonovits and Sós [1].

The graph K_t, with $t \in \mathbb{N}$, denotes the complete graph on t vertices, e.g., K_3 is a triangle. All results in this paper apply to finite, undirected, and simple graphs.

The first significant progress towards proving the Simonovits–Sós conjecture came from an information-theoretic approach [3]. Using the combinatorial Shearer lemma, a simple and elegant upper bound on the size of $\mathcal{G}$ was derived in [3]. That bound is equal to $2^{\binom{n}{2}-2}$, falling short of the Simonovits–Sós conjecture by a factor of 2.

Proposition 2.1 (Chung, Graham, Frankl, and Shearer, [3])**.** Let $\mathcal{G}$ be a family of K_3-intersecting graphs on a common vertex set $[n]$. Then, $|\mathcal{G}| \leq 2^{\binom{n}{2}-2}$.

We next consider more general intersecting families of graphs.

Definition 2.2 (H-intersecting Families of Graphs)**.** Let $\mathcal{G}$ be a family of graphs on a common vertex set. Then, it is said that $\mathcal{G}$ is H-intersecting if for every two graphs $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}$, the graph $\mathsf{G}_1 \cap \mathsf{G}_2$ contains a subgraph isomorphic to H.

The combinatorial version of Shearer's lemma, presented next, was essential in [3] for deriving Proposition 2.1. It is also used later in this work to establish a nontrivial extension of that result, providing a new upper bound on the maximum cardinality of a family of graphs with a fixed number of vertices that is H-intersecting for an arbitrary nonempty graph H.

Theorem 2.2 (Combinatorial version of Shearer's lemma, [3])**.** Let $\mathscr{F}$ be a finite multiset of subsets of $[n]$ (allowing repetitions of some subsets), where each element $i \in [n]$ is included in at least $k \geq 1$ sets of $\mathscr{F}$, and let $\mathscr{M}$ be a set of subsets of $[n]$. For every set $\mathcal{S} \in \mathscr{F}$, let the trace of $\mathscr{M}$ on $\mathcal{S}$, denoted by $\text{trace}_{\mathcal{S}}(\mathscr{M})$, be the set of all possible intersections of elements of $\mathscr{M}$ with $\mathcal{S}$, i.e.,

$$\text{trace}_{\mathcal{S}}(\mathscr{M}) \triangleq \{\mathcal{A} \cap \mathcal{S} : \mathcal{A} \in \mathscr{M}\}, \quad \forall \mathcal{S} \in \mathscr{F}. \tag{1}$$

Then,

$$|\mathscr{M}| \leq \prod_{\mathcal{S} \in \mathscr{F}} \left|\text{trace}_{\mathcal{S}}(\mathscr{M})\right|^{\frac{1}{k}}. \tag{2}$$

An open problem in extremal combinatorics is, given H and n, what is the maximum size of an H-intersecting family of graphs on n labeled vertices? It was conjectured by Ellis, Filmus, and Friedgut in [4] that every K_t-intersecting family of graphs on a common vertex set $[n]$ has size at most $2^{\binom{n}{2}-\binom{t}{2}}$, with equality for the family of all graphs containing a fixed clique on t vertices. This conjecture was proved in [4] for $t = 3$, and was recently proved by Berger and Zhao [6] for $t = 4$, while this problem is left open for $t \geq 5$.

3 Intersecting Families of Graphs

The following result generalizes Proposition 2.1 and it extends the concept of proof in [3] to hold for every family of H-intersecting graphs on a common vertex set.

Proposition 3.1 (An upper bound on the cardinality of H-intersecting graphs, [8])**.** Let H be a non-empty graph, and let $\mathcal{G}$ be a family of H-intersecting graphs on a common vertex set $[n]$. Then,

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-(\chi(\mathsf{H})-1)}. \tag{3}$$

Proof.

- Identify every graph $\mathsf{G} \in \mathcal{G}$ with its edge set $\mathsf{E}(\mathsf{G})$, and let $\mathscr{M} = \{\mathsf{E}(\mathsf{G}) : \mathsf{G} \in \mathcal{G}\}$ (all these graphs have the common vertex set $[n]$).
- Let $\mathcal{U} = \mathsf{E}(\mathsf{K}_n)$. For every $\mathsf{G} \in \mathcal{G}$, we have $\mathsf{E}(\mathsf{G}) \subseteq \mathcal{U}$, and $|\mathcal{U}| = \binom{n}{2}$.
- Let $t \triangleq \chi(\mathsf{H})$. For every unordered equipartition of $[n]$ into $t-1$ disjoint subsets, i.e., $\bigcup_{j=1}^{t-1} \mathcal{A}_j = [n]$, which satisfies $\big||\mathcal{A}_i| - |\mathcal{A}_j|\big| \leq 1$ for all $1 \leq i < j \leq t-1$, let $\mathcal{U}(\{\mathcal{A}_j\}_{j=1}^{t-1})$ be the subset of $\mathcal{U}$ consisting of all those edges that lie entirely inside one of the subsets $\{\mathcal{A}_j\}_{j=1}^{t-1}$.

- We apply the combinatorial version of Shearer's lemma (Theorem 2.2) with
$$\mathscr{F} = \{\mathcal{U}(\{\mathcal{A}_j\}_{j=1}^{t-1})\}, \tag{4}$$
taken over all unordered equipartitions of $[n]$, $\{\mathcal{A}_j\}_{j=1}^{t-1}$, as described above.

- Let $m = |\mathcal{U}(\{\mathcal{A}_j\}_{j=1}^{t-1})|$, which is independent of the equipartition since
$$m = \begin{cases} (t-1)\binom{n/(t-1)}{2} & \text{if } (t-1)|n, \\ (t-2)\binom{\lfloor n/(t-1)\rfloor}{2} + \binom{\lceil n/(t-1)\rceil}{2} & \text{if } (t-1)|(n-1), \\ \vdots & \\ \binom{\lfloor n/(t-1)\rfloor}{2} + (t-2)\binom{\lceil n/(t-1)\rceil}{2} & \text{if } (t-1)|\bigl(n-(t-2)\bigr). \end{cases} \tag{5}$$

- By (5) with $t \triangleq \chi(\mathsf{H})$, it can be verified that
$$m \leq \frac{1}{\chi(\mathsf{H})-1}\binom{n}{2}. \tag{6}$$
The details of that derivation are omitted and can be found in [8].

- By a simple double-counting argument in regard to the edges of the complete graph K_n (the set $\mathcal{U}$), if k is the number of elements of $\mathscr{F}$ in which each element of $\mathcal{U}$ occurs, then
$$m\,|\mathscr{F}| = \binom{n}{2}k. \tag{7}$$

- Let $\mathcal{S} \in \mathscr{F}$. Observe that $\text{trace}_{\mathcal{S}}(\mathscr{M})$, as defined in (1), forms an intersecting family of subsets of $\mathcal{S}$. Indeed,

  1. Assign to each vertex in $[n]$ the index j of the subset $\mathcal{A}_j$ $(1 \leq j \leq \chi(\mathsf{H})-1)$ in the partition of $[n]$ corresponding to $\mathcal{S}$. Let these assignments be associated with $\chi(\mathsf{H})-1$ color classes of the vertices.
  2. For any $\mathsf{G}, \mathsf{G}' \in \mathcal{G}$, the graph $\mathsf{G} \cap \mathsf{G}'$ contains a subgraph H (by assumption).
  3. By the definition of the chromatic number of H as the smallest number of colors that are required such that any two adjacent vertices in H are assigned different colors, it follows that there exists an edge in H whose two vertices are assigned the same index (color). Hence, that edge belongs to the set $\mathcal{A}_j$, for some $j \in [\chi(\mathsf{H})-1]$, so it belongs to $\mathcal{S}$.
  4. The complement of $\mathcal{S}$ (in $\mathcal{U}$) is therefore H-free (viewed as a graph with the vertex set $[n]$).

  Consequently, since $|\mathcal{S}| = m$, we get
$$|\text{trace}_{\mathcal{S}}(\mathscr{M})| \leq 2^{m-1}. \tag{8}$$

- By Theorem 2.2 (and the one-to-one correspondence between $\mathcal{G}$ and $\mathscr{M}$),

$$\begin{aligned} |\mathcal{G}| &= |\mathscr{M}| \\ &\leq \left(2^{m-1}\right)^{\frac{|\mathscr{F}|}{k}} \qquad (9) \\ &= 2^{\binom{n}{2}\left(1-\frac{1}{m}\right)} \qquad (10) \\ &\leq 2^{\binom{n}{2}-(\chi(\mathsf{H})-1)}, \qquad (11) \end{aligned}$$

where (9) relies on (2) and (8), then (10) relies on (7), and (11) is due to (6).

□

The family $\mathcal{G}$ of H-intersecting graphs on n vertices can be as large as $2^{\binom{n}{2}-|\mathsf{E}(\mathsf{H})|}$. To that end, consider the family $\mathcal{G}$ of all graphs on n vertices that include a particular H subgraph. Combining this lower bound on $|\mathcal{G}|$ with its upper bound in Theorem 3 gives that the largest family $\mathcal{G}$ of H-intersecting graphs on n vertices satisfies

$$2^{\binom{n}{2}-|\mathsf{E}(\mathsf{H})|} \leq |\mathcal{G}| \leq 2^{\binom{n}{2}-(\chi(\mathsf{H})-1)}. \qquad (12)$$

Specialization of Proposition 3.1 to a family $\mathcal{G}$ that is K_t-intersecting graphs, with $t \geq 2$, on a common vertex set $[n]$, gives that $|\mathcal{G}| \leq 2^{\binom{n}{2}-(t-1)}$.

The computational complexity of the chromatic number of a graph is in general NP-hard [9]. This poses a problem in calculating the upper bound in Proposition 3.1 on the cardinality of H-intersecting families of graphs on a fixed number of vertices. This bound can be loosened, expressing it in terms of the Lovász ϑ-function of the complement graph $\overline{\mathsf{H}}$.

Corollary 3.1. Let H be a graph, and let $\mathcal{G}$ be a family of H-intersecting graphs on a common vertex set $[n]$. Then,

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-(\lceil\vartheta(\overline{\mathsf{H}})\rceil-1)}. \qquad (13)$$

Proof. The Lovász ϑ-function of the complement graph $\overline{\mathsf{H}}$ satisfies (see Corollary 3 of [10])

$$\omega(\mathsf{H}) \leq \vartheta(\overline{\mathsf{H}}) \leq \chi(\mathsf{H}), \qquad (14)$$

so it is bounded between the clique and chromatic numbers of H, which are both NP-hard to compute [9]. Since the chromatic number $\chi(\mathsf{H})$ is an integer, we have $\chi(\mathsf{H}) \geq \lceil\vartheta(\overline{\mathsf{H}})\rceil$. Combining (3) and the latter inequality yields (13). □

The Lovász ϑ-function of the complement graph $\overline{\mathsf{H}}$, as presented in Corollary 3.1, can be efficiently computed with a precision of r decimal digits, having a computational complexity that is polynomial in $p \triangleq |\mathsf{V}(\mathsf{H})|$ and r. It is obtained by solving the following semidefinite programming (SDP) problem [11]:

$$\boxed{\begin{array}{l} \text{maximize } \operatorname{Tr}(\mathbf{B}\,\mathbf{J}_p) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^p, \ \operatorname{Tr}(\mathbf{B}) = 1, \\ A_{i,j} = 0 \ \Rightarrow \ B_{i,j} = 0, \quad i,j \in [p], \ i \neq j, \end{cases} \end{array}} \qquad (15)$$

where the following notation is used: $\mathbf{A} = \mathbf{A}(\mathsf{H})$ is the $p \times p$ adjacency matrix of H; $\mathbf{J}_p$ is the all-ones $p \times p$ matrix, and $\mathcal{S}_+^p$ is the set of all $p \times p$ positive semidefinite matrices. The reader is referred to an account of interesting properties of the Lovász ϑ-function in [12], Chapter 11 of [13], and more recently in Section 2.5 of [14].

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