Tightened Exponential Bounds for Discrete Time, Conditionally Symmetric Martingales with Bounded Jumps

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Abstract

This letter derives some new exponential bounds for discrete time, real valued, conditionally symmetric martingales with bounded jumps. The new bounds are extended to conditionally symmetric sub/ supermartingales, and they are compared to some existing bounds.

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I. Introduction and Main Results

Classes of exponential bounds for discrete-time real-valued martingales have been extensively studied in the literature (see, e.g., Alon and Spencer (2008), Azuma (1967), Burkholder (1991), Chung and Lu (2006), Dembo and Zeitouni (1997), Dzhaparide and van Zanten (2001), Freedman (1975), Grama and E. Haeusler (2000), Hoeffding (1963), McDiarmid (1989, 1998), de la Peña (1999), de la Peña, Klass and Lai (2004), Pinelis (1994) and Steiger (1969)). This letter further assumes conditional symmetry of these martingales, as is defined in the following.

Definition 1: Let $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$, where $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$, be a discrete-time and real-valued martingale, and let $\xi_k \triangleq X_k - X_{k-1}$ for every $k \in \mathbb{N}$ designate the jumps of the martingale. Then $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ is called a *conditionally symmetric martingale* if, conditioned on $\mathcal{F}_{k-1}$, the random variable ξ_k is symmetrically distributed around zero.

Our goal in this letter is to demonstrate how the assumption of the conditional symmetry improves the existing exponential inequalities for discrete-time real-valued martingales with bounded increments. Earlier results, serving as motivation, appear in Section 4 of Dzhaparide and J. H. van Zanten (2001) and Section 6 of de la Peña (1999). The new exponential bounds are also extended to conditionally symmetric submartingales or supermartingales, where the construction of these objects is exemplified later in this section. The relation of some of the exponential bounds derived in this work with some existing bounds is discussed later in this letter. Additional results addressing weak-type inequalities, maximal inequalities and ratio inequalities for conditionally symmetric martingales were derived in Osękowski (2010a,b) and Wang (1991).

A. Main Results

Our main results for conditionally symmetric martingales with bounded jumps are introduced in Theorems 1, 3 and 4. Theorems 2 and 5 are existing bounds, for general martingales without the conditional symmetry assumption, that are introduced in connection to the new theorems. Corollaries 2 and 3 provide an extension of the new results to conditionally symmetric sub/ supermartingales with bounded jumps. Our first result is the following theorem.

Theorem 1: Let $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ be a discrete-time real-valued and conditionally symmetric martingale. Assume that, for some fixed numbers $d, \sigma > 0$, the following two requirements are satisfied a.s.:

$$|X_k - X_{k-1}| \leq d, \qquad \text{Var}(X_k|\mathcal{F}_{k-1}) = \mathbb{E}\big[(X_k - X_{k-1})^2 \,|\, \mathcal{F}_{k-1}\big] \leq \sigma^2 \tag{1}$$

for every $k \in \mathbb{N}$. Then, for every $\alpha \geq 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} |X_k - X_0| \geq \alpha n\right) \leq 2\exp\big(-nE(\gamma, \delta)\big) \tag{2}$$

where

$$\gamma \triangleq \frac{\sigma^2}{d^2}, \quad \delta \triangleq \frac{\alpha}{d} \tag{3}$$

and for $\gamma \in (0, 1]$ and $\delta \in [0, 1)$

$$E(\gamma, \delta) \triangleq \delta x - \ln\Big(1 + \gamma\big[\cosh(x) - 1\big]\Big) \tag{4}$$

$$x \triangleq \ln\left(\frac{\delta(1-\gamma) + \sqrt{\delta^2(1-\gamma)^2 + \gamma^2(1-\delta^2)}}{\gamma(1-\delta)}\right). \tag{5}$$

If $\delta > 1$, then the probability on the left-hand side of (2) is zero (so $E(\gamma, \delta) \triangleq +\infty$), and $E(\gamma, 1) = \ln\big(\frac{2}{\gamma}\big)$. Furthermore, the exponent $E(\gamma, \delta)$ is asymptotically optimal in the sense that there exists a conditionally symmetric martingale, satisfying the conditions in (1) a.s., that attains this exponent in the limit where $n \to \infty$.

Remark 1: From the above conditions, without any loss of generality, $\sigma^2 \leq d^2$ and therefore $\gamma \in (0, 1]$. This implies that Theorem 1 characterizes the exponent $E(\gamma, \delta)$ for all values of γ and δ.

Corollary 1: Let $\{U_k\}_{k=1}^{\infty} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ be i.i.d. and bounded random variables with a symmetric distribution around their mean value. Assume that $|U_1 - \mathbb{E}[U_1]| \leq d$ a.s. for some $d > 0$, and $\text{Var}(U_1) \leq \gamma d^2$ for some $\gamma \in [0, 1]$. Let $\{S_n\}$ designate the sequence of partial sums, i.e., $S_n \triangleq \sum_{k=1}^n U_k$ for every $n \in \mathbb{N}$. Then, for every $\alpha \geq 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \big|S_k - k\,\mathbb{E}(U_1)\big| \geq \alpha n\right) \leq 2\exp\big(-nE(\gamma, \delta)\big), \quad \forall\, n \in \mathbb{N} \tag{6}$$

where $\delta \triangleq \frac{\alpha}{d}$, and $E(\gamma, \delta)$ is introduced in (4) and (5).

Theorem 1 should be compared to the statement in Theorem 6.1 of McDiarmid (1989) (see also Corollary 2.4.7 in Dembo and Zeitouni (1997)), which does not require the conditional symmetry property. It gives the following result.

Theorem 2: Let $\{X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a discrete-time real-valued martingale with bounded jumps. Assume that the two conditions in (1) are satisfied a.s. for every $k \in \mathbb{N}$. Then, for every $\alpha \geq 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |X_k - X_0| \geq \alpha n\right) \leq 2\exp\left(-n\, D\Big(\frac{\delta + \gamma}{1 + \gamma}\Big|\Big|\frac{\gamma}{1 + \gamma}\Big)\right) \tag{7}$$

where γ and δ are introduced in (3), and

$$D(p||q) \triangleq p \ln\Big(\frac{p}{q}\Big) + (1-p)\ln\Big(\frac{1-p}{1-q}\Big), \quad \forall\, p, q \in [0, 1] \tag{8}$$

is the divergence (also known as relative entropy or Kullback-Leibler distance) to the natural base between the two probability distributions $(p, 1-p)$ and $(q, 1-q)$ (with the convention that $0 \log 0$ is zero in the case where p or q take on the values zero or one). If $\delta > 1$, then the probability on the left-hand side of (7) is zero. Furthermore, the exponent on the right-hand side of (7) is asymptotically optimal under the assumptions of this theorem.

Remark 2: The two exponents in Theorems 1 and 2 are both discontinuous at $\delta = 1$. This is consistent with the assumption of the bounded jumps that implies that $\mathbb{P}(|X_n - X_0| \geq nd\delta)$ is equal to zero if $\delta > 1$.

If $\delta \to 1^-$ then, from (4) and (5), for every $\gamma \in (0, 1]$,

$$\lim_{\delta \to 1^-} E(\gamma, \delta) = \lim_{x \to \infty} \big[x - \ln\big(1 + \gamma(\cosh(x) - 1)\big)\big] = \ln\left(\frac{2}{\gamma}\right). \tag{9}$$

On the other hand, the right limit at $\delta = 1$ is infinity since $E(\gamma, \delta) = +\infty$ for every $\delta > 1$. The same discontinuity also exists for the exponent in Theorem 2 where the right limit at $\delta = 1$ is infinity, and the left limit is equal to

$$\lim_{\delta \to 1^-} D\Big(\frac{\delta + \gamma}{1 + \gamma}\Big|\Big|\frac{\gamma}{1 + \gamma}\Big) = \ln\left(1 + \frac{1}{\gamma}\right) \tag{10}$$

where the last equality follows from (8). A comparison of the limits in (9) and (10) is consistent with the improvement that is obtained in Theorem 1 as compared to Theorem 2 due to the additional assumption of the conditional symmetry that is relevant if $\gamma \in (0,1)$. It can be verified that the two exponents coincide if $\gamma = 1$ (which is equivalent to removing the constraint on the conditional variance), and their common value is equal to

$$f(\delta) = \begin{cases} \ln(2)\Big[1 - h_2\left(\frac{1-\delta}{2}\right)\Big], & 0 \leq \delta \leq 1 \\ +\infty, & \delta > 1 \end{cases} \tag{11}$$

where $h_2(x) \triangleq -x\log_2(x) - (1-x)\log_2(1-x)$ for $0 \leq x \leq 1$ denotes the binary entropy function to the base 2 (with the convention that it is defined to be zero at $x = 0$ or $x = 1$).

Theorem 1 provides an improvement over the bound in Theorem 2 for conditionally symmetric martingales with bounded jumps. The bounds in Theorems 1 and 2 depend on the conditional variance of the martingale, but they do not take into consideration conditional moments of higher orders. The following bound generalizes the bound in Theorem 1, but it does not admit in general a closed-form expression.

Theorem 3: Let $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ be a discrete-time and real-valued conditionally symmetric martingale. Let $m \in \mathbb{N}$ be an even number, and assume that the following conditions hold a.s. for every $k \in \mathbb{N}$

$$|X_k - X_{k-1}| \leq d, \qquad \mathbb{E}\big[(X_k - X_{k-1})^l \,|\, \mathcal{F}_{k-1}\big] \leq \mu_l, \;\; \forall\, l \in \{2, 4, \dots, m\}$$

for some $d > 0$ and non-negative numbers $\{\mu_2, \mu_4, \dots, \mu_m\}$. Then, for every $\alpha \geq 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} |X_k - X_0| \geq \alpha n\right) \leq 2\left\{\min_{x\geq 0} e^{-\delta x}\left[1 + \sum_{l=1}^{\frac{m}{2}-1} \frac{(\gamma_{2l} - \gamma_m)\, x^{2l}}{(2l)!} + \gamma_m\big(\cosh(x) - 1\big)\right]\right\}^n \tag{12}$$

where

$$\delta \triangleq \frac{\alpha}{d}, \quad \gamma_{2l} \triangleq \frac{\mu_{2l}}{d^{2l}}, \;\; \forall\, l \in \Big\{1, \dots, \frac{m}{2}\Big\}. \tag{13}$$

We consider in the following a different type of exponential inequalities for conditionally symmetric martingales with bounded jumps.

Theorem 4: Let $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a discrete-time real-valued and conditionally symmetric martingale. Assume that there exists a fixed number $d > 0$ such that $\xi_k \triangleq X_k - X_{k-1} \leq d$ a.s. for every $k \in \mathbb{N}$. Let

$$Q_n \triangleq \sum_{k=1}^{n} \mathbb{E}[\xi_k^2 \,|\, \mathcal{F}_{k-1}] \tag{14}$$

with $Q_0 \triangleq 0$, be the predictable quadratic variation of the martingale up to time n. Then, for every $z, r > 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} (X_k - X_0) \geq z,\; Q_n \leq r \;\text{ for some } n \in \mathbb{N}\right) \leq \exp\left(-\frac{z^2}{2r} \cdot C\left(\frac{zd}{r}\right)\right) \tag{15}$$

where

$$C(u) \triangleq \frac{2[u\sinh^{-1}(u) - \sqrt{1+u^2} + 1]}{u^2}, \quad \forall\, u > 0. \tag{16}$$

Theorem 4 should be compared to Theorem 1.6 in Freedman (1975) (see also Exercise 2.4.21(b) in Dembo and Zeitouni (1997)) that was stated without the requirement for the conditional symmetry of the martingale. It provides the following result:

Theorem 5: Let $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a discrete-time real-valued martingale. Assume that there exists a fixed number $d > 0$ such that $\xi_k \triangleq X_k - X_{k-1} \leq d$ a.s. for every $k \in \mathbb{N}$. Then, for every $z, r > 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} (X_k - X_0) \geq z,\; Q_n \leq r \;\text{ for some } n \in \mathbb{N}\right) \leq \exp\left(-\frac{z^2}{2r} \cdot B\left(\frac{zd}{r}\right)\right) \tag{17}$$

where

$$B(u) \triangleq \frac{2[(1+u)\ln(1+u) - u]}{u^2}, \quad \forall\, u > 0. \tag{18}$$

The proof of Theorem 1.6 in Freedman (1975) is modified by using Bennett's inequality (see Bennett (1962)) for the derivation of the original bound in Theorem 5 (without the conditional symmetry requirement). Furthermore, this modified proof serves to derive the improved bound in Theorem 4 under the conditional symmetry assumption.

In the following, the inequalities are extended to discrete-time, real-valued, and conditionally symmetric sub/ supermartingales.

Definition 2: Let $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ be a discrete-time real-valued sub or supermartingale, and $\eta_k \triangleq X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]$ for every $k \in \mathbb{N}$. Then the martingale $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ is called, respectively, a conditionally symmetric sub or supermartingale if, conditioned on $\mathcal{F}_{k-1}$, the random variable η_k is symmetrically distributed around zero.

Remark 3: For martingales, $\eta_k = \xi_k$ for every $k \in \mathbb{N}$, so we obtain consistency with Definition 1.

An extension of Theorem 1 to conditionally symmetric sub and supermartingales is introduced in the following.

Corollary 2: Let $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ be a discrete-time, real-valued and conditionally symmetric supermartingale. Assume that, for some constants $d, \sigma > 0$, the following two requirements are satisfied a.s.

$$\eta_k \le d, \qquad \text{Var}(X_k|\mathcal{F}_{k-1}) \triangleq \mathbb{E}\big[\eta_k^2 \,|\, \mathcal{F}_{k-1}\big] \le \sigma^2 \tag{19}$$

for every $k \in \mathbb{N}$. Then, for every $\alpha \ge 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\Big(\max_{1\le k\le n} (X_k - X_0) \ge \alpha n\Big) \le \exp\big(-n\, E(\gamma, \delta)\big) \tag{20}$$

where γ and δ are defined in (3), and $E(\gamma, \delta)$ is introduced in (4). Alternatively, if $\{X_k, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ is a conditionally symmetric submartingale, the same bound holds for $\mathbb{P}\big(\min_{1\le k\le n}(X_k - X_0) \le -\alpha n\big)$ provided that $\eta_k \ge -d$ and the second condition in (19) hold a.s. for every $k \in \mathbb{N}$. If $\delta > 1$, then these two probabilities are zero.

The following statement extends Theorem 4 to conditionally symmetric supermartingales.

Corollary 3: Let $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a discrete-time, real-valued supermartingale. Assume that there exists a fixed number $d > 0$ such that $\eta_k \le d$ a.s. for every $k \in \mathbb{N}$. Let $\{Q_n\}_{n\in\mathbb{N}_0}$ be the predictable quadratic variations of the supermartingale, i.e., $Q_n \triangleq \sum_{k=1}^n \mathbb{E}[\eta_k^2 \,|\, \mathcal{F}_{k-1}]$ for every $n \in \mathbb{N}$ with $Q_0 \triangleq 0$. Then, the result in (17) holds. Furthermore, if the supermartingale is conditionally symmetric, then the improved bound in (15) holds.

B. Construction of Discrete-Time, Real-Valued and Conditionally Symmetric Sub/ Supermartingales

Before proving the tightened inequalities for discrete-time conditionally symmetric sub/ supermartingales, it is worth exemplifying the construction of these objects.

Example 1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{U_k\}_{k\in\mathbb{N}} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a sequence of independent random variables with zero mean. Let $\{\mathcal{F}_k\}_{k\ge 0}$ be the natural filtration of sub σ-algebras of $\mathcal{F}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(U_1, \ldots, U_k)$ for $k \ge 1$. Furthermore, for $k \in \mathbb{N}$, let $A_k \in L^\infty(\Omega, \mathcal{F}_{k-1}, \mathbb{P})$ be an $\mathcal{F}_{k-1}$-measurable random variable with a finite essential supremum. Define a new sequence of random variables in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ where

$$X_n = \sum_{k=1}^n A_k U_k, \quad \forall\, n \in \mathbb{N}$$

and $X_0 = 0$. Then, $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is a martingale. Let us assume that the random variables $\{U_k\}_{k\in\mathbb{N}}$ are symmetrically distributed around zero. Note that $X_n = X_{n-1} + A_n U_n$ where A_n is $\mathcal{F}_{n-1}$-measurable and U_n is independent of the σ-algebra $\mathcal{F}_{n-1}$ (due to the independence of the random variables $U_1, \ldots, U_n$). It therefore follows that for every $n \in \mathbb{N}$, given $\mathcal{F}_{n-1}$, the random variable X_n is symmetrically distributed around its conditional expectation X_{n-1}. Hence, the martingale $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is conditionally symmetric.

Example 2: In continuation of Example 1, let $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a martingale, and define $Y_0 = 0$ and

$$Y_n = \sum_{k=1}^n A_k (X_k - X_{k-1}), \quad \forall\, n \in \mathbb{N}.$$

The sequence $\{Y_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is a martingale. If $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is a conditionally symmetric martingale then also the martingale $\{Y_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is conditionally symmetric (since $Y_n = Y_{n-1} + A_n(X_n - X_{n-1})$, and by assumption A_n is $\mathcal{F}_{n-1}$-measurable).

Example 3: In continuation of Example 1, let $\{U_k\}_{k\in\mathbb{N}}$ be independent random variables with a symmetric distribution around their expected value, and also assume that $\mathbb{E}(U_k) \leq 0$ for every $k \in \mathbb{N}$. Furthermore, let $A_k \in L^\infty(\Omega, \mathcal{F}_{k-1}, \mathbb{P})$, and assume that a.s. $A_k \geq 0$ for every $k \in \mathbb{N}$. Let $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a martingale as defined in Example 1. Note that $X_n = X_{n-1} + A_n U_n$ where A_n is non-negative and $\mathcal{F}_{n-1}$-measurable, and U_n is independent of $\mathcal{F}_{n-1}$ and symmetrically distributed around its average. This implies that $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is a conditionally symmetric supermartingale.

Example 4: In continuation of Examples 2 and 3, let $\{X_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a conditionally symmetric supermartingale. Define $\{Y_n\}_{n\in\mathbb{N}_0}$ as in Example 2 where A_k is non-negative a.s. and $\mathcal{F}_{k-1}$-measurable for every $k \in \mathbb{N}$. Then $\{Y_n, \mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is a conditionally symmetric supermartingale.

II. PROOFS

A. *Proof of Theorem 1*

We rely here on the proof of the existing bound that is stated in Theorem 2, for discrete-time real-valued martingales with bounded jumps (see Theorem 6.1 in McDiarmid (1989) and Corollary 2.4.7 in Dembo and Zeitouni (1997)), and then deviate from this proof at the point where the additional property of the conditional symmetry of the martingale is taken into consideration for the derivation of the improved exponential inequality in Theorem 1.

Write $X_n - X_0 = \sum_{k=1}^n \xi_k$ where $\xi_k \triangleq X_k - X_{k-1}$ for $k \in \mathbb{N}$. Since $\{X_k - X_0, \mathcal{F}_k\}_{k\in\mathbb{N}_0}$ is a martingale, $h(x) = \exp(tx)$ is a convex function on $\mathbb{R}$ for every $t \in \mathbb{R}$, and a composition of a convex function with a martingale gives a submartingale w.r.t. the same filtration, $\big\{\exp(t(X_k - X_0)), \mathcal{F}_k\big\}_{k\in\mathbb{N}_0}$ is a sub-martingale for every $t \in \mathbb{R}$. By applying the maximal inequality for submartingales, then for every $\alpha \geq 0$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\leq k\leq n} (X_k - X_0) \geq \alpha n\Big) \\
&\leq \exp(-\alpha n t)\, \mathbb{E}\Big[\exp\big(t(X_n - X_0)\big)\Big] \qquad \forall\, t \geq 0 \\
&= \exp(-\alpha n t)\, \mathbb{E}\left[\exp\left(t \sum_{k=1}^n \xi_k\right)\right].
\end{aligned} \tag{21}$$

Furthermore,

$$\mathbb{E}\left[\exp\left(t\sum_{k=1}^n \xi_k\right)\right] = \mathbb{E}\left[\exp\left(t\sum_{k=1}^{n-1}\xi_k\right)\mathbb{E}\big[\exp(t\xi_n) \,|\, \mathcal{F}_{n-1}\big]\right] \tag{22}$$

where this equality holds since $\exp\big(t\sum_{k=1}^{n-1}\xi_k\big)$ is $\mathcal{F}_{n-1}$-measurable.

In order to prove Theorem 1 for a discrete-time, real-valued and conditionally symmetric martingale with bounded jumps, we deviate from the proof of Theorem 2. This is done by a replacement of Bennett's inequality (see Bennett (1962)) for the conditional expectation with a tightened bound under the conditional symmetry assumption. The following lemma appears in several probability textbooks on stochastic ordering (see, e.g., Denuit et al. (2005)), and will be useful in our analysis.

Lemma 1: Let X be a real-valued random variable with a symmetric distribution around zero, and a support $[-d, d]$, and assume that $\mathbb{E}[X^2] = \text{Var}(X) \leq \gamma d^2$ for some $d > 0$ and $\gamma \in [0, 1]$. Let h be a real-valued convex function, and assume that $h(d^2) \geq h(0)$. Then

$$\mathbb{E}[h(X^2)] \leq (1-\gamma)h(0) + \gamma h(d^2) \tag{23}$$

where equality holds for the symmetric distribution

$$\mathbb{P}(X = d) = \mathbb{P}(X = -d) = \frac{\gamma}{2}, \quad \mathbb{P}(X = 0) = 1 - \gamma. \tag{24}$$

Proof: Since h is convex and $\text{supp}(X) = [-d, d]$, then a.s. $h(X^2) \leq h(0) + \left(\frac{X}{d}\right)^2 \big(h(d^2) - h(0)\big)$. Taking expectations on both sides gives (23), which holds with equality for the symmetric distribution in (24). ■

Corollary 4: If X is a random variable that satisfies the three requirements in Lemma 1 then, for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}\big[\exp(\lambda X)\big] \leq 1 + \gamma\big[\cosh(\lambda d) - 1\big] \tag{25}$$

and (25) holds with equality for the symmetric distribution in Lemma 1, independently of the value of λ.

Proof: For every $\lambda \in \mathbb{R}$, due to the symmetric distribution of X, $\mathbb{E}\big[\exp(\lambda X)\big] = \mathbb{E}\big[\cosh(\lambda X)\big]$. The claim now follows from Lemma 1 since, for every $x \in \mathbb{R}$, $\cosh(\lambda x) = h(x^2)$ where $h(x) \triangleq \sum_{n=0}^{\infty} \frac{\lambda^{2n}|x|^n}{(2n)!}$ is a convex function (h is convex since it is a linear combination, with non-negative coefficients, of convex functions), and $h(d^2) = \cosh(\lambda d) \geq 1 = h(0)$. ■

We continue with the proof of Theorem 1. Under the assumption of this theorem, for every $k \in \mathbb{N}$, the random variable $\xi_k \triangleq X_k - X_{k-1}$ satisfies a.s. $\mathbb{E}[\xi_k \,|\, \mathcal{F}_{k-1}] = 0$ and $\mathbb{E}[(\xi_k)^2 \,|\, \mathcal{F}_{k-1}] \leq \sigma^2$. Applying Corollary 4 for the conditional law of ξ_k given $\mathcal{F}_{k-1}$, it follows that for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$\mathbb{E}\left[\exp(t\xi_k) \,|\, \mathcal{F}_{k-1}\right] \leq 1 + \gamma\big[\cosh(td) - 1\big] \tag{26}$$

holds a.s., and therefore it follows from (22) and (26) that for every $t \in \mathbb{R}$

$$\mathbb{E}\left[\exp\left(t \sum_{k=1}^{n} \xi_k\right)\right] \leq \Big(1 + \gamma\big[\cosh(td) - 1\big]\Big)^n. \tag{27}$$

Therefore, from (21), for every $t \geq 0$,

$$\mathbb{P}\Big(\max_{1 \leq k \leq n} (X_k - X_0) \geq \alpha n\Big) \leq \exp(-\alpha n t)\Big(1 + \gamma\big[\cosh(td) - 1\big]\Big)^n. \tag{28}$$

From (3), and by using a replacement of td with x, then for an arbitrary $\alpha \geq 0$ and $n \in \mathbb{N}$

$$\mathbb{P}\Big(\max_{1 \leq k \leq n} (X_k - X_0) \geq \alpha n\Big) \leq \inf_{x \geq 0} \left\{\exp\left(-n\Big[\delta x - \ln\big(1 + \gamma\big[\cosh(x) - 1\big]\big)\Big]\right)\right\}. \tag{29}$$

An optimization over the non-negative parameter x gives the solution for the optimized parameter in (5). Applying (29) to the martingale $\{-X_k, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ gives the same bound on $\mathbb{P}(\min_{1 \leq k \leq n}(X_k - X_0) \leq -\alpha n)$. This completes the proof of Theorem 1.

Proof for the asymptotic optimality of the exponents in Theorems 1 and 2: In the following, we show that under the conditions of Theorem 1, the exponent $E(\gamma, \delta)$ in (4) and (5) is asymptotically optimal. To show this, let $d > 0$ and $\gamma \in (0, 1]$, and let $U_1, U_2, \ldots$ be i.i.d. random variables whose probability distribution is given by

$$\mathbb{P}(U_i = d) = \mathbb{P}(U_i = -d) = \frac{\gamma}{2}, \quad \mathbb{P}(U_i = 0) = 1 - \gamma, \quad \forall\, i \in \mathbb{N}. \tag{30}$$

Consider the particular case of the conditionally symmetric martingale $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ in Example 1 (see Section I-B) where $X_n \triangleq \sum_{i=1}^{n} U_i$ for $n \in \mathbb{N}$, and $X_0 \triangleq 0$. It follows that $|X_n - X_{n-1}| \leq d$ and $\text{Var}(X_n | \mathcal{F}_{n-1}) = \gamma d^2$ a.s. for every $n \in \mathbb{N}$. From Cramér's theorem in $\mathbb{R}$, for every $\alpha \geq \mathbb{E}[U_1] = 0$,

$$\begin{aligned}
&\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}(X_n - X_0 \geq \alpha n) \\
&= \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} U_i \geq \alpha\right) \\
&= -I(\alpha)
\end{aligned} \tag{31}$$

where the rate function is given by

$$I(\alpha) = \sup_{t \geq 0} \left\{t\alpha - \ln \mathbb{E}[\exp(tU_1)]\right\} \tag{32}$$

(see, e.g., Theorem 2.2.3 in Dembo and Zeitouni (1997) and Lemma 2.2.5(b) in Dembo and Zeitouni (1997) for the restriction of the supremum to the interval $[0, \infty)$). From (30) and (32), for every $\alpha \geq 0$,

$$I(\alpha) = \sup_{t \geq 0} \Big\{t\alpha - \ln\big(1 + \gamma[\cosh(td) - 1]\big)\Big\}$$

but this is equivalent to the optimized exponent on the right-hand side of (28), giving the exponent of the bound in Theorem 1. Hence, $I(\alpha) = E(\gamma, \delta)$ in (4) and (5). This proves that the exponent of the bound in Theorem 1 is indeed asymptotically optimal in the sense that there exists a discrete-time real-valued and conditionally symmetric martingale, satisfying the conditions in (1) a.s., that attains this exponent in the limit where $n \to \infty$. The proof for the asymptotic optimality of the exponent in Theorem 2 (see the right-hand side of (7)) is similar to the proof for Theorem 1, except that the i.i.d. random variables $U_1, U_2, \ldots$ are now distributed as follows:

$$\mathbb{P}(U_i = d) = \frac{\gamma}{1+\gamma}, \quad \mathbb{P}(U_i = -\gamma d) = \frac{1}{1+\gamma}, \quad \forall\, i \in \mathbb{N}$$

and, as before, the martingale $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is defined by $X_n = \sum_{i=1}^n U_i$ and $\mathcal{F}_n = \sigma(U_1, \ldots, U_n)$ for every $n \in \mathbb{N}$ with $X_0 = 0$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (in this case, it is not a conditionally symmetric martingale unless $\gamma = 1$).

B. Proof of Theorem 3

The starting point of the proof of Theorem 3 relies on (21) and (22). For every $k \in \mathbb{N}$ and $t \in \mathbb{R}$, since $\mathbb{E}\bigl[\xi_k^{2l-1} \,|\, \mathcal{F}_{k-1}\bigr] = 0$ for every $l \in \mathbb{N}$ (due to the conditionally symmetry property of the martingale),

$$\begin{aligned}
&\mathbb{E}\bigl[\exp(t\xi_k) | \mathcal{F}_{k-1}\bigr] \\
&= 1 + \sum_{l=1}^{\frac{m}{2}-1} \frac{t^{2l}\, \mathbb{E}\bigl[\xi_k^{2l} \,|\, \mathcal{F}_{k-1}\bigr]}{(2l)!} + \sum_{l=\frac{m}{2}}^{\infty} \frac{t^{2l}\, \mathbb{E}\bigl[\xi_k^{2l} \,|\, \mathcal{F}_{k-1}\bigr]}{(2l)!} \\
&= 1 + \sum_{l=1}^{\frac{m}{2}-1} \frac{(td)^{2l}\, \mathbb{E}\bigl[\bigl(\frac{\xi_k}{d}\bigr)^{2l} \,|\, \mathcal{F}_{k-1}\bigr]}{(2l)!} + \sum_{l=\frac{m}{2}}^{\infty} \frac{(td)^{2l}\, \mathbb{E}\bigl[\bigl(\frac{\xi_k}{d}\bigr)^{2l} \,|\, \mathcal{F}_{k-1}\bigr]}{(2l)!} \\
&\leq 1 + \sum_{l=1}^{\frac{m}{2}-1} \frac{(td)^{2l}\, \gamma_{2l}}{(2l)!} + \sum_{l=\frac{m}{2}}^{\infty} \frac{(td)^{2l}\, \gamma_m}{(2l)!} \\
&= 1 + \sum_{l=1}^{\frac{m}{2}-1} \frac{(td)^{2l}\, \bigl(\gamma_{2l} - \gamma_m\bigr)}{(2l)!} + \gamma_m \bigl(\cosh(td) - 1\bigr)
\end{aligned} \tag{33}$$

where the inequality above holds since $|\frac{\xi_k}{d}| \leq 1$ a.s., so that $0 \leq \ldots \leq \gamma_m \leq \ldots \leq \gamma_4 \leq \gamma_2 \leq 1$, and the last equality in (33) holds since $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ for every $x \in \mathbb{R}$. Therefore, from (22),

$$\mathbb{E}\left[\exp\left(t \sum_{k=1}^n \xi_k\right)\right] \leq \left(1 + \sum_{l=1}^{\frac{m}{2}-1} \frac{(td)^{2l}\, \bigl(\gamma_{2l} - \gamma_m\bigr)}{(2l)!} + \gamma_m \bigl[\cosh(td) - 1\bigr]\right)^n \tag{34}$$

for an arbitrary $t \in \mathbb{R}$. The inequality then follows from (21). This completes the proof of Theorem 3.

C. A Proof of Theorem 4

The proof of Theorem 4 relies on the proof of the known result in Theorem 5, where the latter dates back to Freedman's paper (see Theorem 1.6 in Freedman (1975), and also Exercise 2.4.21(b) in Dembo and Zeitouni (1997)). The original proof of Theorem 5 (see Section 3 in Freedman (1975)) is modified in a way that facilitates realizing how the bound can be improved for conditionally symmetric martingales with bounded jumps. This improvement is obtained via the refinement of Bennett's inequality for conditionally symmetric distributions.

Without any loss of generality, let us assume that $d = 1$ (otherwise, $\{X_k\}$ and z are divided by d, and $\{Q_k\}$ and r are divided by d^2; this normalization extends the bound to the case of an arbitrary $d > 0$). Let $S_n \triangleq X_n - X_0$ for every $n \in \mathbb{N}_0$; then $\{S_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a martingale with $S_0 = 0$.

Lemma 2: Under the assumptions of Theorem 4, let

$$U_n \triangleq \exp(\lambda S_n - \theta Q_n), \quad \forall\, n \in \{0, 1, \ldots\} \tag{35}$$

where $\lambda \geq 0$ and $\theta \geq \cosh(\lambda) - 1 \triangleq \theta_{\min}(\lambda)$ are arbitrary constants. Then, $\{U_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a supermartingale.

Proof: It is easy to verify that $U_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ for $\lambda, \theta \geq 0$ (note that $S_n \leq n$ a.s.). It is required to show that $\mathbb{E}[U_n|\mathcal{F}_{n-1}] \leq U_{n-1}$ holds a.s. for every $n \in \mathbb{N}$, under the above assumptions on λ and θ in (35). We have

$$\begin{aligned}
&\mathbb{E}[U_n|\mathcal{F}_{n-1}] \\
&\stackrel{\text{(a)}}{=} \exp(-\theta Q_n)\, \exp(\lambda S_{n-1})\, \mathbb{E}\big[\exp(\lambda \xi_n) \,|\, \mathcal{F}_{n-1}\big] \\
&\stackrel{\text{(b)}}{=} \exp(\lambda S_{n-1})\, \exp\big(-\theta(Q_{n-1} + \mathbb{E}[\xi_n^2|\mathcal{F}_{n-1}])\big)\, \mathbb{E}\big[\exp(\lambda \xi_n) \,|\, \mathcal{F}_{n-1}\big] \\
&\stackrel{\text{(c)}}{=} U_{n-1} \left(\frac{\mathbb{E}\big[\exp(\lambda \xi_n) \,|\, \mathcal{F}_{n-1}\big]}{\exp(\theta \mathbb{E}[\xi_n^2 \,|\, \mathcal{F}_{n-1}])} \right)
\end{aligned} \tag{36}$$

where (a) follows from (35) and because Q_n and S_{n-1} are $\mathcal{F}_{n-1}$-measurable and $S_n = S_{n-1} + \xi_n$, (b) follows from (14), and (c) follows from (35).

By assumption $\xi_n = S_n - S_{n-1} \leq 1$ a.s., and ξ_n is conditionally symmetric around zero, given $\mathcal{F}_{n-1}$, for every $n \in \mathbb{N}$. By applying Corollary 4 to the conditional expectation of $\exp(\lambda \xi_n)$ given $\mathcal{F}_{n-1}$, it follows from (36) that

$$\mathbb{E}[U_n|\mathcal{F}_{n-1}] \leq U_{n-1} \left(\frac{1 + \mathbb{E}[\xi_n^2 \,|\, \mathcal{F}_{n-1}] \, \big(\cosh(\lambda) - 1\big)}{\exp\big(\theta \mathbb{E}[\xi_n^2|\mathcal{F}_{n-1}]\big)} \right). \tag{37}$$

Let $\lambda \geq 0$. In order to ensure that $\{U_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ forms a supermartingale, it is sufficient (based on (37)) that the following condition holds:

$$\frac{1 + \alpha\big(\cosh(\lambda) - 1\big)}{\exp(\theta \alpha)} \leq 1, \quad \forall\, \alpha \geq 0. \tag{38}$$

Calculus shows that, for $\lambda \geq 0$, the condition in (38) is satisfied if and only if

$$\theta \geq \cosh(\lambda) - 1 \triangleq \theta_{\min}(\lambda). \tag{39}$$

From (37), $\{U_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a supermartingale if $\lambda \geq 0$ and $\theta \geq \theta_{\min}(\lambda)$. This proves Lemma 2. ■

Let $z, r > 0$, $\lambda \geq 0$, and $\theta \geq \cosh(\lambda) - 1$. In the following, we rely on Doob's sampling theorem. To this end, let $M \in \mathbb{N}$, and define two stopping times adapted to $\{\mathcal{F}_n\}$. The first stopping time is $\alpha = 0$, and the second stopping time β is the minimal value of $n \in \{0, \ldots, M\}$ (if any) such that $S_n \geq z$ and $Q_n \leq r$ (note that S_n is $\mathcal{F}_n$-measurable and Q_n is $\mathcal{F}_{n-1}$-measurable, so the event $\{\beta \leq n\}$ is $\mathcal{F}_n$-measurable); if such a value of n does not exist, let $\beta \triangleq M$. Hence $\alpha \leq \beta$ are two bounded stopping times. From Lemma 2, $\{U_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a supermartingale for the corresponding set of parameters λ and θ, and from Doob's sampling theorem

$$\mathbb{E}[U_\beta] \leq \mathbb{E}[U_0] = 1 \tag{40}$$

($S_0 = Q_0 = 0$, so from (35), $U_0 = 1$ a.s.). Hence, this implies the following chain of inequalities:

$$\begin{aligned}
&\mathbb{P}(\exists\, n \leq M :\ S_n \geq z, Q_n \leq r) \\
&\stackrel{\text{(a)}}{=} \mathbb{P}(S_\beta \geq z, Q_\beta \leq r) \\
&\stackrel{\text{(b)}}{\leq} \mathbb{P}(\lambda S_\beta - \theta Q_\beta \geq \lambda z - \theta r) \\
&\stackrel{\text{(c)}}{\leq} \frac{\mathbb{E}[U_\beta]}{\exp(\lambda z - \theta r)} \\
&\stackrel{\text{(d)}}{\leq} \exp\big(-(\lambda z - \theta r)\big)
\end{aligned} \tag{41}$$

where equality (a) follows from the definition of the stopping time $\beta \in \{0, \ldots, M\}$, (b) holds since $\lambda, \theta \geq 0$, (c) follows from Chernoff's bound and the definition in (35), and finally (d) follows from (40). Since (41) holds for every $M \in \mathbb{N}$, from the continuity theorem for non-decreasing events and (41)

$$\begin{aligned}
&\mathbb{P}(\exists\, n \in \mathbb{N} :\ S_n \geq z, Q_n \leq r) \\
&= \lim_{M \to \infty} \mathbb{P}(\exists\, n \leq M :\ S_n \geq z, Q_n \leq r) \\
&\leq \exp\big(-(\lambda z - \theta r)\big).
\end{aligned} \tag{42}$$

The choice of the non-negative parameter θ as the minimal value for which (42) is valid provides the tightest bound within this form. Hence, for a fixed $\lambda \geq 0$ and $\theta = \theta_{\min}(\lambda)$, the bound in (42) gives the inequality

$$\mathbb{P}(\exists\, n \in \mathbb{N} :\, S_n \geq z, Q_n \leq r) \leq \exp\Bigl(-\bigl[\lambda z - r\,\theta_{\min}(\lambda)\bigr]\Bigr), \quad \forall\, \lambda \geq 0.$$

The optimized λ is equal to $\lambda = \sinh^{-1}\Bigl(\frac{z}{r}\Bigr)$. Its substitution in (39) gives that $\theta_{\min}(\lambda) = \sqrt{1 + \frac{z^2}{r^2}} - 1$, and

$$\mathbb{P}(\exists\, n \in \mathbb{N} :\, S_n \geq z, Q_n \leq r) \leq \exp\left(-\frac{z^2}{2r} \cdot C\Bigl(\frac{z}{r}\Bigr)\right) \tag{43}$$

with C in (16). Finally, the proof of Theorem 4 is completed by showing that the following equality holds:

$$\begin{aligned} A &\triangleq \{\exists\, n \in \mathbb{N} :\, S_n \geq z, Q_n \leq r\} \\ &= \{\exists\, n \in \mathbb{N} :\, \max_{1 \leq k \leq n} S_k \geq z, Q_n \leq r\} \triangleq B. \end{aligned} \tag{44}$$

Clearly $A \subseteq B$. To show that $B \subseteq A$, assume that event B is satisfied. Then, there exists some $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$ such that $S_k \geq z$ and $Q_n \leq r$. Since the predictable quadratic variation process $\{Q_n\}_{n \in \mathbb{N}_0}$ in (14) is monotonic non-decreasing, this implies that $S_k \geq z$ and $Q_k \leq r$; therefore, event A is also satisfied and $B \subseteq A$. The combination of (43) and (44) completes the proof of Theorem 4.

D. Proof of Corollary 2

The proof of Corollary 2 is similar to the proof of Theorem 1. The only difference is that for a supermartingale, $X_k - X_0 = \sum_{j=1}^k (X_j - X_{j-1}) \leq \sum_{j=1}^k \eta_j$ a.s., where $\eta_j \triangleq X_j - \mathbb{E}[X_j \,|\, \mathcal{F}_{j-1}]$ is $\mathcal{F}_j$-measurable. Hence $\mathbb{P}\Bigl(\max_{1 \leq k \leq n} X_k - X_0 \geq \alpha n\Bigr) \leq \mathbb{P}\Bigl(\max_{1 \leq k \leq n} \sum_{j=1}^k \eta_j \geq \alpha n\Bigr)$ where a.s. $\eta_j \leq d$, $\mathbb{E}[\eta_j \,|\, \mathcal{F}_{j-1}] = 0$, and $\text{Var}(\eta_j \,|\, \mathcal{F}_{j-1}) \leq \sigma^2$. The continuation coincides with the proof of Theorem 1 for the martingale $\{\sum_{j=1}^k \eta_j, \mathcal{F}_k\}_{k \in \mathbb{N}_0}$ (starting from (21) and (27)). The passage to submartingales is trivial.

E. Proof of Corollary 3

Consider the martingale $\{Y_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ where $Y_n \triangleq \sum_{j=1}^n \eta_j$ with $\eta_j \triangleq X_j - \mathbb{E}[X_j \,|\, \mathcal{F}_{j-1}]$, and $Y_0 \triangleq 0$. Since $Y_k - Y_{k-1} = \eta_k$ for every $k \in \mathbb{N}$, the predictable quadratic variation process $\{Q_n\}_{n \in \mathbb{N}_0}$ which corresponds to the martingale $\{Y_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is, from (14), the same process as the one which corresponds to the supermartingale $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$. Furthermore, $X_k - X_0 \leq \sum_{j=1}^n \xi_j = Y_k - Y_0$ for every $k \in \mathbb{N}$. Hence, for every $z, r > 0$,

$$\mathbb{P}\Bigl(\exists\, n \in \mathbb{N} :\, \max_{1 \leq k \leq n} (X_k - X_0) \geq z,\, Q_n \leq r\Bigr) \leq \mathbb{P}\Bigl(\exists\, n \in \mathbb{N} :\, \max_{1 \leq k \leq n} (Y_k - Y_0) \geq z,\, Q_n \leq r\Bigr).$$

If $\{X_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a conditionally symmetric supermartingale, then $\{Y_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ is a conditionally symmetric martingale. Then, applying Theorem 4 to the martingale $\{Y_n, \mathcal{F}_n\}_{n \in \mathbb{N}_0}$ gives the improved bound in (15) and (16).

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