# Observations on Lovász Function, Shannon Capacity of Graphs, and Strong Products

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June 22, 2023

## Graph Spectrum

Throughout this presentation,

- $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ is a finite, undirected, and simple graph of order $|\mathsf{V}(\mathsf{G})| = n$ and size $|\mathsf{E}(\mathsf{G})| = m$.
- $\mathbf{A} = \mathbf{A}(\mathsf{G})$ is the *adjacency matrix* of the graph.
- The eigenvalues of $\mathbf{A}$ are given in decreasing order by

$$\lambda_{\max}(\mathsf{G}) = \lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}) = \lambda_{\min}(\mathsf{G}). \tag{1}$$

- The *spectrum* of G consists of the eigenvalues of $\mathbf{A}$, including their multiplicities.

## Orthogonal Representation of Graphs

### Definition

Let G be a finite, undirected and simple graph.
An orthogonal representation of G in $\mathbb{R}^d$

$$i \in \mathsf{V}(\mathsf{G}) \;\mapsto\; \mathbf{u}_i \in \mathbb{R}^d$$

such that

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0, \quad \forall \{i, j\} \notin \mathsf{E}(\mathsf{G}).$$

An orthonormal representation of G: $\|\mathbf{u}_i\| = 1$ for all $i \in \mathsf{V}(\mathsf{G})$.

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In an orthogonal representation of a graph G:

- non-adjacent vertices: mapped to orthogonal vectors;
- adjacent vertices: not necessarily mapped to non-orthogonal vectors.

# An Orthonormal Representation of a Pentagon

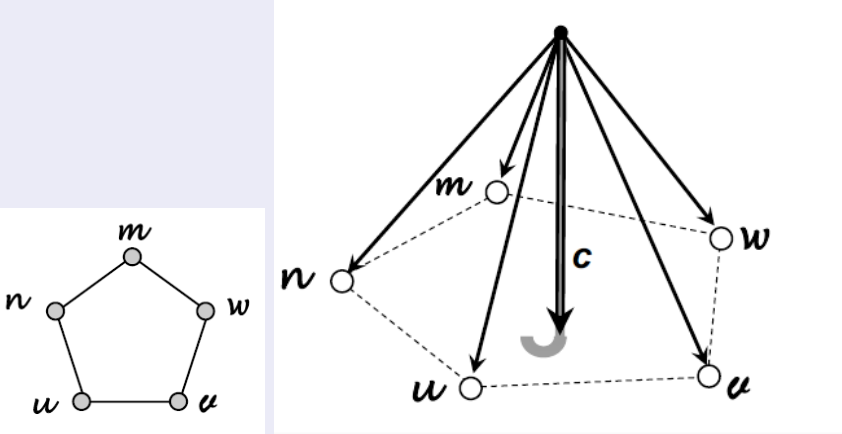


Figure: A 5-cycle graph and its orthonormal representation (Lovász umbrella).

## Lovász $\theta$-function

Let G be a finite, undirected and simple graph.

The Lovász $\theta$-function of G is defined as

$$\theta(\mathsf{G}) \triangleq \min_{\mathbf{u},\mathbf{c}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right)^2}, \tag{2}$$

where the minimum is taken over

- all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G, and
- all unit vectors $\mathbf{c}$.

The unit vector $\mathbf{c}$ is called the *handle* of the orthonormal representation.

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$$\left|\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right| \leq \|\mathbf{c}\| \, \|\mathbf{u}_i\| = 1 \implies \theta(\mathsf{G}) \geq 1,$$

with equality if and only if G is a complete graph.

## Lovász $\theta$-function (Cont.)

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\theta(\mathsf{G})$:

$$\begin{array}{l} \text{maximize } \ \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \ \ \mathrm{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \ \Rightarrow \ B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\theta(\mathsf{G})$, for every graph G, with precision of $r$ decimal digits, and polynomial-time in $n$ and $r$.

## Lovász $\theta$-function (Cont.)

1. Sandwich theorem:

$$\alpha(\mathsf{G}) \le \theta(\mathsf{G}) \le \chi(\overline{\mathsf{G}}), \tag{3}$$

$$\omega(\mathsf{G}) \le \theta(\overline{\mathsf{G}}) \le \chi(\mathsf{G}). \tag{4}$$

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2. Computational complexity:
   - $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems.
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   - $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems.
   - However, the numerical computation of $\theta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

3. Hoffman-Lovász inequality: Let G be $d$-regular of order $n$. Then,
$$\theta(\mathsf{G}) \le -\frac{n\,\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}, \tag{5}$$
with equality if G is edge-transitive.

## Strongly Regular Graphs

Let G be a $d$-regular graph of order $n$. It is a *strongly regular* graph (SRG) if there exist nonnegative integers $\lambda$ and $\mu$ such that

- Every pair of adjacent vertices have exactly $\lambda$ common neighbors;
- Every pair of distinct and non-adjacent vertices have exactly $\mu$ common neighbors.

Such a strongly regular graph is denoted by $\text{srg}(n, d, \lambda, \mu)$.

## Bounds on the Lovász function of Regular Graphs

Theorem (I.S., '23):

Let G be a $d$-regular graph of order $n$, which is a non-complete and non-empty graph. Then, the following bounds hold for the Lovász $\theta$-function of G and its complement $\overline{\mathsf{G}}$:

1)

$$\frac{n-d+\lambda_2(\mathsf{G})}{1+\lambda_2(\mathsf{G})} \leq \theta(\mathsf{G}) \leq -\frac{n\lambda_n(\mathsf{G})}{d-\lambda_n(\mathsf{G})}. \tag{6}$$

- Equality holds in the leftmost inequality if $\overline{\mathsf{G}}$ is both vertex-transitive and edge-transitive, or if G is a strongly regular graph;
- Equality holds in the rightmost inequality if G is edge-transitive, or if G is a strongly regular graph.

## Bounds (cont.)

2)

$$1 - \frac{d}{\lambda_n(\mathsf{G})} \leq \theta(\overline{\mathsf{G}}) \leq \frac{n\big(1 + \lambda_2(\mathsf{G})\big)}{n - d + \lambda_2(\mathsf{G})}. \tag{7}$$

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## A Common Sufficient Condition

All inequalities hold with equality if G is strongly regular. (Recall that the graph G is strongly regular if and only if $\overline{\mathsf{G}}$ is so).

## Lovász Function of Strongly Regular Graphs (I.S., '23)

Let G be a strongly regular graph with parameters $\text{srg}(n, d, \lambda, \mu)$. Then,

$$\theta(\mathsf{G}) = \frac{n\,(t + \mu - \lambda)}{2d + t + \mu - \lambda}, \tag{8}$$

$$\theta(\overline{\mathsf{G}}) = 1 + \frac{2d}{t + \mu - \lambda}, \tag{9}$$

where

$$t \triangleq \sqrt{(\mu - \lambda)^2 + 4(d - \mu)}. \tag{10}$$

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## New Relation for Strongly Regular Graphs

$$\theta(\mathsf{G})\,\theta(\overline{\mathsf{G}}) = n, \tag{11}$$

holding not only for all vertex-transitive graphs (Lovász '79), but also for all strongly regular graphs (that are not necessarily vertex-transitive).

## Corollary: Bounds on Parameters of SRGs (I.S., '23)

Let G be a strongly regular graph with parameters srg$(n, d, \lambda, \mu)$. Then,

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rfloor \tag{12}$$

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{2d}{t+\mu-\lambda} \right\rfloor, \tag{13}$$

$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{2d}{t+\mu-\lambda} \right\rceil, \tag{14}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rceil, \tag{15}$$

with

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{16}$$

## Examples: Bounds on Parameters of SRGs

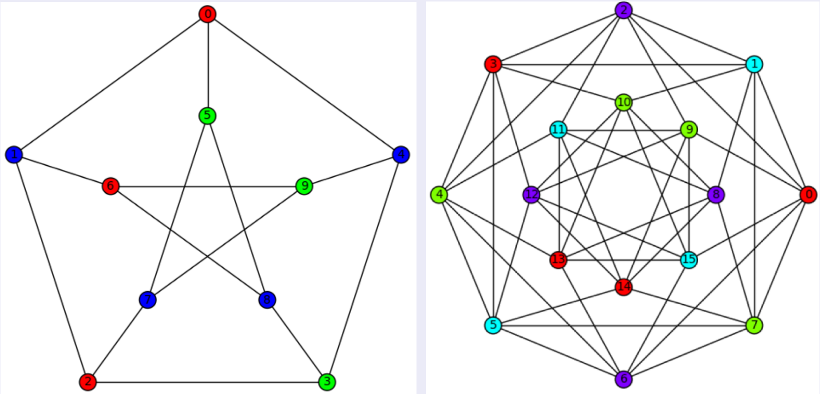


Figure: The Petersen graph is srg$(10, 3, 0, 1)$ (left), and the Shrikhande graph is srg$(16, 6, 2, 2)$ (right). Their chromatic numbers are 3 and 4, respectively.

# Schläfli Graph

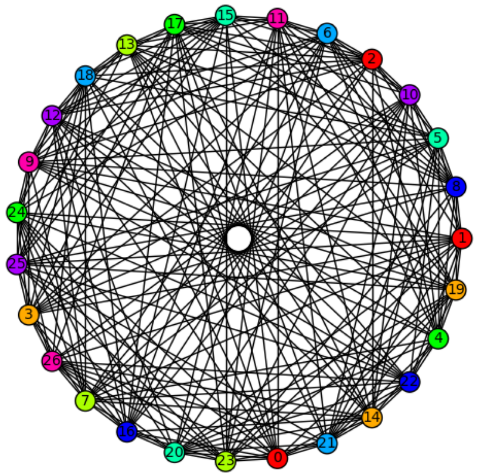


Figure: Schläfli graph is srg$(27, 16, 10, 8)$ with chromatic number $\chi(\mathsf{G}) = 9$.

## Examples: Bounds on Parameters of SRGs (Cont.)

1. Let $G_1$ be the Petersen graph. Then, the bounds on the independence, clique, and chromatic numbers of G are tight:

$$\alpha(G_1) = 4, \quad \omega(G_1) = 2, \quad \chi(G_1) = 3. \tag{17}$$

2. The bounds on the chromatic numbers of the Schläfli graph ($G_2$), Shrikhande graph ($G_3$) and Hall-Janko graph ($G_4$) are tight:

$$\chi(G_2) = 9, \quad \chi(G_3) = 4, \quad \chi(G_4) = 10. \tag{18}$$

3. For the Shrikhande graph ($G_3$),
   - the bound on its independence number is also tight: $\alpha(G_3) = 4$,
   - its upper bound on its clique number is, however, not tight (it is equal to 4, and $\omega(G_3) = 3$).

## Strong Product of Graphs

Let G and H be two graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with

- vertex set: $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$,
- two distinct vertices $(g, h)$ and $(g', h')$ in $\mathsf{G} \boxtimes \mathsf{H}$ are adjacent if the following two conditions hold:
    1. $g = g'$ or $\{g, g'\} \in \mathsf{E}(\mathsf{G})$,
    2. $h = h'$ or $\{h, h'\} \in \mathsf{E}(\mathsf{H})$.

Strong products are commutative and associative.

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## Strong Powers of Graphs

Let

$$\mathrm{G}^{\boxtimes k} \triangleq \underbrace{\mathrm{G} \boxtimes \ldots \boxtimes \mathrm{G}}_{\mathrm{G} \text{ appears } k \text{ times}}, \quad k \in \mathbb{N} \tag{19}$$

denote the $k$-fold strong power of a graph G.

## Properties of the Lovász $\theta$-Function with Strong Products

1. Factorization for strong product graphs: For all graphs G and H,

$$\theta(\mathsf{G} \boxtimes \mathsf{H}) = \theta(\mathsf{G})\, \theta(\mathsf{H}), \tag{20}$$

$$\theta(\overline{\mathsf{G} \boxtimes \mathsf{H}}) = \theta(\overline{\mathsf{G}})\, \theta(\overline{\mathsf{H}}). \tag{21}$$

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2. The equality

$$\sup_{\mathsf{H}} \frac{\alpha(\mathsf{G} \boxtimes \mathsf{H})}{\theta(\mathsf{G} \boxtimes \mathsf{H})} = 1, \tag{22}$$

holds for every simple, finite, and undirected graph G, where the supremum is taken over all such graphs H.

## Independence Numbers of Strong Powers of Graphs

Proposition: Let G be a finite, undirected, and simple graph.
If $\alpha(\mathsf{G}^{\boxtimes \ell}) = \theta(\mathsf{G})^{\ell}$ for some $\ell \in \mathbb{N}$, then for every $k$ that is an integral multiple of $\ell$,

$$\alpha(\mathsf{G}^{\boxtimes k}) = \theta(\mathsf{G})^{k}. \tag{23}$$

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## Proof

Let $k = \ell p$ with $p \in \mathbb{N}$. Then, since $\alpha(\mathsf{G} \boxtimes \mathsf{H}) \geq \alpha(\mathsf{G})\, \alpha(\mathsf{H})$ for all graphs G and H,

$$\theta(\mathsf{G})^{k} = \alpha(\mathsf{G}^{\boxtimes \ell})^{p} \leq \alpha(\mathsf{G}^{\boxtimes k}) \leq \theta(\mathsf{G}^{\boxtimes k}) = \theta(\mathsf{G})^{k}.$$

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Corollary 1: If $\alpha(\mathsf{G}) = \theta(\mathsf{G})$, then for all $k \in \mathbb{N}$, the $k$-fold strong power of G satisfies

$$\alpha(\mathsf{G}^{\boxtimes k}) = \theta(\mathsf{G})^{k}, \quad \forall\, k \in \mathbb{N}. \tag{24}$$

## Example: the Tietze Graph (I.S., '23)

Let G be the Tietze graph, which is a 3-regular graph on 12 vertices that is not strongly regular, nor vertex- or edge-transitive.

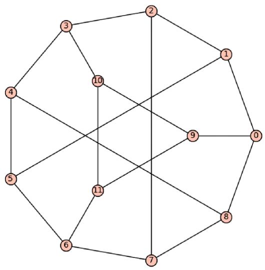


Figure: Tietze graph.

It can be verified that

$$\alpha(\mathsf{G}) = 5 = \theta(\mathsf{G}) \implies \alpha(\mathsf{G}^{\boxtimes k}) = 5^k, \; \forall \, k \in \mathbb{N}.$$

## Example: the Tietze Graph (Cont.)

1. A largest independent set of G is $\{0, 3, 5, 7, 11\}$, so $\alpha(\mathsf{G}) = 5$.
2. The result $\theta(\mathsf{G}) = 5$ is obtained by solving the SDP problem:

$$\begin{array}{l} \text{maximize} \;\; \text{Trace}(\mathbf{B}\,\mathbf{J}_{12}) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_{+}^{12}, \;\; \text{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in \{1, \ldots, 12\}. \end{cases} \end{array}$$

$$\Rightarrow \; B = \frac{1}{15} \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \; \theta(\mathsf{G}) = \text{Trace}(\mathbf{B}\,\mathbf{J}_{12}) = 5.$$

## Example: the Tietze Graph (Cont.)

For comparison, since the Tietze graph is 3-regular on 12 vertices with $\lambda_{\min}(\mathsf{G}) = -2.30278$, the Hoffman-Lovász bound on $\theta(\mathsf{G})$ is equal to

$$\theta(\mathsf{G}) \leq -\frac{n\lambda_{\min}(\mathsf{G})}{d - \lambda_{\min}(\mathsf{G})} = 5.21110, \tag{25}$$

so it is not tight. The fact that the bound is not tight is consistent with the fact that G is not an edge-transitive graph.

By Corollary 1 and our closed-form expression for the Lovász $\theta$-function of SRGs, we calculate $\alpha(\mathsf{G}^{\boxtimes k})$ for some SRGs.

## Independence Numbers of All Strong Powers of SRGs

1. The Hall-Janko graph G is $\text{srg}(100, 36, 14, 12)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 10^k$.
2. The Hoffman-Singleton graph G is $\text{srg}(50, 7, 0, 1)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 15^k$.
3. The Janko-Kharaghani graphs of orders 936 and 1800 are $\text{srg}(936, 375, 150, 150)$ and $\text{srg}(1800, 1029, 588, 588)$, respectively. For both graphs $\alpha(\mathsf{G}^{\boxtimes k}) = 36^k$.
4. Janko-Kharaghani-Tonchev: $\mathsf{G} = \text{srg}(324, 153, 72, 72), \alpha(\mathsf{G}^{\boxtimes k}) = 18^k$.
5. The graphs introduced by Makhnev are $\mathsf{G} = \text{srg}(64, 18, 2, 6)$ and $\overline{\mathsf{G}} = \text{srg}(64, 45, 32, 30)$. We have $\alpha(\mathsf{G}^{\boxtimes k}) = 16^k$, and $\alpha(\overline{\mathsf{G}}^{\boxtimes k}) = 4^k$.
6. The Mathon-Rosa graph G is $\text{srg}(280, 117, 44, 52)$: $\alpha(\mathsf{G}^{\boxtimes k}) = 28^k$.
7. The Schläfli graph G is $\text{srg}(27, 16, 10, 8)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 3^k$.
8. The Shrikhande graph is $\text{srg}(16, 6, 2, 2)$; its capacity is $\alpha(\mathsf{G}^{\boxtimes k}) = 4^k$.
9. The Sims-Gewirtz graph G is $\text{srg}(56, 10, 0, 2)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 16^k$.
10. The graph G by Tonchev is $\text{srg}(220, 84, 38, 28)$, and $\alpha(\mathsf{G}^{\boxtimes k}) = 10^k$.

## Corollary: Clique and Chromatic Numbers of Ramanujan Graphs (I.S.)

Let G be a Ramanujan $d$-regular graph on $n$ vertices. Then,

$$\omega(\mathsf{G}) \leq \left\lfloor \frac{n\left(1+2\sqrt{d-1}\right)}{n-d+2\sqrt{d-1}} \right\rfloor, \tag{26}$$

$$\theta(\mathsf{G}) \geq \frac{n-d+2\sqrt{d-1}}{1+2\sqrt{d-1}}, \tag{27}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n-d+2\sqrt{d-1}}{1+2\sqrt{d-1}} \right\rceil. \tag{28}$$

## Corollary: Asymptotic Bounds (I.S., '23)

Let $\{\mathsf{G}_\ell\}_{\ell \in \mathbb{N}}$ be a sequence of Ramanujan $d$-regular graphs where $d \in \mathbb{N}$ is fixed, $\mathsf{G}_\ell$ is a graph on $n_\ell$ vertices, and $\lim_{\ell \to \infty} n_\ell = \infty$. Then,

$$\limsup_{\ell \to \infty} \omega(\mathsf{G}_\ell) \leq 1 + \lfloor 2\sqrt{d-1} \rfloor, \tag{29}$$

$$\liminf_{\ell \to \infty} \frac{\chi(\overline{\mathsf{G}}_\ell)}{n_\ell} \geq \frac{1}{1 + 2\sqrt{d-1}}. \tag{30}$$

## Theorem: Second-Largest and Least Eigenvalues (I.S., '23)

Let G be a $d$-regular graph of order $n$, which is non-complete and non-empty. Then,

$$\lambda_n(\mathsf{G}) \leq -\frac{d\,(n-d+\lambda_2(\mathsf{G}))}{d+(n-1)\,\lambda_2(\mathsf{G})}, \tag{31}$$

or equivalently,

$$\lambda_2(\mathsf{G}) \geq -\frac{d\,(n-d+\lambda_n(\mathsf{G}))}{d+(n-1)\,\lambda_n(\mathsf{G})}. \tag{32}$$

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1. From our earlier bounds, it follows that the inequality holds with equality if G is a strongly regular graph.
2. We prove that if G is regular, then equality holds if and only if G is strongly regular (I.S., '23).

## Theorem: Chromatic Numbers of Strong Product of SRGs (I.S., '23)

Let $\mathsf{G}_1, \ldots, \mathsf{G}_k$ be strongly regular graphs $\text{srg}(n_\ell, d_\ell, \lambda_\ell, \mu_\ell)$ for $\ell \in [k]$ (they need not be distinct). Then, the chromatic number of their strong product satisfies

$$\left\lceil \prod_{\ell=1}^{k} \left(1 + \frac{2d_\ell}{t_\ell + \mu_\ell - \lambda_\ell}\right) \right\rceil \leq \chi(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_k) \leq \prod_{\ell=1}^{k} \chi(\mathsf{G}_k), \quad (33)$$

where $\{t_\ell\}_{\ell=1}^{k}$ in the leftmost term is given by

$$t_\ell \triangleq \sqrt{(\lambda_\ell - \mu_\ell)^2 + 4(d_\ell - \mu_\ell)}, \quad \ell \in [k]. \quad (34)$$

The above lower bound is also larger than or equal to the product of the clique numbers of the factors $\{\mathsf{G}_\ell\}_{\ell=1}^{k}$.

## Example: Chromatic Numbers of Strong Products

Let

$$\mathsf{G} = \mathsf{srg}(27, 16, 10, 8), \quad \mathsf{H} = \mathsf{srg}(16, 6, 2, 2), \quad \mathsf{J} = \mathsf{srg}(100, 36, 14, 12)$$

be the Schläfli, Shrikhande, and Hall-Janko graphs, respectively.
The upper and lower bounds (in the previous slide) coincide here: for all integers $k_1, k_2, k_3 \geq 0$,

$$\chi(\mathsf{G}^{\boxtimes\, k_1} \boxtimes \mathsf{H}^{\boxtimes\, k_2} \boxtimes \mathsf{J}^{\boxtimes\, k_3}) = 9^{k_1}\, 4^{k_2}\, 10^{k_3}. \tag{35}$$

For comparison, the lower bound that is given by the product of the clique numbers of each factor is looser, and it is equal to $6^{k_1} 3^{k_2} 4^{k_3}$.

## Shannon Capacity of a Graph (1956)

- The capacity of a graph G was introduced by Claude E. Shannon (1956) to represent the maximum information rate that can be obtained with zero-error communication.
- A channel is represented by a proper graph G, and the Shannon capacity of a graph G is given by

$$\begin{aligned}\Theta(\mathsf{G}) &= \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(G^{\boxtimes k})} \\ &= \lim_{k \to \infty} \sqrt[k]{\alpha(G^{\boxtimes k})}. \end{aligned} \tag{36}$$

- The last equality holds by Fekete's Lemma since the sequence $\{\log \alpha(G^{\boxtimes k})\}_{k=1}^{\infty}$ is super-additive, i.e.,

$$\alpha(G^{\boxtimes (k_1+k_2)}) \geq \alpha(G^{\boxtimes k_1})\, \alpha(G^{\boxtimes k_2}). \tag{37}$$

## On the Computability of the Shannon Capacity of Graphs

- The Shannon capacity of a graph can be rarely computed exactly. ☹
- However, the Lovász $\theta$-function of a graph is a computable (and sometimes tight) upper bound on the Shannon capacity. ☺

## Lovász Bound on the Shannon Capacity of Graphs (1979)

Theorem: For every finite, simple and undirected graph G,

$$\Theta(\mathsf{G}) \leq \theta(\mathsf{G}). \tag{38}$$

## Capacity of Graphs

Proposition: Let G be a finite, undirected, and simple graph. If $\alpha(\mathsf{G}^{\boxtimes \ell}) = \theta(\mathsf{G})^{\ell}$ for some $\ell \in \mathbb{N}$, then

$$\Theta(\mathsf{G}) = \theta(\mathsf{G}), \quad \forall\, k \in \mathbb{N}. \tag{39}$$

## Capacity of Graphs

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Corollary 1: If $\alpha(\mathsf{G}) = \theta(\mathsf{G})$, then for all $k \in \mathbb{N}$, the $k$-fold strong power of G satisfies

$$\alpha(\mathsf{G})^k = \alpha(\mathsf{G}^{\boxtimes k}) = \Theta(\mathsf{G}^{\boxtimes k}) = \theta(\mathsf{G}^{\boxtimes k}) = \theta(\mathsf{G})^k, \quad \forall\, k \in \mathbb{N}. \tag{40}$$

By Corollary 1, we calculate the Shannon capacity of some regular graphs.

## Shannon Capacities of Some Strongly Regular Graphs

1. The Hall-Janko graph G is $\text{srg}(100, 36, 14, 12)$, and $\Theta(\text{G}) = 10$.
2. The Hoffman-Singleton graph G is $\text{srg}(50, 7, 0, 1)$, and $\Theta(\text{G}) = 15$.
3. The Janko-Kharaghani graphs of orders 936 and 1800 are $\text{srg}(936, 375, 150, 150)$ and $\text{srg}(1800, 1029, 588, 588)$, respectively. The capacity of both graphs is $36$.
4. Janko-Kharaghani-Tonchev: $\text{G} = \text{srg}(324, 153, 72, 72), \Theta(\text{G}) = 18$.
5. The graphs introduced by Makhnev are $\text{G} = \text{srg}(64, 18, 2, 6)$ and $\overline{\text{G}} = \text{srg}(64, 45, 32, 30)$. Capacities: $\Theta(\text{G}) = 16$, and $\Theta(\overline{\text{G}}) = 4$.
6. The Mathon-Rosa graph G is $\text{srg}(280, 117, 44, 52)$, and $\Theta(\text{G}) = 28$.
7. The Schläfli graph G is $\text{srg}(27, 16, 10, 8)$, and $\Theta(\text{G}) = 3$.
8. The Shrikhande graph is $\text{srg}(16, 6, 2, 2)$; its capacity is $\Theta(\text{G}) = 4$.
9. The Sims-Gewirtz graph G is $\text{srg}(56, 10, 0, 2)$, and $\Theta(\text{G}) = 16$.
10. The graph G by Tonchev is $\text{srg}(220, 84, 38, 28)$, and $\Theta(\text{G}) = 10$.

## Journal Paper

I. Sason, "Observations on the Lovász $\theta$-function, graph capacity, eigenvalues, and strong products," *Entropy*, vol. 25, no. 1, paper 104, pp. 1–40, January 2023 (https://doi.org/10.3390/e25010104).

https://www.mdpi.com/journal/entropy/special_issues/Combinatorial_Aspects

Submission Deadline: Dec. 31st, 2023