

Moderate Deviations Analysis of Binary Hypothesis Testing

Igal Sason

sason@ee.technion.ac.il

Department of Electrical Engineering

Technion, Haifa 32000, Israel

Abstract—This work refers to moderate-deviations analysis of binary hypothesis testing. It relies on a concentration inequality for discrete-parameter martingales with bounded jumps, which forms a refinement to the Azuma-Hoeffding inequality. Relations of the analysis to the moderate deviations principle for i.i.d. random variables and the relative entropy are considered.

Index Terms—Concentration inequalities, hypothesis testing, moderate deviations principle.

I. INTRODUCTION

The moderate deviations analysis in the context of source and channel coding has recently attracted some interest among information theorists (see [1], [4], [11], [16], [19] and [22]). The purpose of this paper is to consider moderate deviations analysis for binary hypothesis testing.

In the following, related literature on moderate deviations analysis in information-theoretic aspects is shortly reviewed. Moderate deviations were analyzed in [1, Section 4.3] for a channel model that gets noisier as the block length is increased. Due to the dependence of the channel parameter in the block length, the usual notion of capacity for these channels is zero. Hence, the issue of increasing the block length for the considered type of degrading channels was examined in [1, Section 4.3] via moderate deviations analysis when the number of codewords increases sub-exponentially with the block length. In another recent work [4], the moderate deviations behavior of channel coding for discrete memoryless channels was studied by Altug and Wagner with a derivation of direct and converse results which explicitly characterize the rate function of the moderate deviations principle (MDP). In [4], the authors studied the interplay between the probability of error, code rate and block length when the communication takes place over discrete memoryless channels, having the interest to figure out how the decoding error probability of the best code scales when simultaneously the block length tends to infinity and the code rate approaches the channel capacity. The novelty in the setup of their analysis was the consideration of the scenario mentioned above, in contrast to the case where the rate is kept fixed below capacity, and the study is reduced to a characterization of the dependence between the two remaining parameters (i.e., the block length n and the average/ maximal error probability of the best code). As opposed to the latter case when the code rate is kept fixed, which then corresponds to large deviations analysis and characterizes the error exponents as a function of the rate, the analysis in [4] (via the introduction of direct and converse theorems) demonstrated a sub-exponential scaling of the maximal error probability in the considered moderate

deviations regime. This work was followed by a work by Polynaskiy and Verdú where they show that a DMC satisfies the MDP if and only if its channel dispersion is non-zero, and also that the AWGN channel satisfies the MDP with a constant that is equal to the channel dispersion. The approach used in [4] was based on the method of types, whereas the approach used in [17] borrowed some tools from a recent work by the same authors in [16].

In [11], the moderate deviations analysis of the Slepian-Wolf problem for lossless source coding was studied. More recently, moderate deviations analysis for lossy source coding of stationary memoryless sources was studied in [22].

These works, including this paper, indicate a recent interest in moderate deviations analysis in the context of information-theoretic problems. In the literature on probability theory, the moderate deviations analysis was extensively studied (see, e.g., [10, Section 3.7]), and in particular the MDP was studied in [9] for continuous-time martingales with bounded jumps.

This paper has the following structure: Section II introduces briefly some preliminary material related to martingales and Azuma's inequality. It then follows by introducing a refined version of Azuma's inequality, and a study of its relation to the moderate deviations principle for i.i.d. random variables. Section III considers the relation of Azuma's inequality and the refined version of this inequality (from Section II) to moderate deviations analysis of binary hypothesis testing. Section IV concludes the paper, followed by a discussion on the MDP that is relegated to an appendix.

II. CONCENTRATION AND ITS RELATION TO THE MODERATE DEVIATIONS PRINCIPLE

We present here some essential material that is related to the martingale approach used in this paper for the moderate-deviations analysis of binary hypothesis testing. A background on martingales is provided in, e.g., [23] where we only rely here on basic knowledge on martingales.

A. Azuma's Inequality

Azuma's inequality¹ forms a useful concentration inequality for bounded-difference martingales [5]. In the following, this inequality is introduced. The reader is referred to, e.g., [6] and [15] for surveys on concentration inequalities for martingales (including a proof of this inequality).

Theorem 1: [Azuma's inequality] Let $\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}$ be a discrete-parameter real-valued martingale sequence (where

¹Azuma's inequality is also known as the Azuma-Hoeffding inequality. It will be named from this point as Azuma's inequality for the sake of brevity.

$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$ is called a filtration). Assume that for every $k \in \mathbb{N}$, the condition $|X_k - X_{k-1}| \leq d_k$ holds a.s. for some non-negative constants $\{d_k\}_{k=1}^\infty$. Then

$$\mathbb{P}(|X_n - X_0| \geq r) \leq 2 \exp\left(-\frac{r^2}{2 \sum_{k=1}^n d_k^2}\right) \quad \forall r \geq 0. \quad (1)$$

The concentration inequality stated in Theorem 1 was proved in [12] for independent bounded random variables, and it was later derived in [5] for bounded-difference martingales.

B. A Refined Version of Azuma's Inequality

Theorem 2: Let $\{X_k, \mathcal{F}_k\}_{k=0}^\infty$ be a discrete-parameter real-valued martingale. Assume that, for some constants $d, \sigma > 0$, the following two requirements are satisfied a.s.

$$|X_k - X_{k-1}| \leq d, \\ \text{Var}(X_k | \mathcal{F}_{k-1}) = \mathbb{E}[(X_k - X_{k-1})^2 | \mathcal{F}_{k-1}] \leq \sigma^2$$

for every $k \in \{1, \dots, n\}$. Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|X_n - X_0| \geq \alpha n) \leq 2 \exp\left(-n D\left(\frac{\delta + \gamma}{1 + \gamma} \parallel \frac{\gamma}{1 + \gamma}\right)\right) \quad (2)$$

where

$$\gamma \triangleq \frac{\sigma^2}{d^2}, \quad \delta \triangleq \frac{\alpha}{d} \quad (3)$$

and $D(p||q) \triangleq p \ln\left(\frac{p}{q}\right) + (1-p) \ln\left(\frac{1-p}{1-q}\right)$ for $p, q \in [0, 1]$ is the divergence (a.k.a. relative entropy or Kullback-Leibler distance) between the two probability distributions $(p, 1-p)$ and $(q, 1-q)$. If $\delta > 1$, then the probability on the left-hand side of (2) is equal to zero.

Proof: See [14], [10, Corollary 2.4.7] or [19, Section III]. ■

C. Relation of Theorem 2 with the Moderate Deviations Principle for i.i.d. RVs

According to the moderate deviations theorem (see, e.g., [10, Theorem 3.7.1]) in $\mathbb{R}$, let $\{X_i\}_{i=1}^n$ be a sequence of i.i.d. real-valued RVs such that $\Lambda_X(\lambda) = \mathbb{E}[e^{\lambda X_i}] < \infty$ in some neighborhood of zero, and also assume that $\mathbb{E}[X_i] = 0$ and $\sigma^2 = \text{Var}(X_i) > 0$. Let $\{a_n\}_{n=1}^\infty$ be a non-negative sequence such that $a_n \rightarrow 0$ and $na_n \rightarrow \infty$ as $n \rightarrow \infty$, and let

$$Z_n \triangleq \sqrt{\frac{a_n}{n}} \sum_{i=1}^n X_i, \quad \forall n \in \mathbb{N}. \quad (4)$$

Then, for every measurable set $\Gamma \subseteq \mathbb{R}$,

$$\begin{aligned} & -\frac{1}{2\sigma^2} \inf_{x \in \Gamma^0} x^2 \\ & \leq \liminf_{n \rightarrow \infty} a_n \ln \mathbb{P}(Z_n \in \Gamma) \\ & \leq \limsup_{n \rightarrow \infty} a_n \ln \mathbb{P}(Z_n \in \Gamma) \\ & \leq -\frac{1}{2\sigma^2} \inf_{x \in \bar{\Gamma}} x^2 \end{aligned} \quad (5)$$

where Γ^0 and $\bar{\Gamma}$ designate, respectively, the interior and closure sets of Γ .

Let $\eta \in (\frac{1}{2}, 1)$ be an arbitrary fixed number, and let $\{a_n\}_{n=1}^\infty$ be the non-negative sequence

$$a_n = n^{1-2\eta}, \quad \forall n \in \mathbb{N}$$

so that $a_n \rightarrow 0$ and $na_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $\alpha \in \mathbb{R}^+$, and $\Gamma \triangleq (-\infty, -\alpha] \cup [\alpha, \infty)$. Note that, from (4),

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \alpha n^\eta\right) = \mathbb{P}(Z_n \in \Gamma)$$

so from the moderate deviations principle (MDP)

$$\lim_{n \rightarrow \infty} n^{1-2\eta} \ln \mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \alpha n^\eta\right) = -\frac{\alpha^2}{2\sigma^2}, \quad \forall \alpha \geq 0. \quad (6)$$

It is demonstrated in Appendix A that, in contrast to Azuma's inequality, Theorem 2 gives an upper bound on the probability $\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \alpha n^\eta\right)$ (where $n \in \mathbb{N}$ and $\alpha \geq 0$) which coincides with the exact asymptotic limit in (6). The analysis in Appendix A provides another interesting link between Theorem 2 and a classical result in probability theory, which also emphasizes the significance of the refinements of Azuma's inequality.

III. MODERATE DEVIATIONS ANALYSIS FOR BINARY HYPOTHESIS TESTING

Binary hypothesis testing for finite alphabet models was analyzed via the method of types, e.g., in [7, Chapter 11] and [8]. It is assumed that the data sequence is of a fixed length (n), and one wishes to make the optimal decision based on the received sequence and the Neyman-Pearson ratio test.

Let the RVs $X_1, X_2, \dots$ be i.i.d. $\sim Q$, and consider two hypotheses:

- $H_1 : Q = P_1$.
- $H_2 : Q = P_2$.

For the simplicity of the analysis, let us assume that the RVs are discrete, and take their values on a finite alphabet $\mathcal{X}$ where $P_1(x), P_2(x) > 0$ for every $x \in \mathcal{X}$.

In the following, let

$$L(X_1, \dots, X_n) \triangleq \ln \frac{P_1^n(X_1, \dots, X_n)}{P_2^n(X_1, \dots, X_n)} = \sum_{i=1}^n \ln \frac{P_1(X_i)}{P_2(X_i)}$$

designate the log-likelihood ratio. By the strong law of large numbers (SLLN), if hypothesis H_1 is true, then a.s.

$$\lim_{n \rightarrow \infty} \frac{L(X_1, \dots, X_n)}{n} = D(P_1||P_2) \quad (7)$$

and otherwise, if hypothesis H_2 is true, then a.s.

$$\lim_{n \rightarrow \infty} \frac{L(X_1, \dots, X_n)}{n} = -D(P_2||P_1) \quad (8)$$

where the above assumptions on the probability mass functions P_1 and P_2 imply that the relative entropies, $D(P_1||P_2)$ and $D(P_2||P_1)$, are both finite. Consider the case where for some fixed constants $\bar{\lambda}, \underline{\lambda} \in \mathbb{R}$ that satisfy

$$-D(P_2||P_1) < \underline{\lambda} \leq \bar{\lambda} < D(P_1||P_2)$$

one decides on hypothesis H_1 if $L(X_1, \dots, X_n) > n\bar{\lambda}$, and on hypothesis H_2 if $L(X_1, \dots, X_n) < n\underline{\lambda}$. Note that if

$\bar{\lambda} = \underline{\lambda} \triangleq \lambda$ then a decision on the two hypotheses is based on comparing the normalized log-likelihood ratio (w.r.t. n) to a single threshold (λ), and deciding on hypothesis H_1 or H_2 if this normalized log-likelihood ratio is, respectively, above or below λ . If $\underline{\lambda} < \bar{\lambda}$ then one decides on H_1 or H_2 if the normalized log-likelihood ratio is, respectively, above the upper threshold $\bar{\lambda}$ or below the lower threshold $\underline{\lambda}$. Otherwise, if the normalized log-likelihood ratio is between the upper and lower thresholds, then an erasure is declared and no decision is taken in this case.

Let

$$\alpha_n^{(1)} \triangleq P_1^n \left(L(X_1, \dots, X_n) \leq n\bar{\lambda} \right) \quad (9)$$

$$\alpha_n^{(2)} \triangleq P_1^n \left(L(X_1, \dots, X_n) \leq n\underline{\lambda} \right) \quad (10)$$

and

$$\beta_n^{(1)} \triangleq P_2^n \left(L(X_1, \dots, X_n) \geq n\underline{\lambda} \right) \quad (11)$$

$$\beta_n^{(2)} \triangleq P_2^n \left(L(X_1, \dots, X_n) \geq n\bar{\lambda} \right) \quad (12)$$

then $\alpha_n^{(1)}$ and $\beta_n^{(1)}$ are the probabilities of either making an error or declaring an erasure under, respectively, hypotheses H_1 and H_2 ; similarly $\alpha_n^{(2)}$ and $\beta_n^{(2)}$ are the probabilities of making an error under hypotheses H_1 and H_2 , respectively.

Let $\pi_1, \pi_2 \in (0, 1)$ denote the a-priori probabilities of the hypotheses H_1 and H_2 , respectively, so

$$P_{e,n}^{(1)} = \pi_1 \alpha_n^{(1)} + \pi_2 \beta_n^{(1)} \quad (13)$$

is the probability of having either an error or an erasure, and

$$P_{e,n}^{(2)} = \pi_1 \alpha_n^{(2)} + \pi_2 \beta_n^{(2)} \quad (14)$$

is the probability of error.

Based on the asymptotic results in (7) and (8), which hold a.s. under hypotheses H_1 and H_2 respectively, the large deviations analysis refers to upper and lower thresholds $\bar{\lambda}$ and $\underline{\lambda}$ which are *kept fixed* (i.e., these thresholds do not depend on the block length n of the data sequence) where

$$-D(P_2||P_1) < \underline{\lambda} \leq \bar{\lambda} < D(P_1||P_2).$$

Suppose that instead of having some fixed upper and lower thresholds, one is interested to set these thresholds such that as the block length n tends to infinity, they tend simultaneously to their asymptotic limits in (7) and (8), i.e.,

$$\lim_{n \rightarrow \infty} \bar{\lambda}^{(n)} = D(P_1||P_2), \quad \lim_{n \rightarrow \infty} \underline{\lambda}^{(n)} = -D(P_2||P_1).$$

Specifically, let $\eta \in (\frac{1}{2}, 1)$, and $\varepsilon_1, \varepsilon_2 > 0$ be arbitrary fixed numbers, and consider the case where one decides on hypothesis H_1 if $L(X_1, \dots, X_n) > n\bar{\lambda}^{(n)}$, and on hypothesis H_2 if $L(X_1, \dots, X_n) < n\underline{\lambda}^{(n)}$ where these upper and lower thresholds are set to

$$\begin{aligned} \bar{\lambda}^{(n)} &= D(P_1||P_2) - \varepsilon_1 n^{-(1-\eta)} \\ \underline{\lambda}^{(n)} &= -D(P_2||P_1) + \varepsilon_2 n^{-(1-\eta)} \end{aligned}$$

so that they approach, respectively, the relative entropies $D(P_1||P_2)$ and $-D(P_2||P_1)$ in the asymptotic case where the block length n of the data sequence tends to infinity.

Accordingly, the conditional probabilities in (9)–(12) are modified so that the fixed thresholds $\bar{\lambda}$ and $\underline{\lambda}$ are replaced with the above block-length dependent thresholds $\bar{\lambda}^{(n)}$ and $\underline{\lambda}^{(n)}$, respectively. The moderate deviations analysis for binary hypothesis testing studies the probability of an error event and the probability of a joint error and erasure event under the two hypotheses, and it studies the interplay between each of these probabilities, the block length n , and the related thresholds that tend asymptotically to the limits in (7) and (8) when the block length tends to infinity.

In light of the discussion in Section II-C on the MDP for i.i.d. RVs and the discussion of its relation to Theorem 2 (see Appendix A), and also motivated by the three recent works in [1, Section 4.3], [4] and [11], we proceed to consider in the following moderate deviations analysis for binary hypothesis testing. Our approach for this kind of analysis is different, and it relies on concentration inequalities for martingales.

In the following, we analyze the probability of a joint error and erasure event under hypothesis H_1 , i.e., derive an upper bound on $\alpha_n^{(1)}$ in (9). The same kind of analysis can be adapted easily for the other probabilities in (10)–(12).

Under hypothesis H_1 , let us construct the martingale sequence $\{U_k, \mathcal{F}_k\}_{k=0}^n$ where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \mathcal{F}_n$ is the filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_k = \sigma(X_1, \dots, X_k), \quad \forall k \in \{1, \dots, n\}$$

and

$$U_k = \mathbb{E}_{P_1^n} [L(X_1, \dots, X_n) | \mathcal{F}_k]. \quad (15)$$

For every $k \in \{0, \dots, n\}$

$$\begin{aligned} U_k &= \mathbb{E}_{P_1^n} \left[\sum_{i=1}^n \ln \frac{P_1(X_i)}{P_2(X_i)} \middle| \mathcal{F}_k \right] \\ &= \sum_{i=1}^k \ln \frac{P_1(X_i)}{P_2(X_i)} + \sum_{i=k+1}^n \mathbb{E}_{P_1^n} \left[\ln \frac{P_1(X_i)}{P_2(X_i)} \right] \\ &= \sum_{i=1}^k \ln \frac{P_1(X_i)}{P_2(X_i)} + (n-k)D(P_1||P_2). \end{aligned}$$

In particular

$$U_0 = nD(P_1||P_2), \quad (16)$$

$$U_n = \sum_{i=1}^n \ln \frac{P_1(X_i)}{P_2(X_i)} = L(X_1, \dots, X_n) \quad (17)$$

and, for every $k \in \{1, \dots, n\}$,

$$U_k - U_{k-1} = \ln \frac{P_1(X_k)}{P_2(X_k)} - D(P_1||P_2). \quad (18)$$

Let

$$d_1 \triangleq \max_{x \in \mathcal{X}} \left| \ln \frac{P_1(x)}{P_2(x)} - D(P_1||P_2) \right| \quad (19)$$

so $d_1 < \infty$ since by assumption the alphabet set $\mathcal{X}$ is finite, and $P_1(x), P_2(x) > 0$ for every $x \in \mathcal{X}$. From (18) and (19), $|U_k - U_{k-1}| \leq d_1$ a.s. for every $k \in \{1, \dots, n\}$, and due to the statistical independence of $\{X_i\}$

$$\begin{aligned} &\mathbb{E}_{P_1^n} [(U_k - U_{k-1})^2 | \mathcal{F}_{k-1}] \\ &= \sum_{x \in \mathcal{X}} \left\{ P_1(x) \left(\ln \frac{P_1(x)}{P_2(x)} - D(P_1||P_2) \right)^2 \right\} \triangleq \sigma_1^2. \quad (20) \end{aligned}$$

Let $\varepsilon_1 > 0$ and $\eta \in (\frac{1}{2}, 1)$ be two arbitrarily fixed numbers. Then, under hypothesis H_1 , it follows from Theorem 2 and the above construction of a martingale that

$$\begin{aligned} P_1^n(L(X_1, \dots, X_n) \leq n\bar{\lambda}^{(n)}) \\ = P_1^n(U_n - U_0 \leq -\varepsilon_1 n^\eta) \\ \leq \exp\left(-nD\left(\frac{\delta_1^{(\eta, n)} + \gamma_1}{1 + \gamma_1} \parallel \frac{\gamma_1}{1 + \gamma_1}\right)\right) \end{aligned} \quad (21)$$

where

$$\delta_1^{(\eta, n)} \triangleq \frac{\varepsilon_1 n^{-(1-\eta)}}{d_1}, \quad \gamma_1 \triangleq \frac{\sigma_1^2}{d_1^2} \quad (22)$$

with d_1 and σ_1^2 from (19) and (20).

In the following, we will make use of the following lemma:

Lemma 1:

$$(1 + u) \ln(1 + u) \geq \begin{cases} u + \frac{u^2}{2}, & u \in [-1, 0] \\ u + \frac{u^2}{2} - \frac{u^3}{6}, & u \geq 0 \end{cases} \quad (23)$$

where at $u = -1$, the left-hand side is defined to be zero (it is the limit of this function when $u \rightarrow -1$ from above).

Proof: The proof follows by elementary calculus. ■

From (22) and the inequality in Lemma 1, it follows that

$$\begin{aligned} D\left(\frac{\delta_1^{(\eta, n)} + \gamma_1}{1 + \gamma_1} \parallel \frac{\gamma_1}{1 + \gamma_1}\right) \\ \geq \frac{\gamma_1}{1 + \gamma_1} \left[\left(\frac{\delta_1^{(\eta, n)}}{\gamma_1} + \frac{(\delta_1^{(\eta, n)})^2}{2\gamma_1^2} - \frac{(\delta_1^{(\eta, n)})^3}{6\gamma_1^3} \right) \right. \\ \left. + \frac{1}{\gamma_1} \left(-\delta_1^{(\eta, n)} + \frac{(\delta_1^{(\eta, n)})^2}{2} \right) \right] \\ = \frac{(\delta_1^{(\eta, n)})^2}{2\gamma_1} - \frac{(\delta_1^{(\eta, n)})^3}{6\gamma_1^2(1 + \gamma_1)} \\ = \frac{\varepsilon_1^2 n^{-2(1-\eta)}}{2\sigma_1^2} \left(1 - \frac{\varepsilon_1 d_1}{3\sigma_1^2(1 + \gamma_1)} \frac{1}{n^{1-\eta}} \right) \end{aligned}$$

provided that $\delta_1^{(\eta, n)} < 1$ (which holds for $n \geq n_0$ for some $n_0 \triangleq n_0(\eta, \varepsilon_1, d_1) \in \mathbb{N}$ that is determined from (22)). By substituting this lower bound on the divergence into (21), it follows that

$$\begin{aligned} \alpha_n^{(1)} = P_1^n(L(X_1, \dots, X_n) \leq nD(P_1 \parallel P_2) - \varepsilon_1 n^\eta) \\ \leq \exp\left(-\frac{\varepsilon_1^2 n^{2\eta-1}}{2\sigma_1^2} \left(1 - \frac{\varepsilon_1 d_1}{3\sigma_1^2(1 + \gamma_1)} \frac{1}{n^{1-\eta}} \right)\right). \end{aligned} \quad (24)$$

Consequently, in the limit where n tends to infinity,

$$\lim_{n \rightarrow \infty} n^{1-2\eta} \ln \alpha_n^{(1)} \leq -\frac{\varepsilon_1^2}{2\sigma_1^2} \quad (25)$$

with σ_1^2 in (20). From the analysis in Section II-C and Appendix A, it follows that the inequality for the asymptotic limit in (25) holds in fact with equality. To verify this, consider the real-valued sequence of i.i.d. RVs

$$Y_i \triangleq \ln\left(\frac{P_1(X_i)}{P_2(X_i)}\right) - D(P_1 \parallel P_2), \quad i = 1, \dots, n$$

that, under hypothesis H_1 , have zero mean and variance σ_1^2 . Since, by assumption, the sequence $\{X_i\}_{i=1}^n$ are i.i.d., then

$$L(X_1, \dots, X_n) - nD(P_1 \parallel P_2) = \sum_{i=1}^n Y_i, \quad (26)$$

and it follows from the one-sided version of the MDP in (6) that indeed (25) holds with equality. Moreover, Theorem 2 provides, via the inequality in (24), a finite-length result that enhances the asymptotic result for $n \rightarrow \infty$.

In the considered setting of moderate deviations analysis for binary hypothesis testing, the upper bound on the probability $\alpha_n^{(1)}$ in (24), which refers to the probability of either making an error or declaring an erasure (i.e., making no decision) under the hypothesis H_1 , decays to zero sub-exponentially with the length n of the sequence. As mentioned above, based on the analysis in Section II-C and Appendix A, the asymptotic upper bound in (25) is tight. A completely similar moderate-deviations analysis can be also performed under the hypothesis H_2 . Hence, a sub-exponential scaling of the probability $\beta_n^{(1)}$ in (11) of either making an error or declaring an erasure (where the lower threshold $\underline{\lambda}$ is replaced with $\underline{\lambda}^{(n)}$) also holds under the hypothesis H_2 . These two sub-exponential decays to zero for the probabilities $\alpha_n^{(1)}$ and $\beta_n^{(1)}$, under hypothesis H_1 or H_2 respectively, improve as the value of $\eta \in (\frac{1}{2}, 1)$ is increased. On the other hand, the two *exponential decays* to zero of the probabilities of error (i.e., $\alpha_n^{(2)}$ and $\beta_n^{(2)}$ under hypothesis H_1 or H_2 , respectively) improve as the value of $\eta \in (\frac{1}{2}, 1)$ is decreased; this is due to the fact that, for a fixed value of n , the margin which serves to protect us from making an error (either under hypothesis H_1 or H_2) is increased by decreasing the value of η as above (note that by reducing the value of η for a fixed n , the upper and lower thresholds $\bar{\lambda}^{(n)}$ and $\underline{\lambda}^{(n)}$ are made closer to $D(P_1 \parallel P_2)$ from below and to $-D(P_2 \parallel P_1)$ from above, respectively, which therefore increases the margin that is used for protecting one from making an erroneous decision). This shows the existence of a tradeoff, in the choice of the parameter $\eta \in (\frac{1}{2}, 1)$, between the probability of error and the joint probability of error and erasure under either hypothesis H_1 or H_2 (where this tradeoff exists symmetrically for each of the two hypotheses).

In [4] and [17], the authors consider moderate deviations analysis for channel coding over memoryless channels. In particular, [4, Theorem 2.2] and [17, Theorem 6] indicate on a tight lower bound (i.e., a converse) to the asymptotic result in (25) for binary hypothesis testing. This tight converse is indeed consistent with the asymptotic result of the MDP in (6) for real-valued i.i.d. random variables, which implies that the asymptotic upper bound in (25), obtained via the martingale approach with the refined version of Azuma's inequality in Theorem 2, holds indeed with equality. Note that this equality does not follow from Azuma's inequality, so its refinement was essential for obtaining this equality. The reason is that, due to Appendix A, the upper bound in (25) that is equal to $-\frac{\varepsilon_1^2}{2\sigma_1^2}$ is replaced via Azuma's inequality by the looser bound $-\frac{\varepsilon_1^2}{2d_1^2}$ (note that, from (19) and (20), $\sigma_1 \leq d_1$ where in general σ_1 may be significantly smaller than d_1).

IV. SUMMARY

This paper is focused on the moderate deviations analysis of binary hypothesis testing. The analysis is based on a concentration inequality for discrete-parameter martingales with bounded jumps, which forms a refined version of Azuma's inequality (see [10, Corollary 2.4.7]). The relation of this concentration inequality to the moderate deviations principle for i.i.d. random variables is considered. This paper presents in part the work in [19], and it exemplifies the use of a refinement of Azuma's inequality in an information-theoretic aspect. Further information-theoretic applications are considered in, e.g., [20] and [24]. The slides are available in [21].

Acknowledgment: One of the reviewers pointed out that the moderate deviations analysis in this work can be done alternatively by relying on results, e.g., from [3] or [18]. We thank the reviewer for this note, and we currently study this line of work.

APPENDIX A

ANALYSIS RELATED TO THE MODERATE DEVIATIONS PRINCIPLE FOR I.I.D. RVs (SEE SECTION II-C)

It is demonstrated in the following that, in contrast to Azuma's inequality, Theorem 2 provides an upper bound on $\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \alpha n^\eta\right)$ for $\alpha \geq 0$, which coincides with the correct asymptotic result in (6). It is proved under the further assumption that there exists some constant $d > 0$ such that $|X_k| \leq d$ a.s. for every $k \in \mathbb{N}$ (since the RVs $\{X_k\}$ are assumed to be i.i.d., it is sufficient to require it for $k = 1$). Let us define the martingale sequence $\{S_k, \mathcal{F}_k\}_{k=0}^n$ where $S_k \triangleq \sum_{i=1}^k X_i$ and $\mathcal{F}_k \triangleq \sigma(X_1, \dots, X_k)$ for every $k \in \{1, \dots, n\}$ with $S_0 = 0$ and $\mathcal{F}_0 = \{\emptyset, \mathcal{F}\}$.

1) *Analysis related to Azuma's inequality:* The martingale sequence $\{S_k, \mathcal{F}_k\}_{k=0}^n$ has uniformly bounded jumps, where $|S_k - S_{k-1}| = |X_k| \leq d$ a.s. for every $k \in \{1, \dots, n\}$. Hence it follows from Azuma's inequality that, for every $\alpha \geq 0$,

$$\mathbb{P}(|S_n| \geq \alpha n^\eta) \leq 2 \exp\left(-\frac{\alpha^2 n^{2\eta-1}}{2d^2}\right)$$

and therefore

$$\lim_{n \rightarrow \infty} n^{1-2\eta} \ln \mathbb{P}(|S_n| \geq \alpha n^\eta) \leq -\frac{\alpha^2}{2d^2}. \quad (27)$$

This differs from the limit in (6) where σ^2 is replaced by d^2 , so Azuma's inequality does not provide the correct asymptotic result in (6) (unless $\sigma^2 = d^2$, i.e., $|X_k| = d$ a.s. for every k).

2) *Analysis related to Theorem 2:* From Theorem 2, it follows that for every $\alpha \geq 0$,

$$\mathbb{P}(|S_n| \geq \alpha n^\eta) \leq 2 \exp\left(-n D\left(\frac{\delta' + \gamma}{1 + \gamma} \parallel \frac{\gamma}{1 + \gamma}\right)\right)$$

where γ is introduced in (3), and δ' is given by

$$\delta' \triangleq \frac{\alpha}{n^{1-\eta}} = \delta n^{-(1-\eta)} \quad (28)$$

due to the definition of δ in (3). Hence, it follows that

$$\begin{aligned} & \mathbb{P}(|S_n| \geq \alpha n^\eta) \\ & \leq 2 \exp\left(-\frac{\delta^2 n^{2\eta-1}}{2\gamma} \left[1 + \frac{\alpha(1-\gamma)}{3\gamma d} \cdot n^{-(1-\eta)} + \dots\right]\right) \end{aligned}$$

for every $n \in \mathbb{N}$, and therefore (since, from (3), $\frac{\delta^2}{\gamma} = \frac{\alpha^2}{\sigma^2}$)

$$\lim_{n \rightarrow \infty} n^{1-2\eta} \ln \mathbb{P}(|S_n| \geq \alpha n^\eta) \leq -\frac{\alpha^2}{2\sigma^2}. \quad (29)$$

Hence, this bound coincides with the exact limit in (6).

REFERENCES

- [1] E. A. Abbe, *Local to Global Geometric Methods in Information Theory*, Ph.D. dissertation, MIT, Boston, MA, USA, June 2008.
- [2] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley Series in Discrete Mathematics and Optimization, Third Edition, 2008.
- [3] A. N. Arkhangel'skii, "Lower bounds for probabilities of large deviations for sums of independent random variables," *Theory of Probability and Applications*, vol. 34, no. 4, pp. 565-575, 1989.
- [4] Y. Altuğ and A. B. Wagner, "Moderate deviations analysis of channel coding: discrete memoryless case," *Proceedings 2010 IEEE International Symposium on Information Theory (ISIT 2010)*, pp. 265-269, Austin, Texas, USA, June 2010.
- [5] K. Azuma, "Weighted sums of certain dependent random variables," *Tohoku Mathematical Journal*, vol. 19, pp. 357-367, 1967.
- [6] F. Chung and L. Lu, "Concentration inequalities and martingale inequalities: a survey," *Internet Mathematics*, vol. 3, no. 1, pp. 79-127, March 2006.
- [7] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley and Sons, second edition, 2006.
- [8] I. Csiszár and P. C. Shields, *Information Theory and Statistics: A Tutorial*, Foundations and Trends in Communications and Information Theory, vol. 1, no. 4, pp. 417-528, 2004.
- [9] A. Dembo, "Moderate deviations for martingales with bounded jumps," *Electronic Communications in Probability*, vol. 1, no. 3, pp. 11-17, March 1996.
- [10] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer, second edition, 1997.
- [11] D. He, L. A. Lastras-Montaño, E. Yang, A. Jagmohan and J. Chen, "On the redundancy of Slepian-Wolf coding," *IEEE Trans. on Information Theory*, vol. 55, no. 12, pp. 5607-5627, December 2009.
- [12] W. Hoeffding, "Probability inequalities for sums of bounded random variables," *Journal of the American Statistical Association*, vol. 58, no. 301, pp. 13-30, March 1963.
- [13] F. den Hollander, *Large Deviations*, Fields Institute Monographs, American Mathematical Society, 2000.
- [14] C. McDiarmid, "On the method of bounded differences," *Surveys in Combinatorics*, vol. 141, pp. 148-188, Cambridge University Press, Cambridge, 1989.
- [15] C. McDiarmid, "Concentration," *Probabilistic Methods for Algorithmic Discrete Mathematics*, pp. 195-248, Springer, 1998.
- [16] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in finite blocklength regime," *IEEE Trans. on Information Theory*, vol. 56, no. 5, pp. 2307-2359, May 2010.
- [17] Y. Polyanskiy and S. Verdú, "Channel dispersion and moderate deviations limits of memoryless channels," *Proceedings Forty-Eighth Annual Allerton Conference*, pp. 1334-1339, UIUC, Illinois, USA, October 2010.
- [18] L. V. Rozovsky, "Estimate from below for large-deviation probabilities of a sum of independent random variables with finite variances," *Journal of Mathematical Sciences*, vol. 109, no. 6, May 2002.
- [19] I. Sason, "On refined versions of the Azuma-Hoeffding inequality with applications in information theory," last updated on February 2012. [Online]. Available: <http://arxiv.org/abs/1111.1977>.
- [20] I. Sason, "On the concentration of the crest factor for OFDM signals," *Proceedings of the 2011 8th International Symposium on Wireless Communication Systems (ISWCS '11)*, pp. 784-788, Aachen, Germany, November 2011. [Online]. Available: <http://arxiv.org/abs/1111.1982>.
- [21] I. Sason, "On Concentration of measures and moderate deviations analysis of binary hypothesis testing," presentation is online available at http://web.ee.technion.ac.il/people/sason/ISIT2012a_presentation.pdf.
- [22] V. Y. F. Tan, "Moderate-deviations of lossy source coding for discrete and Gaussian sources," <http://arxiv.org/abs/1111.2217>, November 2011.
- [23] D. Williams, *Probability with Martingales*, Cambridge University Press, 1991.
- [24] K. Xenoulis, N. Kalouptsidis and I. Sason, "New achievable rates for nonlinear Volterra channels via martingale inequalities," *Proceedings of the 2012 International Symposium on Information Theory*, MIT, Boston, USA, July 2012.