Contributions in Algebraic Graph Theory

Research Thesis

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Computer Science

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Submitted to the Senate of
the Technion — Israel Institute of Technology
Av 5786 Haifa July 2026

The research thesis was done under the supervision of Professor Igal Sason and Professor Emeritus Abraham Berman in the Computer Science Department.

The author of this thesis states that the research, including the collection, processing and presentation of data, addressing and comparing to previous research, etc., was done entirely in an honest way, as expected from scientific research that is conducted according to the ethical standards of the academic world. Also, reporting the research and its results in this thesis was done in an honest and complete manner, according to the same standards.

List of Publications

[1] N. Krupnik and A. Berman, "The graphs of pyramids are determined by their spectrum," *Linear Algebra and its Applications*, 2024.

[2] I. Sason, N. Krupnik, S. Hamud, and A. Berman, "On spectral graph determination," *Mathematics*, 2025, **13**, 549.

[3] N. Krupnik, I. Sason, and A. Berman, "On the transitivity of generalized-Hamming graphs and their complements," submitted April 2026, Available at: https://doi.org/10.48550/arXiv.2603.21627.

The generous financial support of the Technion is gratefully acknowledged.

Acknowledgments

I would like to express my sincere gratitude to my supervisors, Professor Igal Sason and Professor Emeritus Abraham Berman, for their invaluable guidance throughout my research journey. Their expertise, encouragement, and unwavering support were instrumental in the completion of this thesis. I am deeply grateful for the opportunity to work under their supervision and to learn from them. Additionally, I would like to thank Professor Ronny Roth and Professor Raphael Yuster for carefully reading my thesis and serving on my examination committee.

I am also grateful to Professor Eitan Yaakobi and Ms. Melinda Margolis for their continued support throughout my undergraduate and graduate studies, and for believing in me even when I doubted myself.

This thesis is dedicated to my grandmother, Dr. Dafna Shasha, and to the memory of my grandfather, Professor Shaul Shasha, whose influence on my life and academic path has been profound. My grandfather was a true mentor to me. We could talk about anything, exchanging ideas, stories, insights, and jokes. He was the wisest person I have ever known. The mathematical discussions that my grandmother and I have shared from my childhood to this day have always been a source of inspiration and fascination.

I would also like to thank my siblings, Carmel and Inbal. Carmel, thank you for the countless discussions about functions, the Dirac delta distribution, and the philosophy of mathematics. Inbal, thank you for listening when I shared my work with you, even though mathematics is not your favorite subject.

Finally, I extend my deepest gratitude to my parents, Haya and Hagai, who made this journey possible. You have always given me unconditional love and support through both the good times and the difficult ones. I consider myself fortunate to be your son.

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Abstract

This thesis investigates two central directions in algebraic graph theory, with an emphasis on spectral methods: spectral determination of graphs and transitivity properties of generalized-Hamming graphs and their complements.

The first part focuses on graphs that are determined by the spectra of associated matrices. We study spectral determination with respect to the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices, with particular emphasis on the adjacency spectrum. We survey existing results on graphs determined by their spectrum and develop new proof techniques for establishing spectral uniqueness. In particular, we present new proofs for the spectral characterization of complete bipartite graphs and Turán graphs, as well as some new results related to the spectral characterization of the important family of strongly regular graphs. In addition, we introduce a new family of graphs, called *the graphs of pyramids*, and prove that they are determined by their adjacency spectrum using tools from matrix analysis, such as Cauchy's interlacing theorem and Schur complements.

The second part of the thesis studies generalized-Hamming graphs, a family of Cayley graphs that generalize the sub-family of Hamming graphs, and their complements. We classify the parameters for which these graphs are edge-transitive or even distance-transitive. Our analysis combines spectral methods, group-theoretic arguments, and techniques from the theory of association schemes. As an application, we derive closed-form expressions for the Lovász ϑ-function of generalized-Hamming graphs and their complements whenever either the graph or its complement is edge-transitive.

Overall, the results demonstrate how spectral methods provide powerful tools for understanding the structure and symmetry of graphs, and they suggest several directions for further research in spectral graph theory.

Abbreviations and Notations

$\mathbb{N}$	—	The set of natural numbers
$\mathbb{R}$	—	The set of real numbers
$\mathbb{R}^n$	—	The set of all n-dimensional column vectors with real entries
$\mathbb{R}^{n\times m}$	—	The set of all $n \times m$ matrices with real entries
Sym_n	—	The set of all $n \times n$ symmetric matrices with real entries
$[n]$	—	The set $\{1, 2, \ldots, n\}$
$\mathbf{I}_n$	—	The $n \times n$ identity matrix
$\mathbf{0}_{k,m}$	—	The $k \times m$ all-zero matrix
$\mathbf{J}_{k,m}$	—	The $k \times m$ all-ones matrix
$\mathbf{1}_n$	—	The n-dimensional column vector of ones
G	—	A simple finite graph
$\overline{\mathsf{G}}$	—	The complement graph of G
$\mathsf{G}_1 \dot\cup \mathsf{G}_2$	—	The disjoint union of graphs G_1 and G_2
$\mathsf{G}_1 \vee \mathsf{G}_2$	—	The join of graphs G_1 and G_2
$\mathsf{V}(\mathsf{G})$	—	The vertex set of a graph G
$\mathsf{E}(\mathsf{G})$	—	The edge set of a graph G
$\mathbf{A}(\mathsf{G})$	—	The adjacency matrix of a graph G
$\mathbf{L}(\mathsf{G})$	—	The Laplacian matrix of a graph G
$\mathbf{Q}(\mathsf{G})$	—	The signless Laplacian matrix of a graph G
$\mathcal{L}(\mathsf{G})$	—	The normalized Laplacian matrix of a graph G
$\mathbf{D}(\mathsf{G})$	—	The degrees matrix of a graph G
$\mathrm{d}(v)$	—	The degree of a vertex v
$\sigma(\mathsf{G})$	—	The adjacency spectrum of a graph G
$\sigma_X(\mathsf{G})$	—	The spectrum of $X(\mathsf{G})$
$\ell(\mathsf{G})$	—	The line graph of a graph G
$\mathbf{M}/\mathbf{D}$	—	The Schur complement of a matrix $\mathbf{M}$ with respect to a submatrix $\mathbf{D}$
X-DS	—	Determined by its X spectrum
X-NICS	—	Non-isomorphic cospectral with respect to its X spectrum

K_n	—	The complete graph on n vertices
$\mathsf{K}_{k,m}$	—	The complete bipartite graph on parts with sizes k and m
$\mathsf{K}_{n_1,\ldots,n_k}$	—	The complete k-partite graph on parts with sizes $n_1, \ldots, n_k$
C_n	—	The cycle graph on n vertices
P_n	—	The path graph on n vertices
S_n	—	The star graph on n vertices
F_q	—	The friendship graph on $2q + 1$ vertices
$\mathsf{T}_{n,k}$	—	The graph of pyramids
$T(n, k)$	—	The Turán graph
$\mathsf{srg}(n, d, \lambda, \mu)$	—	The strongly regular graph with parameters n, d, λ, μ
$\omega(\mathsf{G})$	—	The clique number of a graph G
$\alpha(\mathsf{G})$	—	The independence number of a graph G
$\chi(\mathsf{G})$	—	The chromatic number of a graph G
$\chi_{\mathrm{f}}(\mathsf{G})$	—	The fractional chromatic number of a graph G
$\Theta(\mathsf{G})$	—	The Shannon capacity of a graph G
$\vartheta(\mathsf{G})$	—	The Lovász theta function of a graph G
$\mathrm{AM}(a, b)$	—	The arithmetic mean of a and b
$\mathrm{GM}(a, b)$	—	The geometric mean of a and b
$\mathrm{Du}(G)$	—	The duplication graph of a graph G
$\mathsf{G}_1 \circ \mathsf{G}_2$	—	The corona of graphs G_1 and G_2
$\mathsf{G}_1 \diamond \mathsf{G}_2$	—	The edge corona of graphs G_1 and G_2
$\mathsf{G}_1 \boxminus \mathsf{G}_2$	—	The duplication corona of graphs G_1 and G_2
$\mathsf{G}_1 \boxbar \mathsf{G}_2$	—	The duplication neighborhood corona of graphs G_1 and G_2
$\mathsf{G}_1 \boxplus \mathsf{G}_2$	—	The duplication edge corona of graphs G_1 and G_2
$\mathsf{G}_1 \underline{\boxtimes} \mathsf{G}_2$	—	The closed neighborhood corona of graphs G_1 and G_2
$\mathrm{S}(\mathsf{G})$	—	The subdivision graph of a graph G
$\mathrm{B}(\mathsf{G})$	—	The bipartite incidence graph of a graph G
$\mathrm{S}(\mathsf{G}_1) \ddot{\vee} \mathrm{B}(\mathsf{G}_2)$	—	The subdivision-vertex-bipartite-vertex join of graphs G_1 and G_2
$\mathrm{S}(\mathsf{G}_1) \overset{=}{\vee} \mathrm{B}(\mathsf{G}_2)$	—	The subdivision-edge-bipartite-edge join of graphs G_1 and G_2
$\mathrm{S}(\mathsf{G}_1) \overline{\dot{\vee}} \mathrm{B}(\mathsf{G}_2)$	—	The subdivision-edge-bipartite-vertex join of graphs G_1 and G_2
$\mathrm{S}(\mathsf{G}_1) \overline{\dot{\vee}} \mathrm{B}(\mathsf{G}_2)$	—	The subdivision-vertex-bipartite-edge join of graphs G_1 and G_2
$\mathsf{G}_1 \underline{\vee} \mathsf{G}_2$	—	The neighbors splitting join of graphs G_1 and G_2
$\mathsf{G}_1 \underline{\underline{\vee}} \mathsf{G}_2$	—	The nonneighbors splitting join of graphs G_1 and G_2
$I(n)$	—	The numbers of distinct graphs on n vertices
$\alpha_{\mathcal{X}}(n)$	—	The number of $\mathcal{X}$-DS graphs on n vertices

$\mathbb{F}_q$	—	The field with q elements
$\mathbb{F}_q^n$	—	The n-dimensional vector space over $\mathbb{F}_q$
$\mathcal{G}_{q,n,d}$	—	The generalized-Hamming graph with parameters q, n, and d
$w_H(\mathbf{x})$	—	The weight of a vector $\mathbf{x}$
$\mathrm{d}_H(\mathbf{x}, \mathbf{y})$	—	The Hamming distance between vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathrm{d}_{\mathsf{G}}(u, v)$	—	The graph distance between two vertices u and v
$\mathcal{N}(v)$	—	The neighborhood of a vertex v
$\mathrm{Cay}\,(G, \mathcal{S})$	—	The Cayley graph of a group G with respect to a generating set $\mathcal{S}$
$\mathrm{Aut}(\mathsf{G})$	—	The automorphism group of a graph G
$\mathcal{B}(\mathcal{R})$	—	The Bose-Mesner algebra of an association scheme $\mathcal{R}$
$K_\ell(i; n, q)$	—	The q-ary Krawtchouk polynomial
$\mathbf{E}_i$	—	The i-th element in the dual basis of a Bose-Mesner algebra
$\chi_{\mathbf{y}}$	—	The additive character of an abelian group associated with a vector $\mathbf{y}$
$\langle \mathbf{x}\,,\, \mathbf{y} \rangle$	—	The inner product of vectors $\mathbf{x}$ and $\mathbf{y}$
$\mathrm{Tr}(a)$	—	The trace of an element a in an extension field
$\mathrm{mc}(\mathsf{G})$	—	The maximum cut of a graph G
$\mathrm{sp}(\mathsf{G})$	—	The surplus of a graph G

Chapter 1

Introduction

Algebraic graph theory studies graphs through algebraic tools such as matrices, polynomials, and groups, enabling structural insights that often go beyond purely combinatorial methods. Every graph can be associated with various algebraic objects, such as its adjacency matrix, automorphism group, Lovász theta function, and many others. These connections allow one to translate combinatorial problems into algebraic ones, where powerful analytical techniques become available.

Spectral graph theory is a subfield of algebraic graph theory that focuses on the study of the eigenvalues and eigenvectors of graph-related matrices, such as the adjacency matrix $\mathbf{A}$, the Laplacian $\mathbf{L}$, the signless Laplacian $\mathbf{Q}$, and the normalized Laplacian $\mathcal{L}$. Unlike many graph invariants, such as the chromatic number or independence number, spectral invariants can be computed efficiently (e.g., in polynomial time), making them particularly attractive in algorithmic contexts.

Algebraic and spectral graph theory play an important role in several areas of computer science, including network analysis, machine learning, and coding theory. Applications range from constrained systems and error-correcting [103, 133] codes to clustering algorithms [109, 131], graph-based deep learning [83], and ranking methods [115]. Furthermore, many known combinatorial results have been proved via spectral and algebraic methods, such as the friendship theorem [59], the Graham–Pollak theorem [63], the Erdős–Ko–Rado theorem [62], and Kirchhoff's matrix-tree theorem [84].

The spectra of graph-associated matrices encode rich structural information. For example, the largest adjacency eigenvalue of a graph G lies between the average degree and the maximum degree of the graph. The second-smallest Laplacian eigenvalue (the algebraic connectivity) provides both lower and upper bounds on the Cheeger constant of a graph [8]. With respect to both the Laplacian and normalized Laplacian spectra, the multiplicity of 0 equals the number of connected components in G. The multiplicity of the zero eigenvalue of the signless Laplacian equals the number of bipartite connected components in G. The largest eigenvalue of the normalized Laplacian is at most 2, and it equals 2 if and only if G has a nontrivial bipartite connected component. These examples illustrate the extent to which spectral information reflects structural properties of graphs.

In this thesis, we review both classical and recent results regarding the relationship between the spectra of each of these four matrices and the structural properties of the corresponding graphs, with an emphasis on the adjacency matrix. We also prove several new results in this direction, including a spectral characterization of strong regularity.

Each of the matrices $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$ uniquely encodes the structure of a graph, in contrast to their spectra, which may be identical for several non-isomorphic graphs. Since Kac's famous question, "Can one hear the shape of a drum?" in [81], there has been growing interest in studying cases in which the spectrum of a matrix associated with a graph uniquely determines the structure of the graph (up to isomorphism). Two graphs are said to be cospectral if they share the same multiset of eigenvalues, and a graph G is said to be determined by its spectrum (DS) if any graph cospectral with G is isomorphic to it.

The thesis explores the concepts of cospectral and DS graphs. In particular, we show that certain families of graphs, such as Turán graphs, the Petersen graph, and graphs of pyramids, are determined by their spectra, and we also characterize the cospectral mates of complete bipartite graphs. It also surveys a variety of unary and binary graph operations that preserve the spectrum, enabling the construction of cospectral graphs.

The algebraic study of graphs is not limited to their matrices, but extends to other algebraic structures. The automorphism group of a graph is another well-studied object that captures its symmetries. Vertex-, edge-, and distance-transitivity are properties that can be characterized via the structure of the automorphism group. These properties have implications for many graph invariants, including the Lovász ϑ function, the Schrijver ϑ function, and the Shannon capacity.

The thesis studies the transitivity properties of generalized-Hamming graphs $\mathcal{G}_{q,n,d}$ and their complements, which are Cayley graphs that arise in coding theory and information theory. It characterizes the edge- and distance-transitivity of these graphs as functions of their parameters. The proofs rely on techniques from representation theory and the theory of association schemes, yielding a detailed structural and spectral description of these graphs, including a spectral decomposition and a closed-form expression for their eigenvalues. The results are then applied to the computation of the Lovász ϑ function for a subset of these graphs.

The rest of the thesis is structured as follows:

- Chapter 2 surveys classical and recent results on spectral graph determination, with an emphasis on the adjacency matrix. It reviews spectral properties of graphs, the notion of graphs determined by their spectrum, and constructions of cospectral graphs via standard graph operations. This chapter also presents new proofs that highlight spectral implications of strong regularity and characterizes cospectral mates of several graph families, including Turán and complete bipartite graphs. In addition, it discusses open problems in the field, with particular emphasis on Haemers' conjecture. Section 2.2 provides background material used in

Chapters 3 and 4. This chapter reproduces, with minor modifications, the results of [127].

- Chapter 3 introduces a new family of graphs, called graphs of pyramids, and studies their spectral properties. These graphs generalize the graphs T_n, which are related to completely positive matrices. A closed-form expression for their eigenvalues is derived using Schur complements, and it is shown that these graphs are determined by their adjacency spectrum. This chapter reproduces, with minor modifications, the results of [90].

- Chapter 4 investigates the transitivity properties of generalized-Hamming graphs $\mathcal{G}_{q,n,d}$ and their complements, which arise naturally in coding theory. It characterizes edge- and distance-transitivity in terms of the parameters q, n, d. The analysis combines combinatorial arguments with algebraic techniques from representation theory and association schemes, and further explores implications for computing the Lovász ϑ function. This chapter reproduces, with minor modifications, the results of [91].

- Chapter 5 summarizes the main results of the thesis and outlines directions for future research.

Chapter 2

On Spectral Graph Determination

Chapter Overview

The study of spectral graph determination is a fascinating area of research in spectral graph theory and algebraic combinatorics. This field focuses on examining the spectral characterization of various classes of graphs, developing methods to construct or distinguish cospectral nonisomorphic graphs, and analyzing the conditions under which a graph's spectrum uniquely determines its structure. This chapter presents an overview of both classical and recent advancements in these topics, along with newly obtained proofs of some existing results, which offer additional insights.

2.1 Introduction

Spectral graph theory lies at the intersection of combinatorics and matrix theory, exploring the structural and combinatorial properties of graphs through the analysis of the eigenvalues and eigenvectors of matrices associated with these graphs [20, 38, 46, 49, 66]. Spectral properties of graphs offer powerful insights into a variety of useful graph characteristics, enabling the determination or estimation of features such as the independence number, clique number, chromatic number, and the Shannon capacity of graphs, which are NP-hard to compute.

A particularly intriguing topic in spectral graph theory is the study of cospectral graphs, i.e., graphs that share identical multisets of eigenvalues with respect to one or more matrix representations. While isomorphic graphs are always cospectral, non-isomorphic graphs may also share spectra, leading to the study of non-isomorphic cospectral (NICS) graphs. This phenomenon raises profound questions about the extent to which a graph's spectrum encodes its structural properties. Conversely, graphs determined by their spectrum (DS graphs) are uniquely identifiable, up to isomorphism, by their eigenvalues. In other words, a graph is DS if and only if no other non-isomorphic graph shares the same spectrum.

The problem of spectral graph determination and the characterization of DS graphs dates back to the pioneering 1956 paper by Günthard and Primas [68], which explored the interplay between graph theory and chemistry. This paper posed the question of whether graphs can be uniquely determined by their spectra with respect to their adjacency matrix $\mathbf{A}$.

While every graph can be determined by its adjacency matrix, which enables the determination of every graph by its eigenvalues and a basis of corresponding eigenvectors, the characterization of graphs for which eigenvalues alone suffice for identification forms a fertile area of research in spectral graph theory. This research holds both theoretical interest and practical implications.

Subsequent studies have broadened the scope of this question to include determination by the spectra of other significant matrices, such as the Laplacian matrix ($\mathbf{L}$), signless Laplacian matrix ($\mathbf{Q}$), and normalized Laplacian matrix ($\mathcal{L}$), among many other matrices associated with graphs. The study of cospectral and DS graphs with respect to these matrices has become a cornerstone of spectral graph theory. This line of research has far-reaching applications in diverse fields, including chemistry and molecular structure analysis, physics and quantum computing, network communication theory, machine learning, and data science.

One of the most prominent conjectures in this area is Haemers' conjecture [69, 70], which posits that most graphs are determined by the spectrum of their adjacency matrices ($\mathbf{A}$-DS). Despite many efforts in proving this open conjecture, some theoretical and experimental progress on the theme of this conjecture has been recently presented in [89, 145], while also graphs or graph families that are not DS continue to be discovered. Haemers' conjecture has spurred significant interest in classifying DS graphs and understanding the factors that influence spectral determination, particularly among special families of graphs such as regular graphs, strongly regular graphs, trees, graphs of pyramids, as well as the construction of NICS graphs by a variety of graph operations. Studies in these directions of research have been covered in the seminal works by Schwenk [129], and by van Dam and Haemers [50, 51], as well as in more recent studies such as [1–3, 12, 25, 27–32, 54, 58, 65, 72, 74, 80, 82, 87–90, 95, 97, 104, 110, 112, 113, 124, 149–152], and references therein. Specific contributions of these papers to the problem of the spectral determination of graphs are addressed in the continuation of this chapter.

This chapter surveys both classical and recent results on spectral graph determination, also presenting newly obtained proofs of some existing results, which offer additional insights.

The chapter emphasizes the significance of adjacency spectra ($\mathbf{A}$-spectra), and it provides conditions for $\mathbf{A}$-cospectrality, $\mathbf{A}$-NICS, and $\mathbf{A}$-DS graphs, offering examples that support or refute Haemers' conjecture. We furthermore address the cospectrality of graphs with respect to the Laplacian, signless Laplacian, and normalized Laplacian matrices. For regular graphs, cospectrality with respect to any one of these matrices (or the adjacency matrix) implies cospectrality with respect to all the others, enabling a unified framework for studying DS and NICS graphs across different matrix representations. However, for irregular graphs, cospectrality with respect to one matrix does not necessarily imply cospectrality with respect to another. This distinction underscores the complexity of analyzing spectral properties in irregular graphs, where the interplay among different matrix representations becomes more intricate and often necessitates distinct techniques for characterization and comparison.

The structure of the chapter is as follows: Section 2.2 provides preliminary material in matrix theory, graph theory, and graph-associated matrices. Section 2.3 focuses on graphs determined by their spectra (with respect to one or multiple matrices). Section 2.4 examines special families of graphs and their determination by adjacency spectra. Section 2.5 analyzes unitary and binary graph operations, emphasizing their impact on spectral determination and construction of NICS graphs. Finally, Section 2.6 concludes the chapter with open questions and an outlook on spectral graph determination, highlighting areas for further research.

2.2 Preliminaries

The present section provides preliminary material and notation in matrix theory, graph theory, and graph-associated matrices, which serves for the presentation of this chapter.

2.2.1 Matrix Theory Preliminaries

The following standard notation in matrix theory is used in this chapter:

- $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$ matrices with real entries,
- $\mathbb{R}^{n} \triangleq \mathbb{R}^{n\times 1}$ denotes the set of all n-dimensional column vectors with real entries,
- $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ denotes the $n \times n$ identity matrix,
- $\mathbf{0}_{k,m} \in \mathbb{R}^{k\times m}$ denotes the $k \times m$ all-zero matrix,
- $\mathbf{J}_{k,m} \in \mathbb{R}^{k\times m}$ denotes the $k \times m$ all-ones matrix,
- $\mathbf{1}_n \triangleq \mathbf{J}_{n,1} \in \mathbb{R}^{n}$ denotes the n-dimensional column vector of ones.

Throughout this chapter, we deal with real matrices.

The concepts of *Schur complement* and *interlacing of eigenvalues* are useful in papers on spectral graph determination and cospectral graphs, and are also addressed in this chapter.

Definition 2.2.1. Let $\mathbf{M}$ be a block matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \tag{2.2.1}$$

where the block $\mathbf{D}$ is invertible. The *Schur complement of D in M* is

$$\mathbf{M}/\mathbf{D} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}. \tag{2.2.2}$$

Schur proved the following remarkable theorem:

Theorem 2.2.2 (Theorem on the Schur complement [128])**.** *If D is invertible, then*

$$\det \mathbf{M} = \det(\mathbf{M}/\mathbf{D}) \det \mathbf{D}. \tag{2.2.3}$$

Theorem 2.2.3 (Cauchy Interlacing Theorem [116])**.** *Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of a symmetric matrix $\mathbf{M}$ and let $\mu_1 \geq \ldots \geq \mu_m$ be the eigenvalues of a* principal $m \times m$ submatrix of $\mathbf{M}$ *(i.e., a submatrix that is obtained by deleting the same set of rows and columns from M) then $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$.*

Definition 2.2.4 (Completely Positive Matrices)**.** A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called *completely positive* if there exists a matrix $\mathbf{B} \in \mathbb{R}^{n \times m}$ whose all entries are nonnegative such that $\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$.

A completely positive matrix is therefore symmetric and all its entries are nonnegative. The interested reader is referred to the textbook [130] on completely positive matrices, also addressing their connections to graph theory.

Definition 2.2.5 (Positive Semidefinite Matrices)**.** A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called *positive semidefinite* if $\mathbf{A}$ is symmetric, and the inequality $\underline{x}^{\mathrm{T}} \mathbf{A} \, \underline{x} \geq 0$ holds for every column vector $\underline{x} \in \mathbb{R}^n$.

Proposition 2.2.6. *A symmetric matrix is positive semidefinite if and only if either one of the following conditions hold:*

1. *All its eigenvalues are nonnegative (real) numbers.*
2. *There exists a matrix $\mathbf{B} \in \mathbb{R}^{n \times m}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$.*

The next result readily follows.

Corollary 2.2.7. *A completely positive matrix is positive semidefinite.*

Remark 2.2.8. Regarding Corollary 2.2.7, it is natural to ask whether, under certain conditions, a positive semidefinite matrix whose entries are all nonnegative is also completely positive. By [130, Theorem 3.35], this holds for all square matrices of order $n \leq 4$. Moreover, [130, Example 3.45] also presents an explicit example of a matrix of order 5 that is positive semidefinite with all non-negative entries but is not completely positive.

2.2.2 Graph Theory Preliminaries

A graph $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ forms a pair where $\mathsf{V}(\mathsf{G})$ is a set of vertices and $\mathsf{E}(\mathsf{G}) \subseteq \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{G})$ is a set of edges.

In this chapter all the graphs are assumed to be

- *finite* - $\left|\mathsf{V}(\mathsf{G})\right| < \infty$,
- *simple* - G has no parallel edges and no self loops,
- *undirected* - the edges in G are undirected.

We use the following terminology:

- The *degree*, $d(v)$, of a vertex $v \in \mathsf{V}(\mathsf{G})$ is the number of vertices in G that are adjacent to v.
- A *walk* in a graph G is a sequence of vertices in G, where every two consecutive vertices in the sequence are adjacent in G.
- A *path* in a graph is a walk with no repeated vertices.
- A *cycle* C is a closed walk, obtained by adding an edge to a path in G.
- The *length of a path or a cycle* is equal to its number of edges. A *triangle* is a cycle of length 3.
- A *connected graph* is a graph in which every pair of distinct vertices is connected by a path.
- The *distance* between two vertices in a connected graph is the length of a shortest path that connects them.
- The *diameter* of a connected graph is the maximum distance between any two vertices in the graph, and the diameter of a disconnected graph is set to be infinity.
- The *connected component* of a vertex $v \in \mathsf{V}(\mathsf{G})$ is the subgraph whose vertex set $\mathcal{U} \subseteq \mathsf{V}(\mathsf{G})$ consists of all the vertices that are connected to v by any path (including the vertex v itself), and its edge set consists of all the edges in $\mathsf{E}(\mathsf{G})$ whose two endpoints are contained in the vertex set $\mathcal{U}$.
- A *tree* is a connected graph that has no cycles (i.e., it is a connected and *acyclic* graph).
- A *spanning tree* of a connected graph G is a tree with the vertex set $\mathsf{V}(\mathsf{G})$ and some of the edges of G.
- A graph is *regular* if all its vertices have the same degree.

- A *d-regular* graph is a regular graph whose all vertices have degree d.
- A *bipartite graph* is a graph G whose vertex set is a disjoint union of two subsets such that no two vertices in the same subset are adjacent.
- A *complete bipartite graph* is a bipartite graph where every vertex in each of the two partite sets is adjacent to all the vertices in the other partite set.
- A *complete graph* is a graph in which every pair of distinct vertices is connected by an edge. An *empty graph* is a graph with no edges.

Definition 2.2.9 (Complement of a graph)**.** The *complement* of a graph G, denoted by $\overline{\mathsf{G}}$, is a graph whose vertex set is $\mathsf{V}(\mathsf{G})$, and its edge set is the complement set $\overline{\mathsf{E}(\mathsf{G})}$. Every vertex in $\mathsf{V}(\mathsf{G})$ is nonadjacent to itself in G and $\overline{\mathsf{G}}$, so $\{i, j\} \in \mathsf{E}(\overline{\mathsf{G}})$ if and only if $\{i, j\} \notin \mathsf{E}(\mathsf{G})$ with $i \neq j$.

Definition 2.2.10 (Disjoint union of graphs)**.** Let $\mathsf{G}_1, \ldots, \mathsf{G}_k$ be graphs. If the vertex sets in these graphs are not pairwise disjoint, let $\mathsf{G}'_2, \ldots, \mathsf{G}'_k$ be isomorphic copies of $\mathsf{G}_2, \ldots, \mathsf{G}_k$, respectively, such that none of the graphs $\mathsf{G}_1, \mathsf{G}'_2, \ldots \mathsf{G}'_k$ have a vertex in common. The disjoint union of these graphs, denoted by $\mathsf{G} = \mathsf{G}_1 \dot{\cup} \ldots \dot{\cup} \mathsf{G}_k$, is a graph whose vertex and edge sets are equal to the disjoint unions of the vertex and edge sets of $\mathsf{G}_1, \mathsf{G}'_2, \ldots, \mathsf{G}'_k$ (G is defined up to an isomorphism).

Definition 2.2.11. Let $k \in \mathbb{N}$ and let G be a graph. Define $k\mathsf{G} = \mathsf{G} \dot{\cup} \mathsf{G} \dot{\cup} \ldots \dot{\cup} \mathsf{G}$ to be the disjoint union of k copies of G.

Definition 2.2.12 (Join of graphs)**.** Let G and H be two graphs with disjoint vertex sets. The join of G and H is defined to be their disjoint union, together with all the edges that connect the vertices in G with the vertices in H. It is denoted by $\mathsf{G} \vee \mathsf{H}$.

Definition 2.2.13 (Induced subgraphs)**.** Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph, and let $\mathcal{U} \subseteq \mathsf{V}$. The *subgraph of* G *induced by* $\mathcal{U}$ is the graph obtained by the vertices in $\mathcal{U}$ and the edges in G that has both ends on $\mathcal{U}$. We say that H is an *induced subgraph of* G, if it is induced by some $\mathcal{U} \subseteq \mathsf{V}$.

Definition 2.2.14 (Strongly regular graphs)**.** A regular graph G that is neither complete nor empty is called a *strongly regular* graph with parameters (n, d, λ, μ), where λ and μ are nonnegative integers, if the following conditions hold:

(1) G is a d-regular graph on n vertices.

(2) Every two adjacent vertices in G have exactly λ common neighbors.

(3) Every two distinct and nonadjacent vertices in G have exactly μ common neighbors.

The family of strongly regular graphs with these four specified parameters is denoted by $\mathsf{srg}(n, d, \lambda, \mu)$. It is important to note that a family of the form $\mathsf{srg}(n, d, \lambda, \mu)$ may contain multiple nonisomorphic strongly regular graphs. Throughout this work, we refer to a strongly regular graph as $\mathsf{srg}(n, d, \lambda, \mu)$ if it belongs to this family.

Proposition 2.2.15 (Feasible parameter vectors of strongly regular graphs)**.** *The four parameters of a strongly regular graph* $\mathsf{srg}(n, d, \lambda, \mu)$ *satisfy the equality*

$$(n - d - 1)\mu = d(d - \lambda - 1). \tag{2.2.4}$$

Remark 2.2.16. Equality (2.2.4) provides a necessary, but not sufficient, condition for the existence of a strongly regular graph $\mathsf{srg}(n, d, \lambda, \mu)$. For example, as shown in [71], no $(76, 21, 2, 7)$ strongly regular graph exists, even though the condition $(n - d - 1)\mu = 378 = d(d - \lambda - 1)$ is satisfied in this case.

Notation 2.2.17 (Classes of graphs)**.**

- K_n is the complete graph on n vertices.
- P_n is the path graph on n vertices.
- $\mathsf{K}_{\ell,r}$ is the complete bipartite graph whose degrees of partite sets are ℓ and r (with possible equality between ℓ and r).
- S_n is the star graph on n vertices $\mathsf{S}_n = \mathsf{K}_{1,n-1}$.

Definition 2.2.18 (Integer-valued functions of a graph)**.**

- Let $k \in \mathbb{N}$. A *proper k-coloring* of a graph G is a function $c \colon \mathsf{V}(\mathsf{G}) \to \{1, 2, ..., k\}$, where $c(v) \neq c(u)$ for every $\{u, v\} \in \mathsf{E}(\mathsf{G})$. The *chromatic number* of G, denoted by $\chi(\mathsf{G})$, is the smallest k for which there exists a proper k-coloring of G.
- A *clique* in a graph G is a subset of vertices $U \subseteq \mathsf{V}(\mathsf{G})$ where the subgraph induced by U is a complete graph. The *clique number* of G, denoted by $\omega(\mathsf{G})$, is the largest size of a clique in G; i.e., it is the largest order of an induced complete subgraph in G.
- An *independent set* in a graph G is a subset of vertices $U \subseteq \mathsf{V}(\mathsf{G})$, where $\{u, v\} \notin \mathsf{E}(\mathsf{G})$ for every $u, v \in U$. The *independence number* of G, denoted by $\alpha(\mathsf{G})$, is the largest size of an independent set in G.

Definition 2.2.19 (Orthogonal and orthonormal representations of a graph)**.** Let G be a finite, simple, and undirected graph, and let $d \in \mathbb{N}$.

- An *orthogonal representation* of the graph G in the d-dimensional Euclidean space $\mathbb{R}^d$ assigns to each vertex $i \in \mathsf{V}(\mathsf{G})$ a nonzero vector $\mathbf{u}_i \in \mathbb{R}^d$ such that $\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j = 0$ for every $\{i, j\} \notin \mathsf{E}(\mathsf{G})$ with $i \neq j$. In other words, for every two distinct and nonadjacent vertices in the graph, their assigned nonzero vectors should be orthogonal in $\mathbb{R}^d$.

- An *orthonormal representation* of G is additionally represented by unit vectors, i.e., $\|\mathbf{u}_i\| = 1$ for all $i \in \mathsf{V}(\mathsf{G})$.

- In an orthogonal (orthonormal) representation of G, every two nonadjacent vertices in G are mapped (by definition) into orthogonal (orthonormal) vectors, but adjacent vertices may not necessarily be mapped into nonorthogonal vectors. If $\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j \neq 0$ for all $\{i, j\} \in \mathsf{E}(\mathsf{G})$, then such a representation of G is called *faithful*.

Definition 2.2.20 (Lovász ϑ-function [99])**.** Let G be a finite, simple, and undirected graph. Then, the *Lovász ϑ-function of* G is defined as

$$\vartheta(\mathsf{G}) \triangleq \min_{\mathbf{c},\{\mathbf{u}_i\}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{(\mathbf{c}^{\mathrm{T}}\mathbf{u}_i)^2}, \tag{2.2.5}$$

where the minimum on the right-hand side of (2.2.5) is taken over all unit vectors $\mathbf{c}$ and all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G. In (2.2.5), it suffices to consider orthonormal representations in a space of dimension at most $n = |\mathsf{V}(\mathsf{G})|$.

The Lovász ϑ-function of a graph G can be calculated by solving (numerically) a convex optimization problem. Let $\mathbf{A} = (A_{i,j})$ be the $n \times n$ adjacency matrix of G with $n \triangleq |\mathsf{V}(\mathsf{G})|$. The Lovász ϑ-function $\vartheta(\mathsf{G})$ can be expressed as the solution of the following semidefinite programming (SDP) problem:

$$\boxed{\begin{array}{l} \text{maximize} \;\; \mathrm{Tr}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \succeq 0, \\ \mathrm{Tr}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}} \tag{2.2.6}$$

There exist efficient convex optimization algorithms (e.g., interior-point methods) to compute $\vartheta(\mathsf{G})$, for every graph G, with a precision of r decimal digits, and a computational complexity that is polynomial in n and r. The reader is referred to Section 2.5 of [124] for an account of the various interesting properties of the Lovász ϑ-function. Among these properties, the sandwich theorem states that for every graph G, the following inequalities hold:

$$\alpha(\mathsf{G}) \le \vartheta(\mathsf{G}) \le \chi(\overline{\mathsf{G}}), \tag{2.2.7}$$

$$\omega(\mathsf{G}) \le \vartheta(\overline{\mathsf{G}}) \le \chi(\mathsf{G}). \tag{2.2.8}$$

The usefulness of (2.2.7) and (2.2.8) lies in the fact that while the independence, clique, and chromatic numbers of a graph are NP-hard to compute, the Lovász ϑ-function can be efficiently computed as a bound in these inequalities by solving the convex optimization problem in (2.2.6).

2.2.3 Matrices associated with a graph

Four matrices associated with a graph

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph with vertices $\{v_1, ..., v_n\}$. There are several matrices associated with G. In this survey, we consider four of them, all are symmetric matrices in $\mathbb{R}^{n\times n}$: the *adjacency matrix* ($\mathbf{A}$), *Laplacian matrix* (*LM*), *signless Laplacian matrix* ($\mathbf{Q}$), and the *normialized Laplacian matrix* ($\mathcal{L}$).

1. The adjacency matrix of a graph G, denoted by $\mathbf{A} = \mathbf{A}(\mathsf{G})$, has the binary-valued entries

$$(\mathbf{A}(\mathsf{G}))_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in \mathsf{E}(\mathsf{G}), \\ 0 & \text{if } \{v_i, v_j\} \notin \mathsf{E}(\mathsf{G}). \end{cases} \tag{2.2.9}$$

2. The Laplacian matrix of a graph G, denoted by $\mathbf{L} = \mathbf{L}(\mathsf{G})$, is given by

$$\mathbf{L}(\mathsf{G}) = \mathbf{D}(\mathsf{G}) - \mathbf{A}(\mathsf{G}), \tag{2.2.10}$$

   where

$$\mathbf{D}(\mathsf{G}) = \operatorname{diag}\big(d(v_1), d(v_2), \dots, d(v_n)\big) \tag{2.2.11}$$

   is the *diagonal matrix* whose entries in the principal diagonal are the degrees of the n vertices of G.

3. The signless Laplacian martix of a graph G, denoted by $\mathbf{Q} = \mathbf{Q}(\mathsf{G})$, is given by

$$\mathbf{Q}(\mathsf{G}) = \mathbf{D}(\mathsf{G}) + \mathbf{A}(\mathsf{G}). \tag{2.2.12}$$

4. The normalized Laplacian matrix of a graph G, denoted by $\mathcal{L}(\mathsf{G})$, is given by

$$\mathcal{L}(\mathsf{G}) = \mathbf{D}^{-\frac{1}{2}}(\mathsf{G})\ \mathbf{L}(\mathsf{G})\,\mathbf{D}^{-\frac{1}{2}}(\mathsf{G}), \tag{2.2.13}$$

where

$$\mathbf{D}^{-\frac{1}{2}}(\mathsf{G}) = \operatorname{diag}(d^{-\frac{1}{2}}(v_1), d^{-\frac{1}{2}}(v_2), \ldots, d^{-\frac{1}{2}}(v_n)), \tag{2.2.14}$$

with the convention that if $v \in \mathsf{V}(\mathsf{G})$ is an isolated vertex in G (i.e., $d(v) = 0$), then $d^{-\frac{1}{2}}(v) = 0$. The entries of $\mathcal{L} = (\mathcal{L}_{i,j})$ are given by

$$\mathcal{L}_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ and } d(v_i) \neq 0, \\ -\dfrac{1}{\sqrt{d(v_i)\,d(v_j)}}, & \text{if } i \neq j \text{ and } \{v_i, v_j\} \in \mathsf{E}(\mathsf{G}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.15}$$

In the continuation of this section, we also occasionally refer to two other matrices that are associated with undirected graphs.

Definition 2.2.21. Let G be a graph with n vertices and m edges. The *incidence matrix* of G, denoted by $\mathbf{B} = \mathbf{B}(\mathsf{G})$ is an $n \times m$ matrix with binary entries, defined as follows:

$$B_{i,j} = \begin{cases} 1 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is incident to edge } e_j \in \mathsf{E}(\mathsf{G}), \\ 0 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is not incident to edge } e_j \in \mathsf{E}(\mathsf{G}). \end{cases} \tag{2.2.16}$$

For an undirected graph, each edge e_j connects two vertices v_i and v_k, and the corresponding column in $\mathbf{B}$ has exactly two 1's, one for each vertex.

Definition 2.2.22. Let G be a graph with n vertices and m edges. An *oriented incidence matrix* of G, denoted by $\mathbf{N} = \mathbf{N}(\mathsf{G})$ is an $n \times m$ matrix with ternary entries from $\{-1, 0, 1\}$, defined as follows. One first selects an arbitrary orientation to each edge in G, and then define

$$N_{i,j} = \begin{cases} -1 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is the tail (starting vertex) of edge } e_j \in \mathsf{E}(\mathsf{G}), \\ +1 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is the head (ending vertex) of edge } e_j \in \mathsf{E}(\mathsf{G}), \\ 0 & \text{if vertex } v_i \in \mathsf{V}(\mathsf{G}) \text{ is not incident to edge } e_j \in \mathsf{E}(\mathsf{G}). \end{cases} \tag{2.2.17}$$

Consequently, each column of $\mathbf{N}$ contains exactly one entry equal to 1 and one entry equal to -1, representing the head and tail of the corresponding oriented edge in the graph, respectively, with all other entries in the column being zeros.

For $X \in \{A, L, Q, \mathcal{L}\}$, the *X-spectrum* of a graph G, $\sigma_X(G)$, is the multiset of the eigenvalues of $X(G)$. We denote the elements of the multiset of eigenvalues of $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$, respectively, by

$$\lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}), \tag{2.2.18}$$

$$\mu_1(\mathsf{G}) \leq \mu_2(\mathsf{G}) \leq \ldots \leq \mu_n(\mathsf{G}), \tag{2.2.19}$$

$$\nu_1(\mathsf{G}) \geq \nu_2(\mathsf{G}) \geq \ldots \geq \nu_n(\mathsf{G}), \tag{2.2.20}$$

$$\delta_1(\mathsf{G}) \leq \delta_2(\mathsf{G}) \leq \ldots \leq \delta_n(\mathsf{G}). \tag{2.2.21}$$

Example 2.2.23. Consider the complete bipartite graph $\mathsf{G} = \mathsf{K}_{2,3}$ with the adjacency matrix

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{0}_{2,2} & \mathbf{J}_{2,3} \\ \mathbf{J}_{3,2} & \mathbf{0}_{3,3} \end{pmatrix}.$$

The spectra of G can be verified to be given as follows:

1. The $\mathbf{A}$-spectrum of G is

$$\sigma_{\mathbf{A}}(\mathsf{G}) = \left\{-\sqrt{6}, [0]^3,\ \sqrt{6}\right\}, \tag{2.2.22}$$

   with the notation that $[\lambda]^m$ means that λ is an eigenvalue with multiplicity m.

2. The $\mathbf{L}$-spectrum of G is

$$\sigma_{\mathbf{L}}(\mathsf{G}) = \left\{0, [2]^2, 3, 5\right\}. \tag{2.2.23}$$

3. The $\mathbf{Q}$-spectrum of G is

$$\sigma_{\mathbf{Q}}(\mathsf{G}) = \left\{0, [2]^2, 3, 5\right\}. \tag{2.2.24}$$

4. The $\mathcal{L}$-spectrum of G is

$$\sigma_{\mathcal{L}}(\mathsf{G}) = \left\{0, [1]^3, 2\right\}. \tag{2.2.25}$$

Remark 2.2.24. If H is an induced subgraph of a graph G, then $\mathbf{A}(\mathsf{H})$ is a principal submatrix of $A(\mathsf{G})$. However, since the degrees of the remaining vertices are affected by the removal of vertices when forming the induced subgraph H from the graph G, this property does not hold for the other three associated matrices discussed in this chapter (namely, the Laplacian, signless Laplacian, and normalized Laplacian matrices).

Definition 2.2.25. Let G be a graph, and let $\overline{\mathsf{G}}$ be the complement graph of G. Define the following matrices:

1. $\overline{\mathbf{A}}(\mathsf{G}) = \mathbf{A}(\overline{\mathsf{G}})$.

2. $\overline{\mathbf{L}}(\mathsf{G}) = \mathbf{L}(\overline{\mathsf{G}})$.

3. $\overline{\mathbf{Q}}(\mathsf{G}) = \mathbf{Q}(\overline{\mathsf{G}})$.

4. $\overline{\mathcal{L}}(\mathsf{G}) = \mathcal{L}(\overline{\mathsf{G}})$.

Definition 2.2.26. Let $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$. The $\mathcal{X}$-spectrum of a graph G is a list with $\sigma_X(\mathsf{G})$ for every $X \in \mathcal{X}$.

Observe that if $\mathcal{X} = \{X\}$ is a singleton, then the $\mathcal{X}$ spectrum is equal to the X-spectrum.

We now describe some important applications of the four matrices.

Properties of the adjacency matrix

Theorem 2.2.27 (Number of walks of a given length between two fixed vertices)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph with a vertex set $\mathsf{V} = \mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$, and let $\mathbf{A} = \mathbf{A}(\mathsf{G})$ be the adjacency matrix of G. Then, the number of walks of length ℓ, with the fixed endpoints v_i and v_j, is equal to $(\mathbf{A}^\ell)_{i,j}$.*

Corollary 2.2.28 (Number of closed walks of a given length)**.** *Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a simple undirected graph on n vertices with an adjacency matrix $\mathbf{A} = \mathbf{A}(\mathsf{G})$, and let its spectrum (with respect to $\mathbf{A}$) be given by $\{\lambda_j\}_{j=1}^n$. Then, for all $\ell \in \mathbb{N}$, the number of closed walks of length ℓ in G is equal to $\sum_{j=1}^n \lambda_j^\ell$.*

Corollary 2.2.29 (Number of edges and triangles in a graph)**.** *Let G be a simple undirected graph with $n = |\mathsf{V}(\mathsf{G})|$ vertices, $e = |\mathsf{E}(\mathsf{G})|$ edges, and t triangles. Let $\mathbf{A} = \mathbf{A}(\mathsf{G})$ be the adjacency matrix of G, and let $\{\lambda_j\}_{j=1}^n$ be its adjacency spectrum. Then,*

$$\sum_{j=1}^{n} \lambda_j = \mathrm{tr}(\mathbf{A}) = 0, \tag{2.2.26}$$

$$\sum_{j=1}^{n} \lambda_j^2 = \mathrm{tr}(\mathbf{A}^2) = 2e, \tag{2.2.27}$$

$$\sum_{j=1}^{n} \lambda_j^3 = \mathrm{tr}(\mathbf{A}^3) = 6t. \tag{2.2.28}$$

For a d-regular graph, the largest eigenvalue of its adjacency matrix is equal to d. Consequently, by Eq. (2.2.27), for d-regular graphs, $\sum_j \lambda_j^2 = 2e = nd = n\lambda_1$. Interestingly, this turns to be a necessary and sufficient condition for the regularity of a graph, which means that the adjacency spectrum enables to identify whether a graph is regular.

Theorem 2.2.30. *[49, Corollary 3.2.2] A graph* G *on n vertices is regular if and only if*

$$\sum_{i=1}^{n} \lambda_i^2 = n\lambda_1, \tag{2.2.29}$$

where λ_1 *is the largest eigenvalue of the adjacency matrix of* G.

Theorem 2.2.31 (The eigenvalues of strongly regular graphs)**.** *The following spectral properties are satisfied by the family of strongly regular graphs:*

(1) A strongly regular graph has at most three distinct eigenvalues.

(2) Let G *be a connected strongly regular graph, and let its parameters be* $\mathsf{srg}(n, d, \lambda, \mu)$*. Then, the largest eigenvalue of its adjacency matrix is* $\lambda_1(\mathsf{G}) = d$ *with multiplicity 1, and the other two distinct eigenvalues of its adjacency matrix are given by*

$$p_{1,2} = \tfrac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(d - \mu)}\right), \tag{2.2.30}$$

with the respective multiplicities

$$m_{1,2} = \tfrac{1}{2}\left(n - 1 \mp \frac{2d + (n-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(d - \mu)}}\right). \tag{2.2.31}$$

(3) A connected regular graph with exactly three distinct eigenvalues is strongly regular.

(4) Strongly regular graphs for which $2d + (n-1)(\lambda - \mu) \neq 0$ *have integral eigenvalues and the multiplicities of* $p_{1,2}$ *are distinct.*

(5) A connected regular graph is strongly regular if and only if it has three distinct eigenvalues, where the largest eigenvalue is of multiplicity 1.

(6) A disconnected strongly regular graph is a disjoint union of m identical complete graphs K_r*, where* $m \geq 2$ *and* $r \in \mathbb{N}$*. It belongs to the family* $\mathsf{srg}(mr, r-1, r-2, 0)$*, and its adjacency spectrum is* $\{(r-1)^{[m]}, (-1)^{[m(r-1)]}\}$*, where superscripts indicate the multiplicities of the eigenvalues, thus having two distinct eigenvalues.*

Since the Laplacian spectrum, signless Laplacian spectrum, and normalized Laplacian spectrum of a regular graph are all given by affine transformations of its adjacency spectrum, we can also derive conditions for the strong regularity of a graph from the spectra of each of those three matrices.

The following result follows readily from Theorem 2.2.31.

Corollary 2.2.32. *Strongly regular graphs with identical parameters* (n, d, λ, μ) *are cospectral.*

Remark 2.2.33. Strongly regular graphs having identical parameters (n, d, λ, μ) are cospectral but may not be isomorphic. For instance, Chang graphs form a set of three nonisomorphic strongly regular graphs with identical parameters $\mathsf{srg}(28, 12, 6, 4)$ [22, Section 10.11]. Consequently, the three Chang graphs are strongly regular NICS graphs.

An important class of strongly regular graphs, for which $2d + (n-1)(\lambda - \mu) = 0$, is given by the family of conference graphs.

Definition 2.2.34 (Conference graphs)**.** A conference graph on n vertices is a strongly regular graph with the parameters $\mathsf{srg}(n, \frac{1}{2}(n-1), \frac{1}{4}(n-5), \frac{1}{4}(n-1))$, where n must satisfy $n = 4k + 1$ with $k \in \mathbb{N}$.

If G is a conference graph on n vertices, then so is its complement $\overline{\mathsf{G}}$; it is, however, not necessarily self-complementary. By Theorem 2.2.31, the distinct eigenvalues of the adjacency matrix of G are given by $\frac{1}{2}(n-1)$, $\frac{1}{2}(\sqrt{n}-1)$, and $-\frac{1}{2}(\sqrt{n}+1)$ with multiplicities $1, \frac{1}{2}(n-1)$, and $\frac{1}{2}(n-1)$, respectively. In contrast to Item 4 of Theorem 2.2.31, the eigenvalues $\pm\frac{1}{2}(\sqrt{n}+1)$ are not necessarily integers. For instance, the cycle graph C_5, which is a conference graph, has an adjacency spectrum $\{2, [\frac{1}{2}(\sqrt{5}-1)]^{(2)}, [-\frac{1}{2}(\sqrt{5}+1)]^{(2)}\}$. Thus, apart from the largest eigenvalue, the other eigenvalues are irrational numbers.

Properties of the Laplacian matrix

Theorem 2.2.35. *Let* G *be a finite, simple, and undirected graph, and let* $\mathbf{L}$ *be the Laplacian matrix of* G*. Then,*

1. *The Laplacian matrix* $\mathbf{L} = \mathbf{N}\mathbf{N}^{\mathrm{T}}$ *is positive semidefinite, where* $\mathbf{N}$ *is an oriented incidence matrix of* G *(see Definition 2.2.22 and [49, p. 185]).*

2. *The smallest eigenvalue of* $\mathbf{L}$ *is zero, with a multiplicity equal to the number of components in* G *(see [49, Theorem 7.1.2]).*

3. *The size of the graph,* $\big|\mathsf{E}(\mathsf{G})\big|$*, equals one-half of the sum of the eigenvalues of* $\mathbf{L}$*, counted with multiplicities (see [49, Eq. (7.4)]).*

The following celebrated theorem provides an operational meaning of the **L**-spectrum of graphs in counting their number of spanning subgraphs.

Theorem 2.2.36 (Kirchhoff's Matrix-Tree Theorem [84])**.** *The number of spanning trees in a connected and simple graph* G *on n vertices is determined by the* $n-1$ *nonzero eigenvalues of the Laplacian matrix, and it is equal to* $\frac{1}{n}\prod_{\ell=2}^{n}\mu_\ell(\mathsf{G})$.

Corollary 2.2.37 (Cayley's Formula [36])**.** *The number of spanning trees of* K_n *is* n^{n-2}.

Proof. The **L**-spectrum of K_n is given by $\{0, [n]^{n-1}\}$, and the result readily follows from Theorem 2.2.36. □

Properties of the signless Laplacian matrix

Theorem 2.2.38. *Let* G *be a finite, simple, and undirected graph, and let* $\mathbf{Q}$ *be the signless Laplacian matrix of* G*. Then,*

1. *The matrix* $\mathbf{Q}$ *is positive semidefinite. Moreover, it is a completely positive matrix, expressed as* $\mathbf{Q} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$*, where* $\mathbf{B}$ *is the incidence matrix of* G *(see Definition 2.2.21 and [49, Section 2.4]).*

2. *If* G *is a connected graph, then it is bipartite if and only if the least eigenvalue of* $\mathbf{Q}$ *is equal to zero. In this case,* 0 *is a simple* $\mathbf{Q}$*-eigenvalue (see [49, Theorem 7.8.1]).*

3. *The multiplicity of 0 as an eigenvalue of* $\mathbf{Q}$ *is equal to the number of bipartite components in* G *(see [49, Corollary 7.8.2]).*

4. *The size of the graph* $\left|E(\mathsf{G})\right|$ *is equal to one-half the sum of the eigenvalues of* $\mathbf{Q}$*, counted with multiplicities (see [49, Corollary 7.8.9]).*

The interested reader is referred to [111] for bounds on the **Q**-spread (i.e., the difference between the largest and smallest eigenvalues of the signless Laplacian matrix), expressed as a function of the number of vertices in the graph. In regard to Item 2 of Theorem 2.2.38, the interested reader is referred to [35] for a lower bound on the least eigenvalue of signless Laplacian matrix for connected non-bipartite graphs, and to [37] for a lower bound on the least eigenvalue of signless Laplacian matrix for a general simple graph with a fixed number of vertices and edges.

Properties of the normalized Laplacian matrix

The normalized Laplacian matrix of a graph, defined in (2.2.13), exhibits several interesting spectral properties, which are introduced below.

Theorem 2.2.39. *[48, 49] Let* G *be a finite, simple, and undirected graph, and let* $\mathcal{L}$ *be the normalized Laplacian matrix of* G*. Then,*

1. *The eigenvalues of* $\mathcal{L}$ *lie in the interval* $[0, 2]$ *(see [49, Section 7.7]).*
2. *The number of components in* G *is equal to the multiplicity of 0 as an eigenvalue of* $\mathcal{L}$ *(see [49, Theorem 7.7.3]).*
3. *The largest eigenvalue of* $\mathcal{L}$ *is equal to 2 if and only if the graph has a bipartite component (see [49, Theorem 7.7.2(v)]). Furthermore, the number of the bipartite components of* G *is equal to the multiplicity of 2 as an eigenvalue of* $\mathcal{L}$*.*
4. *The sum of its eigenvalues (including multiplicities) is less than or equal to the graph order* (n)*, with equality if and only if the graph has no isolated vertices (see [49, Theorem 7.7.2(i)]).*

More on the spectral properties of the four associated matrices

The following theorem considers equivalent spectral properties of bipartite graphs.

Theorem 2.2.40. *Let* G *be a graph. The following are equivalent:*

1. G *is a bipartite graph.*
2. G *does not have cycles of odd length.*
3. *The* $\mathbf{A}$*-spectrum of* G *is symmetric around zero, and for every eigenvalue* λ *of* $\mathbf{A}(G)$*, the eigenvalue* $-\lambda$ *is of the same multiplicity [49, Theorem 3.2.3].*
4. *The* $\mathbf{L}$*-spectrum and* $\mathbf{Q}$*-spectrum are identical (see [49, Proposition 7.8.4]).*
5. *The* $\mathcal{L}$*-spectrum has the same multiplicity of* 0*'s and* 2*'s as eigenvalues (see [49, Corollary 7.7.4]).*

Remark 2.2.41. Item 3 of Theorem 2.2.40 can be strengthened if G is a connected graph. In that case, G is bipartite if and only if $\lambda_1 = -\lambda_n$ (see [49, Theorem 3.2.4]).

Table 2.1, borrowed from [26], lists properties of a graph that can or cannot be determined by the X-spectrum for $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$. From the $\mathbf{A}$-spectrum of a graph G, one can determine the number of edges and the number of triangles in G (by Eqs. (2.2.27) and (2.2.28), respectively), and whether the graph is bipartite or not (by Item 3 of Theorem 2.2.40). However, the $\mathbf{A}$ spectrum does not indicate the number of components (see Example 2.3.5). From the $\mathbf{L}$-spectrum of a graph G, one can determine the number of edges (by Item 3 of Theorem 2.2.35), the number of spanning

Matrix	# edges	bipartite	# components	# bipartite components	# of closed walks
A	Yes	Yes	No	No	Yes
L	Yes	No	Yes	No	No
Q	Yes	No	No	Yes	No
$\mathcal{L}$	No	Yes	Yes	Yes	No

Table 2.1: Some properties of a finite, simple, and undirected graph that one can or cannot determine by the X-spectrum for $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$

trees (by Theorem 2.2.36), the number of components of G (by Item 2 of Theorem 2.2.35), but not the number of its triangles, and whether the graph G is bipartite. From the **Q**-spectrum, one can determine whether the graph is bipartite, the number of bipartite components, and the number of edges (respectively, by Items 3 and 4 of Theorem 2.2.38), but not the number of components of the graph, and whether the graph is bipartite (see Remark 2.2.42). From the $\mathcal{L}$-spectrum, one can determine the number of components and the number of bipartite components in G (by Theorem 2.2.39), and whether the graph is bipartite (by Items 1 and 5 of Theorem 2.2.40). The number of closed walks in G is determined by the **A**-spectrum (by Corollary 2.2.28), but not by the spectra with respect to the other three matrices.

Remark 2.2.42. By Item 2 of Theorem 2.2.38, a connected graph is bipartite if and only if the least eigenvalue of its signless Laplacian matrix is equal to zero. If the graph is disconnected and it has a bipartite component and a non-bipartite component, then the least eigenvalue of its signless Laplacian matrix is equal to zero, although the graph is not bipartite. According to Table 2.1, the **Q**-spectrum alone does not determine whether a graph is bipartite. This is due to the fact that the **Q**-spectrum does not provide information about the number of components in the graph or whether the graph is connected. It is worth noting that while neither the **L**-spectrum nor the **Q**-spectrum independently determines whether a graph is bipartite, the combination of these spectra does. Specifically, by Item 4 of Theorem 2.2.40, the combined knowledge of both spectra enables to establish this property.

2.3 Graphs determined by their spectra

The spectral determination of graphs has long been a central topic in spectral graph theory. A major open question in this area is: "Which graphs are determined by their spectrum (DS)?" This section begins our survey of both classical and recent results on spectral graph determination. We explore the spectral characterization of various graph classes, methods for constructing or distinguishing cospectral nonisomorphic graphs, and conditions under which a graph's spectrum uniquely determines its structure. Additionally, we present newly obtained proofs of existing results, offering further insights into this field.

Definition 2.3.1. Let G, H be two graphs. A mapping $\phi\colon \mathsf{V}(\mathsf{G}) \to \mathsf{V}(\mathsf{H})$ is a *graph isomorphism* if

$$\{u, v\} \in \mathsf{E}(\mathsf{G}) \iff \{\phi(u), \phi(v)\} \in \mathsf{E}(\mathsf{H}). \tag{2.3.1}$$

If there is an isomorphism between G and H, we say that these graphs are *isomorphic*.

Definition 2.3.2. A *permutation matrix* is a $\{0, 1\}$–matrix in which each row and each column contains exactly one entry equal to 1.

Remark 2.3.3. In terms of the adjacency matrix of a graph, G and H are cospectral graphs if $\mathbf{A}(\mathsf{G})$ and $\mathbf{A}(\mathsf{H})$ are similar matrices, and G and H are isomorphic if the similarity of their adjacency matrices is through a permutation matrix $\mathbf{P}$, i.e.

$$A(\mathsf{G}) = \mathbf{P}\ \mathbf{A}(\mathsf{H})\,\mathbf{P}^{-1}. \tag{2.3.2}$$

2.3.1 Graphs determined by their adjacency spectrum (DS graphs)

Theorem 2.3.4. *[50] All of the graphs with less than five vertices are DS.*

Example 2.3.5. The star graph S_5 and a graph formed by the disjoint union of a length-4 cycle and an isolated vertex, $\mathsf{C}_4 \dot\cup \mathsf{K}_1$, have the same $\mathbf{A}$-spectrum $\{-2, [0]^3, 2\}$. They are, however, not isomorphic since S_5 is connected and $\mathsf{C}_4 \dot\cup \mathsf{K}_1$ is disconnected (see Figure 2.1). It can be verified

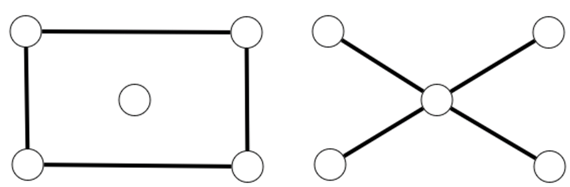


Figure 2.1: The graphs $\mathsf{S}_4 = \mathsf{K}_{1,4}$ and $\mathsf{C}_4 \dot\cup \mathsf{K}_1$ (i.e., a union of a 4-length cycle and an isolated vertex) are cospectral and nonisomorphic graphs ($\mathbf{A}$-NICS graphs) on five vertices. These two graphs therefore cannot be determined by their adjacency matrix.

computationally that all the connected nonisomorphic graphs on five vertices can be distinguished by their $\mathbf{A}$-spectrum (see [49, Appendix A1]).

Theorem 2.3.6. *[50] All the regular graphs with less than ten vertices are DS (and, as will be clarified later, also $\mathcal{X}$-DS for every $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$).*

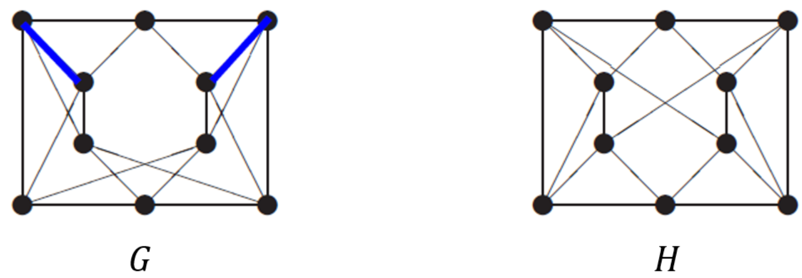


Figure 2.2: $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS regular graphs with 10 vertices. These cospectral graphs are nonisomorphic because each of the two blue edges in G belongs to three triangles, whereas no such an edge exists in H.

Example 2.3.7. [50] The following two regular graphs in Figure 2.2 are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS. The regular graphs G and H in Figure 2.2 can be verified to be cospectral with the common characteristic polynomial

$$P(x) = x^{10} - 20x^8 - 16x^7 + 110x^6 + 136x^5 - 180x^4 - 320x^3 + 9x^2 + 200x + 80.$$

These graphs are also nonisomorphic because each of the two blue edges in G belongs to three triangles, whereas no such an edge exists in H. Furthermore, it is shown in Example 4.18 of [124] that each pair of the regular NICS graphs on 10 vertices, denoted by $\{\mathsf{G}, \mathsf{H}\}$ and $\{\overline{\mathsf{G}}, \overline{\mathsf{H}}\}$, exhibits distinct values of the Lovász ϑ-functions, whereas the graphs G, $\overline{\mathsf{G}}$, H, and $\overline{\mathsf{H}}$ share identical independence numbers (3), clique numbers (3), and chromatic numbers (4). Furthermore, based on these two pairs of graphs, it is constructively shown in Theorem 4.19 of [124] that for every even integer $n \geq 14$, there exist connected, irregular, cospectral, and nonisomorphic graphs on n vertices, being jointly cospectral with respect to their adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices, while also sharing identical independence, clique, and chromatic numbers, but being distinguished by their Lovász ϑ-functions.

Remark 2.3.8. In continuation to Example 2.3.7, it is worth noting that closed-form expressions for the Lovász ϑ-functions of regular graphs, which are edge-transitive or strongly regular, were derived in [99, Theorem 9] and [125, Proposition 1], respectively. In particular, it follows from [125, Proposition 1] that strongly regular graphs with identical four parameters (n, d, λ, μ) are cospectral and they have identical Lovász ϑ-numbers, although they need not be necessarily isomorphic. For such an explicit counterexample, the reader is referred to [125, Remark 3].

We next introduce friendship graphs to address their possible determination by their spectra with respect to several associated matrices.

Definition 2.3.9. Let $p \in \mathbb{N}$. *The friendship graph* F_p, also known as the *windmill graph*, is a graph with $2p+1$ vertices, consisting of a single vertex (the central vertex) that is adjacent to all the other $2p$ vertices. Furthermore, every pair of these $2p$ vertices shares exactly one common neighbor, namely the central vertex (see Figure 2.3). This graph has $3p$ edges and p triangles.

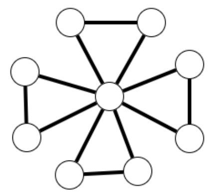


Figure 2.3: The friendship (windmill) graph F_4 has 9 vertices, 12 edges, and 4 triangles.

The term friendship graph in Definition 2.3.9 originates from the *Friendship Theorem* [59]. This theorem states that if G is a finite graph where any two vertices share exactly one common neighbor, then there exists a vertex that is adjacent to all other vertices. In this context, the adjacency of vertices in the graph can be interpreted socially as a representation of friendship between the individuals represented by the vertices (assuming friendship is a mutual relationship). For a nice exposition of the proof of the Friendship Theorem, the interested reader is referred to Chapter 44 of [6].

Theorem 2.3.10. *The following graphs are DS:*

1. *All graphs with less than five vertices, and also all regular graphs with less than 10 vertices [50] (recall Theorems 2.3.4 and 2.3.6).*
2. *The graphs* K_n, C_n, P_n, $\mathsf{K}_{m,m}$ *and* $\overline{\mathsf{K}_n}$ *[50].*
3. *The complement of the path graph* $\overline{\mathsf{P}_n}$ *[57].*
4. *The disjoint union of k path graphs with no isolated vertices, the disjoint union of k complete graphs with no isolated vertices, and the disjoint union of k cycles (i.e., every 2-regular graph) [50].*
5. *The complement graph of a DS regular graph [49].*
6. *Every* $(n-3)$*-regular graph on n vertices [49].*
7. *The friendship graph* F_p *for* $p \neq 16$ *[40].*

8. *Sandglass graphs, which are obtained by appending a triangle to each of the pendant (i.e., degree-1) vertices of a path [100].*

9. *If* H *is a subgraph of a graph* G*, and* $\mathsf{G} \setminus \mathsf{H}$ *denotes the graph obtained from* G *by deleting the edges of* H*, then also the following graphs are DS [32]:*

   - $\mathsf{K}_n \setminus (\ell\, \mathsf{K}_2)$ *and* $\mathsf{K}_n \setminus \mathsf{K}_m$*, where* $m \le n - 2$*,*
   - $\mathsf{K}_n \setminus \mathsf{K}_{\ell,m}$*,*
   - $\mathsf{K}_n \setminus \mathsf{H}$*, where* H *has at most four edges.*

2.3.2 Graphs determined by their spectra with respect to various matrices (X-DS graphs)

In this section, we consider graphs that are determined by the spectra of various associated matrices beyond the adjacency matrix spectrum.

Definition 2.3.11. Let G, H be two graphs and let $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$.

1. G and H are said to be *$\mathcal{X}$-cospectral* if they have the same $\mathcal{X}$-spectrum, i.e. $\sigma_X(\mathsf{G}) = \sigma_X(\mathsf{H})$.

2. Nonisomorphic graphs G and H that are $\mathcal{X}$-cospectral are said to be *$\mathcal{X}$-NICS*, where *NICS* is an abbreviation of *non-isomorphic and cospectral*.

3. A graph G is said to be *determined by its $\mathcal{X}$-spectrum ($\mathcal{X}$-DS)* if every graph that is $\mathcal{X}$-cospectral to G is also isomorphic to G.

Notation 2.3.12. For a singleton $\mathcal{X} = \{X\}$, we abbreviate $\{X\}$-cospectral, $\{X\}$-DS and $\{X\}$-NICS by X-cospectral, X-DS and X-NICS, respectively. For the adjacency matrix, we will abbreviate $\mathbf{A}$-DS by DS.

Remark 2.3.13. Let $\mathcal{X} \subseteq \mathcal{Y} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$. The following holds by definition:

- If two graph G, H are $\mathcal{Y}$-cospectral, then they are $\mathcal{X}$-cospectral.
- If a graph G is $\mathcal{X}$-DS, then it is $\mathcal{Y}$-DS.

Definition 2.3.14. Let G be a graph. The *generalized spectrum* of G is the $\{\mathbf{A}, \overline{\mathbf{A}}\}$-spectrum of G.

The following result on the cospectrality of regular graphs can be readily verified.

Proposition 2.3.15. *Let* G *and* H *be regular graphs that are* $\mathcal{X}$*-cospectral for* some $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$. *Then,* G *and* H *are* $\mathcal{Y}$*-cospectral for* every $\mathcal{Y} \subseteq \{\mathbf{A}, \overline{\mathbf{A}}, \mathbf{L}, \overline{\mathbf{L}}, \mathbf{Q}, \overline{\mathbf{Q}}, \mathcal{L}, \overline{\mathcal{L}}\}$. *In particular, the cospectrality of regular graphs (and their complements) stays unaffected by the chosen matrix among* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$.

Definition 2.3.16. A graph G is said to be *determined by its generalized spectrum (DGS)* if it is uniquely determined by its generalized spectrum. In other words, a graph G is DGS if and only if every graph H with the same $\{\mathbf{A}, \overline{\mathbf{A}}\}$-spectrum as G is necessarily isomorphic to G.

If a graph is not DS, it may still be DGS, as additional spectral information is available. Conversely, every DS graph is also DGS. For further insights into DGS graphs, including various characterizations, conjectures, and studies, we refer the reader to [142, 143, 146].

The continuation of this section characterizes graphs that are X-DS, where $X \in \{\mathbf{L}, \mathbf{Q}, \mathcal{L}\}$, with pointers to various studies. We first consider regular DS graphs.

Theorem 2.3.17. *[50, Proposition 3] For regular graphs, the properties of being DS,* $\mathbf{L}$*-DS, and* $\mathbf{Q}$*-DS are equivalent.*

Remark 2.3.18. To avoid any potential confusion, it is important to emphasize that in statements such as Theorem 2.3.17, the only available information is the spectrum of the graph. There is no indication or prior knowledge that the spectrum corresponds to a regular graph. In such cases, the regularity of the graph is not part of the revealed information and, therefore, cannot be used to determine the graph. This recurring approach — stating that G is stated to be a graph satisfying certain properties (e.g., regularity, strong regularity, etc.) and then examining whether the graph can be determined from its spectrum — appears throughout this chapter. It should be understood that the only available information is the spectrum of the graph, and no additional properties of the graph beyond its spectrum are disclosed.

Remark 2.3.19. The crux of the proof of Theorem 2.3.17 is that there are no two NICS graphs, with respect to either $\mathbf{A}$, $\mathbf{L}$, or $\mathbf{Q}$, where one graph is regular and the other is irregular (see [50, Proposition 2.2]). This, however, does not extend to NICS graphs with respect to the normalized Laplacian matrix $\mathcal{L}$, and regular DS graphs are not necessarily $\mathcal{L}$-DS. For instance, the cycle C_4 and the bipartite complete graph $\mathsf{K}_{1,3}$ (i.e., S_3) share the same $\mathcal{L}$-spectrum, which is given by $\{0, 1^2, 2\}$, but these graphs are nonisomorphic (as C_4 is regular, in contrast to $\mathsf{K}_{1,3}$). It therefore follows that the 2-regular graph C_4 is *not* $\mathcal{L}$-DS, although it is DS (see Item 2 of Theorem 2.3.10). More generally, it is conjectured in [27] that C_n is $\mathcal{L}$-DS if and only if $n > 4$ and $4 \nmid n$.

Theorem 2.3.20. *The following graphs are* $\mathbf{L}$*-DS:*

1. $\mathsf{P}_n, \mathsf{C}_n, \mathsf{K}_n, \mathsf{K}_{m,m}$ *and their complements [50].*

2. *The disjoint union of k paths,* $\mathsf{P}_{n_1} \dot{\cup} \mathsf{P}_{n_2} \dot{\cup} \ldots \dot{\cup} \mathsf{P}_{n_k}$ *each having at least one edge [50].*

3. *The complete bipartite graph* $\mathsf{K}_{m,n}$ *with* $m, n \geq 2$ *and* $\frac{5}{3}n < m$ *[17].*

4. *The star graphs* S_n *with* $n \neq 3$ *[97, 112].*

5. *Trees with a single vertex having a degree greater than 2 (referred to as starlike trees) [97, 112].*

6. *The friendship graph* F_p *[97].*

7. *The path-friendship graphs, where a friendship graph and a starlike tree are joined by merging their vertices of degree greater than 2 [110].*

8. *The wheel graph* $\mathsf{W}_{n+1} \triangleq \mathsf{K}_1 \vee \mathsf{C}_n$ *for* $n \neq 7$ *(otherwise, if* $n = 7$*, then it is not* **L**-*DS) [150].*

9. *The join of a clique and an independent set on n vertices,* $\mathsf{K}_{n-m} \vee \overline{\mathsf{K}_m}$*, where* $m \in [n-1]$ *[53].*

10. *Sandglass graphs (see also Item 8 in Theorem 2.3.10) [100].*

11. *The join graph* $\mathsf{G} \vee \mathsf{K}_m$*, for every* $m \in \mathbb{N}$*, where* G *is a disconnected graph [152].*

12. *The join graph* $\mathsf{G} \vee \mathsf{K}_m$*, for every* $m \in \mathbb{N}$*, where* G *is an* **L**-*DS connected graph on n vertices and m edges with* $m \leq 2n - 6$*,* $\overline{\mathsf{G}}$ *is a connected graph, and either one of the following conditions holds [152]:*
    - $\mathsf{G} \vee \mathsf{K}_1$ *is* **L**-*DS;*
    - *the maximum degree of* G *is smaller than* $\frac{1}{2}(n-2)$.

13. *Specifically, the join graph* $\mathsf{G} \vee \mathsf{K}_m$*, for every* $m \in \mathbb{N}$*, where* G *is an* **L**-*DS tree on* $n \geq 5$ *vertices (since, the equality* $m = n - 1$ *holds for a tree on n vertices and m edges) [152].*

Remark 2.3.21. In general, a disjoint union of complete graphs is not determined by its Laplacian spectrum.

Theorem 2.3.22. *The following graphs are* **Q**-*DS:*

1. *The disjoint union of k paths,* $\mathsf{P}_{n_1} \dot{\cup} \mathsf{P}_{n_2} \dot{\cup} \ldots \dot{\cup} \mathsf{P}_{n_k}$ *each having at least one edge [50].*

2. *The star graphs* S_n *with* $n \geq 3$ *[31, 113].*

3. *Trees with a single vertex having a degree greater than 2 [31, 113].*

4. *The friendship graph* F_k *[144].*

5. *The lollipop graphs, where a lollipop graph, denoted by* $\mathrm{H}_{n,p}$ *where* $n, p \in \mathbb{N}$ *and* $p < n$, *is obtained by appending a cycle* C_p *to a pendant vertex of a path* P_{n-p} *[72, 151].*

6. $\mathsf{G} \vee \mathsf{K}_1$ *where* G *is a either a* 1*-regular graph, an* $(n-2)$*-regular graph of order* n *or a* 2*-regular graph with at least* 11 *vertices [30].*

7. *If* $n \geq 21$ *and* $0 \leq q \leq n-1$, *then* $\mathsf{K}_1 \vee (\mathsf{P}_q \,\dot{\cup}\, (n-q-1)\,\mathsf{K}_1)$ *[149].*

8. *If* $n \geq 21$ *and* $3 \leq q \leq n-1$, *then* $\mathsf{K}_1 \vee (\mathsf{C}_q \,\dot{\cup}\, (n-q-1)\,\mathsf{K}_1)$ *is* **Q***-DS if and only if* $q \neq 3$ *[149].*

9. *The join of a clique and an independent set on n vertices,* $\mathsf{K}_{n-m} \vee \overline{\mathsf{K}_m}$, *where* $m \in [n-1]$ *and* $m \neq 3$ *[53].*

Since the regular graphs K_n, $\overline{\mathsf{K}_n}$, $\mathsf{K}_{m,m}$ and C_n are DS, they are also $\mathcal{X}$-DS for every $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$ (see Theorem 2.3.17). This, however, does not apply to regular $\mathcal{L}$-DS graphs (see Remark 2.3.19), which are next addressed.

Theorem 2.3.23. *The following graphs are* $\mathcal{L}$*-DS:*

- K_n, *for every* $n \in \mathbb{N}$ *[28].*
- *The friendship graph* F_k, *for* $k \geq 2$ *[12, Corollary 1].*
- *More generally,* $\mathrm{F}_{p,q} = \mathsf{K}_1 \vee (p\,\mathsf{K}_q)$ *if* $q \geq 2$, *or* $q = 1$ *and* $p \geq 2$ *[12, Theorem 1].*

2.4 Special families of graphs

This section introduces special families of structured graphs and it states conditions for their unique determination by their spectra.

2.4.1 Stars and graphs of pyramids

Definition 2.4.1. For every $k, n \in \mathbb{N}$ with $k < n$, define the graph $T_{n,k} = \mathsf{K}_k \vee \overline{\mathsf{K}_{n-k}}$. For $k = 1$, the graph $T_{n,k}$ represents the *star graph* S_n. For $k = 2$, it represents a graph comprising $n-2$ triangles sharing a common edge, referred to as a *crown*. For n, k satisfying $1 < k < n$, the graphs $T_{n,k}$ are referred to as *graphs of pyramids* [90].

Theorem 2.4.2. *[90] The graphs of pyramids are DS for every* $1 < k < n$.

Theorem 2.4.3. *[90] The star graph S_n is DS if and only if $n-1$ is prime.*

To prove these theorems, a closed-form expression for the spectrum of $T_{n,k}$ is derived in [90], which also presents a generalized result. Subsequently, using Theorem 2.2.27, the number of edges and triangles in any graph cospectral with $T_{n,k}$ are calculated. Finally, Schur's theorem (Theorem 2.2.2) and Cauchy's interlacing theorem (Theorem 2.2.3) are applied in [90] to prove Theorems 2.4.2 and 2.4.3.

2.4.2 Complete bipartite graphs

By Theorem 2.4.3, the star graph $\mathsf{S}_n = \mathsf{K}_{1,n-1}$ is DS if and only if $n-1$ is prime. By Theorem 2.3.10, the regular complete bipartite graph $\mathsf{K}_{m,m}$ is DS for every $m \in \mathbb{N}$. Here, we generalize these results and provide a characterization for the DS property of $\mathsf{K}_{p,q}$ for every $p, q \in \mathbb{N}$.

Theorem 2.4.4. *[50] The spectrum of the complete bipartite graph $\mathsf{K}_{p,q}$ is $\{-\sqrt{pq}, [0]^{p+q-2}, \sqrt{pq}\}$.*

This theorem can be proved by Theorem 2.2.2. An alternative simple proof is next presented.

Proof. The adjacency matrix of $\mathsf{K}_{p,q}$ is given by

$$\mathbf{A}(\mathsf{K}_{p,q}) = \begin{pmatrix} \mathbf{0}_{p,p} & \mathbf{J}_{p,q} \\ \mathbf{J}_{q,p} & \mathbf{0}_{q,q} \end{pmatrix} \in \mathbb{R}^{(p+q)\times(p+q)} \tag{2.4.1}$$

The rank of $\mathbf{A}(\mathsf{K}_{p,q})$ is equal to 2, so the multiplicity of 0 as an eigenvalue is $p+q-2$. By Corollary 2.2.29, the two remaining eigenvalues are given by $\pm\lambda$ for some $\lambda \in \mathbb{R}$, since the eigenvalues sum to zero. Furthermore,

$$2\lambda^2 = \sum_{i=1}^{p+q} \lambda_i^2 = 2\,|\,\mathsf{E}(\mathsf{K}_{p,q})| = 2pq, \tag{2.4.2}$$

so $\lambda = \sqrt{pq}$. □

For $p, q \in \mathbb{N}$, the arithmetic and geometric means of p, q are, respectively, given by $\mathrm{AM}(p,q) = \frac{1}{2}(p+q)$ and $\mathrm{GM}(p,q) = \sqrt{pq}$. The AM-GM inequality states that for every $p, q \in \mathbb{N}$, we have $\mathrm{GM}(p,q) \leq \mathrm{AM}(p,q)$ with equality if and only if $p = q$.

Definition 2.4.5. Let $p, q \in \mathbb{N}$. The two-elements multiset $\{p, q\}$ is said to be an *AM-minimizer* if it attains the minimum arithmetic mean for their given geometric mean, i.e.,

$$\mathrm{AM}(p,q) = \min\Big\{\mathrm{AM}(a,b) : \; a, b \in \mathbb{N},\, \mathrm{GM}(a,b) = \mathrm{GM}(p,q)\Big\} \tag{2.4.3}$$

$$= \min\Big\{\tfrac{1}{2}(a+b) : \; a, b \in \mathbb{N},\, ab = pq\Big\}. \tag{2.4.4}$$

Example 2.4.6. The following are AM-minimizers:

- $\{k, k\}$ for every $k \in \mathbb{N}$. By the AM-GM inequality, it is the only case where $\text{GM}(p, q) = \text{AM}(p, q)$.
- $\{p, q\}$ where p, q are prime numbers. In this case, the following family of multisets

$$\Big\{\{a, b\} : \ a, b \in \mathbb{N}, \ \text{GM}(a, b) = \text{GM}(p, q)\Big\} \tag{2.4.5}$$

  only contains the two multisets $\{p, q\}, \{pq, 1\}$, and $p + q \leq pq < pq + 1$ since $p, q \geq 2$.
- $\{1, q\}$ where q is a prime number.

Theorem 2.4.7. *The following holds for every $p, q \in \mathbb{N}$:*

1. *Let G be a graph that is cospectral with $\mathsf{K}_{p,q}$. Then, up to isomorphism, $G = \mathsf{K}_{a,b} \cup \mathsf{H}$ (i.e., G is a disjoint union of the two graphs $\mathsf{K}_{a,b}$ and H), where H is an empty graph and $a, b \in \mathbb{N}$ satisfy $GM(a, b) = GM(p, q)$.*
2. *The complete bipartite graph $\mathsf{K}_{p,q}$ is DS if and only if $\{p, q\}$ is an AM-minimizer.*

Remark 2.4.8. Item 2 of Theorem 2.4.7 is equivalent to Corollary 3.1 of [104], for which an alternative proof is presented here.

Proof. (Proof of Theorem 2.4.7):

1. Let G be a graph cospectral with $\mathsf{K}_{p,q}$. The number of edges in G equals the number of edges in $\mathsf{K}_{p,q}$, which is pq. As $\mathsf{K}_{p,q}$ is bipartite, so is G. Since $\mathbf{A}(\mathsf{G})$ is of rank 2, and $\mathbf{A}(\mathsf{P}_3)$ has rank 3, it follows from the Cauchy's Interlacing Theorem (Theorem 2.2.3) that P_3 is not an induced subgraph of G.
   It is claimed that G has a single nonempty connected component. Suppose to the contrary that G has (at least) two nonempty connected components $\mathsf{H}_1, \mathsf{H}_2$. For $i \in \{1, 2\}$, since H_i is a non-empty graph, $\mathbf{A}(\mathsf{H}_i)$ has at least one eigenvalue $\lambda \neq 0$. Since G is a simple graph, the sum of the eigenvalues of $\mathbf{A}(\mathsf{H}_i)$ is $\text{Tr}(\mathbf{A}(\mathsf{H}_i)) = 0$, so H_i has at least one positive eigenvalue. Thus, the induced subgraph $\mathsf{H}_1 \cup \mathsf{H}_2$ has at least two positive eigenvalues, while G has only one positive eigenvalue, contradicting Cauchy's Interlacing Theorem.
   Hence, G can be decomposed as $\mathsf{G} = \mathsf{K}_{a,b} \cup \mathsf{H}$ where H is an empty graph. Since G and $\mathsf{K}_{p,q}$ have the same number of edges, $pq = ab$, so $\text{GM}(p, q) = \text{GM}(a, b)$.
2. First, we will show that if $\{p, q\}$ is not an AM-minimizer, then the graph $\mathsf{K}_{p,q}$ is not $\mathbf{A}$-DS. This is done by finding a nonisomorphic graph to $\mathsf{K}_{p,q}$ that is $\mathbf{A}$-cospectral with it. By assumption, since $\{p, q\}$ is not an AM-minimizer, there exist $a, b \in \mathbb{N}$ satisfying $\text{GM}(a, b) =$

$\mathrm{GM}(p,q)$ and $a+b < p+q$. Define the graph $\mathsf{G} = \mathsf{K}_{a,b} \vee \overline{\mathsf{K}_r}$ where $r = p+q-a-b$. Observe that $r \in \mathbb{N}$. The **A**-spectrum of both of these graphs is given by

$$\sigma_{\mathbf{A}}(\mathsf{G}) = \sigma_{\mathbf{A}}(\mathsf{K}_{p,q}) = \{-\sqrt{pq}, [0]^{pq-2}, \sqrt{pq}\}, \tag{2.4.6}$$

so these two graphs are nonisomorphic and cospectral, which means that G is not **A**-DS.

We next prove that if $\{p,q\}$ is an AM-minimizer, then $\mathsf{K}_{p,q}$ is **A**-DS. Let G be a graph that is cospectral with $\mathsf{K}_{p,q}$. From the first part of this theorem, $\mathsf{G} = \mathsf{K}_{a,b} \cup \mathsf{H}$ where $\mathrm{GM}(a,b) = \mathrm{GM}(p,q)$ and H is an empty graph. Consequently, it follows that $\mathrm{AM}(a,b) = \frac{1}{2}(a+b) \le \frac{1}{2}(p+q) = \mathrm{AM}(p,q)$. Since $\{p,q\}$ is assumed to be an AM-minimizer, it follows that $\mathrm{AM}(a,b) \ge \mathrm{AM}(p,q)$, and thus equality holds. Both equalities $\mathrm{GM}(a,b) = \mathrm{GM}(p,q)$ and $\mathrm{AM}(a,b) = \mathrm{AM}(p,q)$ can be satisfied simultaneously if and only if $\{a,b\} = \{p,q\}$, so $r = p+q-a-b = 0$ and $\mathsf{G} = \mathsf{K}_{p,q}$.

□

Corollary 2.4.9. *Almost all of the complete bipartite graphs are not DS. More specifically, for every $n \in \mathbb{N}$, there exists a single complete bipartite graph on n vertices that is DS.*

Proof. Let $n \in \mathbb{N}$. By the *fundamental theorem of arithmetic*, there is a unique decomposition $n = \prod_{i=1}^{k} p_i$ where $k \in \mathbb{N}$ and $\{p_i\}$ are prime numbers for every $1 \le i \le k$. Consider the family of multisets

$$\mathcal{D} = \Big\{\{a,b\} : a,b \in \mathbb{N}, \mathrm{GM}(a,b) = \sqrt{n}\Big\}. \tag{2.4.7}$$

This family has 2^k members, since every prime factor p_i of n should be in the prime decomposition of a or b. Since the minimization of $\mathrm{AM}(a,b)$ under the equality constraint $\mathrm{GM}(a,b) = \sqrt{n}$ forms a convex optimization problem, only one of the multisets in the family $\mathcal{D}$ is an AM-minimizer. Thus, if $n = \prod_{i=1}^{k} p_i$, then the number of complete bipartite graphs of n vertices is $O(2^k)$, and (by Item 2 of Theorem 2.4.7) only one of them is DS. □

2.4.3 Turán graphs

The Turán graphs are a significant and well-studied class of graphs in extremal graph theory, forming an important family of multipartite complete graphs. Turán graphs are particularly known for their role in Turán's theorem, which provides a solution to the problem of finding the maximum number of edges in a graph that does not contain a complete subgraph of a given order [140]. Before delving into formal definitions, it is noted that the distinction of the Turán graphs as multipartite complete graphs is that they are as balanced as possible, ensuring their vertex sets are divided into parts of nearly equal size.

Definition 2.4.10. Let $n_1, \ldots, n_k$ be natural numbers. Define the *complete k-partite graph*

$$\mathsf{K}_{n_1,\ldots,n_k} = \bigvee_{i=1}^{k} \overline{\mathsf{K}_{n_i}}. \tag{2.4.8}$$

A graph is multipartite if it is k-partite for some $k \geq 2$.

Definition 2.4.11. Let $2 \leq k \leq n$. The *Turán graph* $T(n, k)$ (not to be confused with the graph of pyramids $T_{n,k}$) is formed by partitioning a set of n vertices into k subsets, with sizes as equal as possible, and then every two vertices are adjacent in that graph if and only if they belong to different subsets. It is therefore expressed as the complete k-partite graph $K_{n_1,\ldots,n_k}$, where $|n_i - n_j| \leq 1$ for all $i, j \in [k]$ with $i \neq j$. Let q and s be the quotient and remainder, respectively, of dividing n by k (i.e., $n = qk + s$, $s \in \{0, 1, \ldots, k-1\}$), and let $n_1 \leq \ldots \leq n_k$. Then,

$$n_i = \begin{cases} q, & 1 \leq i \leq k - s, \\ q + 1, & k - s + 1 \leq i \leq k. \end{cases} \tag{2.4.9}$$

By construction, the graph $T(n, k)$ has a clique of order k (any subset of vertices with a single representative from each of the k subsets is a clique of order k), but it cannot have a clique of order $k + 1$ (since vertices from the same subset are nonadjacent). Note also that, by (2.4.9), the Turán graph $T(n, k)$ is an $(n - q)$-regular graph if and only if n is divisible by k, and then $q = \frac{n}{k}$.

Definition 2.4.12. Let $q, k \in \mathbb{N}$. Define the *regular complete multipartite graph*, $\mathrm{K}_q^k := \bigvee_{i=1}^{k} \overline{\mathsf{K}_q}$, to be the k-partite graph with q vertices in each part. Observe that $\mathrm{K}_q^k = T(kq, k)$.

Let G be a simple graph on n vertices that does not contain a clique of order greater than a fixed number $k \in \mathbb{N}$. Turán investigated a fundamental problem in extremal graph theory of determining the maximum number of edges that G can have [140].

Theorem 2.4.13 (Turán's Graph Theorem)**.** *Let* G *be a graph on n vertices with a clique of order at most k for some* $k \in \mathbb{N}$*. Then,*

$$\begin{aligned} |\,\mathsf{E}(\mathsf{G})| &\leq |\,\mathsf{E}(T(n, k))| \qquad (2.4.10) \\ &= \left(1 - \frac{1}{k}\right)\frac{n^2 - s^2}{2} + \binom{s}{2}, \quad s \triangleq n - k\left\lfloor \frac{n}{k} \right\rfloor. \qquad (2.4.11) \end{aligned}$$

For a nice exposition of five different proofs of Turán's Graph Theorem, the interested reader is referred to Chapter 41 of [6].

Corollary 2.4.14. *Let* $k \in \mathbb{N}$*, and let* G *be a graph on n vertices where* $\omega(\mathsf{G}) \leq k$ *and* $|\,\mathsf{E}(\mathsf{G})| = |\,\mathsf{E}(T(n, k))|$*. Let* G_1 *be a graph obtained by adding an arbitrary edge to* G*. Then* $\omega(\mathsf{G}_1) > k$*.*

The spectrum of the Turán graph

Theorem 2.4.15. *[60] Let $k \in \mathbb{N}$, and let $n_1 \le n_2 \le \ldots \le n_k$ be natural numbers. Let $\mathsf{G} = \mathsf{K}_{n_1,n_2,\ldots,n_k}$ be a complete multipartite graph on $n = n_1 + \ldots n_k$ vertices. Then,*

- G *has one positive eigenvalue, i.e., $\lambda_1(\mathsf{G}) > 0$ and $\lambda_2(\mathsf{G}) \le 0$.*
- G *has 0 as an eigenvalue with multiplicity $n-k$.*
- G *has $k-1$ negative eigenvalues, and*

$$n_1 \le -\lambda_{n-k+2}(\mathsf{G}) \le n_2 \le -\lambda_{n-k+3}(\mathsf{G}) \le n_3 \le \ldots \le n_{k-1} \le -\lambda_n(\mathsf{G}) \le n_k. \tag{2.4.12}$$

Corollary 2.4.16. *The spectrum of the regular complete k-partite graph $\mathsf{K}_{q,\ldots,q} \triangleq \mathsf{K}_q^k$ is given by*

$$\sigma_{\mathbf{A}}(\mathsf{K}_q^k) = \Big\{[-q]^{k-1}, [0]^{(q-1)k}, q(k-1)\Big\}. \tag{2.4.13}$$

Proof. This readily follows from Theorem 2.4.15 by setting $n_1 = \ldots = n_k = q$. □

Lemma 2.4.17. *[24] Let G_i be r_i-regular graphs on n_i vertices for $i \in \{1,2\}$, with the adjacency spectrum $\sigma_{\mathbf{A}}(\mathsf{G}_1) = (r_1 = \mu_1 \ge \mu_2 \ge \ldots \ge \mu_n)$ and $\sigma_A(\mathsf{G}_2) = (r_2 = \nu_1 \ge \nu_2 \ge \ldots \ge \nu_n)$. The $\mathbf{A}$-spectrum of $\mathsf{G}_1 \vee \mathsf{G}_2$ is given by*

$$\sigma_{\mathbf{A}}(\mathsf{G}_1 \vee \mathsf{G}_2) = \{\mu_i\}_{i=2}^{n_1} \cup \{\nu_i\}_{i=2}^{n_2} \cup \left\{\frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4n_1 n_2}}{2}\right\}. \tag{2.4.14}$$

Theorem 2.4.18. *Let $q, s \in \mathbb{N}$ such that $n = kq + s$ and $0 \le s \le k-1$. The following holds with respect to the $\mathbf{A}$-spectrum of $T(n,k)$:*

1. *If $1 \le s \le k-1$, then the $\mathbf{A}$-spectrum of the irregular Turán graph $T(n,k)$ is given by*

$$\begin{aligned}\sigma_{\mathbf{A}}(T(n,k)) = &\Big\{[-q-1]^{s-1}, [-q]^{k-s-1}, [0]^{n-k}\Big\} \\ &\cup \left\{\tfrac{1}{2}\Big[n - 2q - 1 \pm \sqrt{\Big(n - 2(q+1)s + 1\Big)^2 + 4q(q+1)s(k-s)}\Big]\right\}.\end{aligned} \tag{2.4.15}$$

2. *If $s = 0$, then $q = \frac{n}{k}$, and the $\mathbf{A}$-spectrum of the regular Turán graph $T(n,k)$ is given by*

$$\sigma_{\mathbf{A}}(T(n,k)) = \Big\{[-q]^{k-1}, [0]^{n-k}, (k-1)q\Big\}. \tag{2.4.16}$$

Proof. Let $1 \le s \le k-1$, and we next derive the $\mathbf{A}$-spectrum of an irregular Turán graph $T(n,k)$ in Item 1 of this theorem (i.e., its spectrum if n is not divisible by k since $s \ne 0$). By Corollary 2.4.16, the spectra of the regular graphs K_q^{k-s} and K_{q+1}^{s} is

$$\sigma_{\mathbf{A}}(\mathsf{K}_q^{k-s}) = \Big\{[-q]^{k-s-1}, [0]^{(q-1)(k-s)}, q(k-s-1)\Big\}, \tag{2.4.17}$$
$$\sigma_{\mathbf{A}}(\mathsf{K}_{q+1}^{s}) = \Big\{[-q-1]^{s-1}, [0]^{qs}, (q+1)(s-1)\Big\}. \tag{2.4.18}$$

The $(k-s)$-partite graph K_q^{k-s} is r_1-regular with $r_1 = q(k-s-1)$, the s-partite graph K_{q+1}^{s} is r_2-regular with $r_2 = (q+1)(s-1)$, and by Definition 2.4.11, we have $T(n,k) = \mathsf{K}_q^{k-s} \vee \mathsf{K}_{q+1}^{s}$. Hence, by Lemma 2.4.17, the adjacency spectrum of $T(n,k)$ is given by

$$\begin{aligned}\sigma_{\mathbf{A}}(T(n,k)) &= \sigma_{\mathbf{A}}(\mathsf{K}_q^{k-s} \vee \mathsf{K}_{q+1}^{s}) \\ &= \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3,\end{aligned} \tag{2.4.19}$$

where

$$\mathcal{S}_1 = \Big\{[-q]^{k-s-1}, [0]^{(q-1)(k-s)}\Big\}, \tag{2.4.20}$$
$$\mathcal{S}_2 = \Big\{[-q-1]^{s-1}, [0]^{qs}\Big\}, \tag{2.4.21}$$
$$\begin{aligned}\mathcal{S}_3 &= \left\{\frac{r_1 + r_2 \pm \sqrt{(r_1-r_2)^2 + 4n_1n_2}}{2}\right\} \\ &= \left\{\tfrac{1}{2}\Big[n - 2q - 1 \pm \sqrt{\Big(n - 2(q+1)s + 1\Big)^2 + 4q(q+1)s(k-s)}\Big]\right\},\end{aligned} \tag{2.4.22}$$

where the last equality holds since, by the equality $n = kq + s$ and the above expressions of r_1 and r_2, it can be readily verified that $r_1 + r_2 = n - 2q - 1$ and $r_1 - r_2 = n - 2(q+1)s + 1$. Finally, combining (2.4.19)–(2.4.22) gives the $\mathbf{A}$-spectrum in (2.4.15) of an irregular Turán graph $T(n,k)$.

We next prove Item 2 of this theorem, referring to a regular Turán graph $T(n,k)$ (i.e., $k|n$ or equivalently, $s = 0$). In that case, we have $T(n,k) = \mathsf{K}_q^{k}$ where $q = \frac{n}{k}$, so the $\mathbf{A}$-spectrum in (2.4.16) holds by Corollary 2.4.16. □

Remark 2.4.19. In light of Theorem 2.4.18, if $k \geq 2$, then the number of negative eigenvalues (including multiplicities) of the adjacency matrix of the Turán graph $T(n,k)$ is $k-1$ if the graph is regular (i.e., if $k|n$), and it is $k-2$ otherwise (i.e., if the graph is irregular). If $k = 1$, which corresponds to an empty graph (having no edges), then all eigenvalues are zeros (having no negative eigenvalues). Furthermore, the adjacency matrix of $T(n,k)$ always has a single positive eigenvalue, which is of multiplicity 1 irrespectively of the values of n and k. We rely on these properties later in this chapter (see Section 2.4.3).

Example 2.4.20. By Theorem 2.4.18, let us calculate the $\mathbf{A}$-spectrum of the Turán graph $T(17,7)$, and verify it numerically with the SageMath software [122]. Having $n = 17$ and $k = 7$ gives $q = 2$

and $s = 3$, which by Theorem 2.4.18 implies that

$$\sigma_{\mathbf{A}}(T(17,7)) = \Big\{[-3]^2, [-2]^3, [0]^{10}, 6(1+\sqrt{2}), 6(1-\sqrt{2})\Big\}. \tag{2.4.23}$$

That has been numerically verified by programming in the SageMath software.

Turán graphs are DS

The main result of this subsection establishes that all Turán graphs are determined by their **A**-spectrum. This result is equivalent to Theorem 3.3 in [104], while also presenting an alternative proof that offers additional insights.

Theorem 2.4.21. *The Turán graph $T(n,k)$ is* **A***-DS.*

In order to prove Theorem 2.4.21, we first introduce an auxiliary result from [134], followed by several other lemmata.

Theorem 2.4.22. *[134, Theorem 1] Let G be a graph. Then, the following statements are equivalent:*

- G *has exactly one positive eigenvalue.*
- $\mathsf{G} = \mathsf{H} \cup \overline{\mathsf{K}_{\mathsf{m}}}$ *for some m, where H is a nonempty complete multipartite graph. In other words, the non-isolated vertices of G form a complete multipartite graph.*

Proof of Theorem 2.4.21. Let G be a graph that is **A**-cospectral with $T(n,k)$. Denote $n = qk + s$ for $s, q \in \mathbb{N}$ such that $0 \leq s < k$.

Lemma 2.4.23. *The graph G doesn't have a clique of order $k+1$.*

Proof. Suppose to the contrary that the graph G has a clique of order $k+1$, which means that K_{k+1} is an induced subgraph of G. The complete graph K_{k+1} has k negative eigenvalues (-1 with a multiplicity of k). On the other hand, G has at most $k-1$ negative eigenvalues, $n-k$ zero eigenvalues, and exactly one positive eigenvalue; indeed, this follows from Theorem 2.4.18 (see Remark 2.4.19), and since G and $T(n,k)$ are **A**-cospectral graphs. Hence, by Cauchy's Interlacing Theorem, every induced subgraph of G on $k+1$ vertices has at most $k-1$ negative eigenvalues (i.e., those eigenvalues interlaced between the negative and zero eigenvalues of G that are placed at distance $k+1$ apart in a sorted list of the eigenvalues of G in decreasing order). This contradicts our assumption on the existence of a clique of $k+1$ vertices because of the k negative eigenvalues of K_{k+1}. □

Lemma 2.4.24. *The graph G is a complete multipartite graph.*

Proof. Since G has exactly one positive eigenvalue, which is of multiplicity one, we get from Theorem 2.4.22 that $\mathsf{G} = \mathsf{H} \cup \overline{\mathsf{K}_\ell}$ for some $\ell \in \mathbb{N}$, where H is a nonempty multipartite graph. We next show that $\ell = 0$. Suppose to the contrary that $\ell \geq 1$, and let v be an isolated vertex of $\overline{\mathsf{K}_\ell}$. Since H is a nonempty graph, there exists a vertex $u \in \mathsf{V}(\mathsf{H})$. Let G_1 be the graph obtained from G by adding the single edge $\{v, u\}$. By Lemma 2.4.23, G does not have a clique of order $k+1$. Hence, G_1 does not have a clique of order $k + 1$ either, contradicting Corollary 2.4.14. Hence, $\mathsf{G} = \mathsf{H}$. □

Lemma 2.4.25. *The graph* G *is a complete* k*-partite graph.*

Proof. By Lemma 2.4.24, G is a complete multipartite graph. Let r be the number of partite subsets in the vertex set $\mathsf{V}(\mathsf{G})$. We show that $r = k$, which then gives that G is a complete k-partite graph. By Lemma 2.4.23, G doesn't have a clique of order $k + 1$. Hence, $r \leq k$. Suppose to the contrary that $r < k$. Since G is a complete r-partite graph, the largest order of a clique in G is r. Let G_1 be a graph obtained from G by adding an edge between two vertices within the same partite subset. The graph G_1 becomes an $(r + 1)$-partite graph. Consequently, the maximum order of a clique in G_1 is at most $r + 1 \leq k$. The graph G_1 has exactly one more edge than G. Since G is $\mathbf{A}$-cospectral to $T(n, k)$, it has the same number of edges as in $T(n, k)$. Hence, G_1 contains more edges than $T(n, k)$, while also lacking a clique of order $k + 1$. This contradicts Corollary 2.4.14, so we conclude that $r = k$. □

Let $n_1, n_2, \ldots, n_k \in \mathbb{N}$ be the number of vertices in each partite subset of the complete k-partite graph G, i.e., $\mathsf{G} = \mathsf{K}_{n_1, n_2, \ldots, n_k}$. Then, the next two lemmata subsequently hold.

Lemma 2.4.26. *For all* $i \in [k]$, $n_i \leq q + 1$.

Proof. Suppose to the contrary that there exists a partite subset in the complete k-partite graph G with more than $q+1$ vertices. Let $\mathcal{P}_1$ be such a partite subset, and suppose without loss of generality that $n_1 = |\mathcal{P}_1| \geq q + 1$. By the pigeonhole principle, there exists a partite subset of G with at most q vertices (since $\sum_{i \in [k]} n_i = n = kq + s$, where $0 \leq s \leq k - 1$). Let $\mathcal{P}_2$ be such a partite subset of G, and suppose without loss of generality that $n_2 = |\mathcal{P}_2| \leq q$. Let G_1 be the graph obtained from G by removing a vertex $v \in \mathcal{P}_1$, adding a new vertex u to $\mathcal{P}_2$, and connecting u to all the vertices outside its partite subset. The new graph G_1 is also k-partite, so it does not contain a clique of order greater than k. Furthermore, by construction, G_1 has more edges than G, so

$$\left|\mathsf{E}(\mathsf{G}_1)\right| > \left|\mathsf{E}(\mathsf{G})\right| = \left|\mathsf{E}(T(n, k))\right|. \tag{2.4.24}$$

Hence, G_1 is a graph with no clique of order greater than k, and it has more edges than $T(n, k)$. That contradicts Theorem 2.4.13, so $\{n_i\}_{i=1}^{k}$ cannot include any element that is larger than $q+1$. □

Lemma 2.4.27. *For all* $i \in [k]$, $n_i \geq q$.

Proof. The proof of this lemma is analogous to the proof of Lemma 2.4.26. Suppose to the contrary that there exists a partite subset of G with less than q vertices. Let $\mathcal{P}_1$ be such a partite subset, so $p_1 \triangleq \left|\mathcal{P}_1\right| < q$. By the pigeonhole principle, there exists a partition with at least $q+1$ vertices. Let $\mathcal{P}_2$ be such a partite subset set, whose number of vertices is denoted by $p_2 \triangleq |\mathcal{P}_2| \geq q+1$. Let G_1 be the graph obtained by removing a vertex $v \in \mathcal{P}_2$, adding a new vertex u to $\mathcal{P}_1$, and connecting the vertex u to all the vertices outside its partite subset. G_1 is k-partite, so it does not contain a clique of order greater than k, and G_1 has more edges than G so (2.4.24) holds. Hence, G_1 is a graph with no clique of order greater than k, and it has more edges than $T(n,k)$. That contradicts Theorem 2.4.13, so $\{n_i\}_{i=1}^k$ cannot include any element that is smaller than q. □

By Lemmata 2.4.26 and 2.4.27, we conclude that $n_i \in \{q, q+1\}$ for every $1 \leq i \leq k-1$. Let α be the number of partite subsets of q vertices and β be the number of partite subsets of $q+1$ vertices. Since G has n vertices, where $\sum n_i = n$, it follows that $q\alpha + (q+1)\beta = n$. Moreover, G is k-partite, so it follows that $\alpha + \beta = k$. This gives the linear system of equations

$$\begin{pmatrix} q & q+1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}, \tag{2.4.25}$$

which has the single solution

$$\alpha = k - s, \quad \beta = n - qk = s. \tag{2.4.26}$$

Hence, $\mathsf{G} = T(n,k)$, which completes the proof of Theorem 2.4.21. □

Remark 2.4.28. The proof of Theorem 2.4.21 is an alternative proof of Theorem 3.3 in [104]. While both proofs rely on Theorem 2.4.22, which is Theorem 1 of [134], our proof relies on the adjacency spectral characterization in Theorem 2.4.18, noteworthy in its own right, and further builds upon a sequence of results presented in Lemmata 2.4.23–2.4.27. On the other hand, the proof of Theorem 3.3 in [104] relies on Theorem 2.4.22, but then deviates substantially from our proof (see Lemmata 2.4 and 2.5 in [104] and Theorem 3.1 in [104], serving to prove Theorem 2.4.21).

2.4.4 Line graphs

Among the various studied transformations on graphs, the line graphs of graphs are one of the most studied transformations [11]. We first introduce their definition, and then address the spectral graph determination properties.

Definition 2.4.29. The *line graph* of a graph G, denoted by $\ell(\mathsf{G})$, is a graph whose vertices are the edges in G, and two vertices are adjacent in $\ell(\mathsf{G})$ if the corresponding edges are incident in G.

A notable spectral property of line graphs is that all the eigenvalues of their adjacency matrix are greater than or equal to -2 (see, e.g., [11, Theorem 4.6]). For the determination of all graphs whose spectrum is bounded from below by -2, the interested reader is referred to [11, Section 4.5].

The following theorem characterizes some families of line graphs that are DS.

Theorem 2.4.30. *The following line graphs are DS:*

1 *The line graph of the complete graph K_k, where $k \geq 4$ and $k \neq 8$ (see [49, Theorem 4.1.7]),*

2 *The line graph of the complete bipartite graph $\mathsf{K}_{k,k}$, where $k \geq 2$ and $k \neq 4$ (see [49, Theorem 4.1.8]),*

3 *The line graph $\ell(\overline{\mathsf{C}_6})$ (see [49, Proposition 4.1.5]),*

4 *The line graph of the complete bipartite graph $\mathsf{K}_{m,n}$, where $m + n \geq 19$ and $\{m, n\} \neq \{2s^2 + s, 2s^2 - s\}$ with $s \in \mathbb{N}$ (see [49, Proposition 4.1.18]).*

Remark 2.4.31. In regard to Item 1 of Theorem 2.4.30, the line graphs of complete graphs are referred to as triangular graphs. These are strongly regular graphs with the parameters $\mathsf{srg}(\frac{1}{2}k(k-1), 2(k-2), k-2, 4)$, where $k \geq 4$. For $k = 8$, the corresponding triangular graph is cospectral and nonisomorphic to the three Chang graphs (see Remark 2.2.33), which are strongly regular graphs $\mathsf{srg}(28, 12, 6, 4)$.

We next prove the following result in regard to the Petersen graph, which appears in [49, Problem 4.3] and [22, Section 10.3] (without a proof).

Corollary 2.4.32. *The Petersen graph is DS.*

Proof. The Petersen graph is known to be isomorphic to the complement of the line graph of the complete graph K_5 (i.e., it is isomorphic to $\overline{\ell(\mathsf{K}_5)}$. By Item 1 of Theorem 2.4.30, the line graph $\ell(\mathsf{K}_5)$ is DS. It is also a 6-regular graph (as the line graph of a d-regular graph is $(2d-2)$-regular, and K_5 is a 4-regular graph). Consequently, by Item 5 of Theorem 2.3.10, the complement of $\ell(\mathsf{K}_5)$ is also DS. □

The following definition and theorem provide further elaboration on Item 2 of Theorem 2.4.30.

Definition 2.4.33. [22, Section 1.1.8] The *Hamming graph* $\mathrm{H}(2, q)$, where $q \geq 2$, has the vertex set $[q] \times [q]$, and any two vertices are adjacent if and only if they differ in one coordinate (i.e., their Hamming distance is equal to 1). These are also referred to *lattice graphs*, and denoted by $\mathrm{L}_2(q)$. The Lattice graph $\mathrm{L}_2(q)$, where $q \geq 2$, is also the line graph of the complete bipartite graph $\mathsf{K}_{q,q}$, and it is a strongly regular graph with parameters $\mathsf{srg}(q^2, 2(q-1), q-2, 2)$.

Theorem 2.4.34. *[132] The lattice graph* $\mathrm{L}_2(q)$ *is a strongly regular DS graph for all* $q \neq 4$*. For* $q = 4$*, the graph* $\mathrm{L}_2(4)$ *is not DS since it is cospectral and nonisomorphic to the Shrikhande graph, which are nonisomorphic strongly regular graphs with the common parameters* $\mathsf{srg}(16, 6, 4, 2)$*.*

The following result provides an interesting connection between **A** and **Q**-cospectralities of graphs.

Theorem 2.4.35. *[49, Proposition 7.8.5] If two graphs are* **Q***-cospectral, then their line graphs are* **A***-cospectral.*

2.4.5 Nice graphs

The family referred to as "nice graphs" has been recently introduced in [89].

Definition 2.4.36.

- A graph G is *sunlike* if it is connected, and can be obtained from a cycle C by adding some vertices and connecting each of them to some vertex in C.
- Let $l, k \in \mathbb{N}$. A sunlike graph G is (l, k)*-nice* if it can be obtained by a cycle C_ℓ and
  - There is a single vertex $v_1 \in \mathsf{V}(\mathsf{C}_\ell)$ of degree 3.
  - There are k vertices $u_1, \ldots, u_k \in \mathsf{V}(\mathsf{C}_\ell)$ of degree 4. Let $\mathcal{U} = \{u_1, \ldots, u_k\}$.
  - By starting a walk on C_ℓ from v_1 at some orientation, then after 4 or 6 steps we get to a vertex $u_1 \in \mathcal{U}$. Then, after another 4 or 6 steps from u_1 we get to $u_2 \in \mathcal{U}$, and so on until we get to the vertex $u_k \in \mathcal{U}$.

Theorem 2.4.37. *[89] Let* $l, k \in \mathbb{N}$ *such that* $l \equiv 2\,(mod\ 4)$*. Let* G *be an* (l, k)*-nice graph. If the order of* G *is a prime number greater than some* $n_0 \in \mathbb{N}$*, then the line graph* $\ell(\mathsf{G})$ *is DS.*

A more general class of graphs is introduced in [89], where it is shown that for every sufficiently large $n \in \mathbb{N}$, the number of nonisomorphic n-vertex DS graphs is at least e^{cn} for some positive constant c (see [89, Theorem 1.4]). This recent result represents a significant advancement in the study of Haemers' conjecture because the earlier lower bounds on the number of nonisomorphic n-vertex DS graphs were all of the form of $e^{c\sqrt{n}}$, for some positive constant c. As noted in [89], the first form of such a lower bound was derived by van Dam and Haemers [51, Proposition 6], who proved that a graph G is DS if every connected component of G is a complete subgraph, leading to a lower bound that is approximately of the form $e^{c\sqrt{n}}$ with $c = \sqrt{\frac{2}{3}}\,\pi$. Therefore, the transition to a lower bound in [89] that scales exponentially with n, rather than with $\sqrt{n}$, is both remarkable and noteworthy.

2.4.6 Friendship graphs and their generalization

The next theorem considers whether friendship graphs (see Definition 2.3.9) can be uniquely determined by the spectra of four of their associated matrices.

Theorem 2.4.38. *The friendship graph F_p satisfies the following properties: It is DS if and only if $p \neq 16$ (i.e., the friendship graph is DS unless it has 16 triangles) [40],* **L***-DS [96],* **Q***-DS [144], and $\mathcal{L}$-DS [12].*

The friendship graph F_p, where $p \in \mathbb{N}$, can be expressed in the form $\mathsf{F}_p = \mathsf{K}_1 \vee (p\,\mathsf{K}_2)$ (see Figure 2.3). The last observation follows from a property of a generalized friendship graph, which is defined as follows.

Definition 2.4.39. Let $p, q \in \mathbb{N}$. The *generalized friendship graph* is given by $\mathsf{F}_{p,q} = \mathsf{K}_1 \vee (p\,\mathsf{K}_q)$. Note that $\mathsf{F}_{p,2} = \mathsf{F}_p$.

The following theorem addresses the conditions under which generalized friendship graphs can be uniquely determined by the spectra of their normalized Laplacian matrix.

Theorem 2.4.40. *The generalized friendship graph $\mathsf{F}_{p,q}$ is $\mathcal{L}$-DS if and only if $q \geq 2$, or $q = 1$ and $p = 2$ [12].*

Corollary 2.4.41. *The friendship graph F_p is $\mathcal{L}$-DS [12].*

2.4.7 Strongly regular graphs

Strongly regular graphs with an identical vector of parameters (n, d, λ, μ) are cospectral, but may not be isomorphic (see, Corollary 2.2.32, Remark 2.2.33, and Theorem 2.4.34). For that reason, strongly regular graphs are not necessarily determined by their spectrum. There are, however, infinite families of strongly regular DS graphs:

Theorem 2.4.42. *[20, Proposition 14.5.1] If $n \neq 8$, $m \neq 4$, $a \geq 2$, and $\ell \geq 2$, then the disjoint union of identical complete graphs $a\,\mathsf{K}_\ell$, the line graph of a complete graph $\ell(\mathsf{K}_n)$, and the line graph of a complete bipartite graph with partite sets of equal size $\ell(\mathsf{K}_{m,m})$, as well as their complements, are strongly regular DS graphs.*

We next show that, although connected strongly regular graphs are not generally DS, the property of strong regularity, as well as the four parameters that characterize strongly regular graphs can be determined by the spectrum of their adjacency matrix.

Theorem 2.4.43. *Let G be a connected strongly regular graph. Then, its strong regularity, the vector of parameters (n, d, λ, μ), Lovász ϑ-function $\vartheta(\mathsf{G})$, number of edges and triangles, girth, and diameter can be all determined by its* **A***-spectrum.*

Proof. The order of a graph n is determined by the **A**-spectrum, being the number of eigenvalues (including multiplicities). By Theorem 2.2.30, the regularity of a graph is determined by its **A**-spectrum. By Item 5 of Theorem 2.2.31, a connected regular graph is strongly regular if and only if it has three distinct eigenvalues. Hence, the strong regularity property of G is determined by its **A**-spectrum. For such a connected regular, the largest eigenvalue is simple, $\lambda_1 = d$, and the other two distinct eigenvalues of the adjacency matrix of G are given by λ_2 and λ_n with $\lambda_n < \lambda_2$. We next show that the number of common neighbors of any pair of adjacent vertices (λ), and the number of common neighbors of any pair of nonadjacent vertices (μ) in G are, respectively, given by

$$\lambda = \lambda_1 + (1 + \lambda_2)(1 + \lambda_n) - 1, \tag{2.4.27}$$

$$\mu = \lambda_1 + \lambda_2\lambda_n, \tag{2.4.28}$$

so, these parameters are explicitly expressed in terms of the adjacency spectrum of the strongly regular graph. Indeed, by Theorem 2.2.31, the second-largest and least eigenvalues of the adjacency matrix of G are given by

$$\begin{cases} \lambda_2 = \frac{1}{2}\Big(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(d - \mu)}\Big), \\ \lambda_n = \frac{1}{2}\Big(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(d - \mu)}\Big), \end{cases} \tag{2.4.29}$$

from which it follows that (noting that $d = \lambda_1$)

$$\begin{cases} \lambda_2 + \lambda_n = \lambda - \mu, \\ \lambda_2\lambda_n = \mu - \lambda_1. \end{cases} \tag{2.4.30}$$

This gives (2.4.27) and (2.4.28) from, respectively, the second equality in (2.4.30) and by adding the two equalities in (2.4.30).

The Lovász ϑ-function of a strongly regular graph is given by (see [125, Proposition 1])

$$\vartheta(\mathsf{G}) = -\frac{n\lambda_n}{d - \lambda_n}, \tag{2.4.31}$$

so $\vartheta(\mathsf{G})$ is determined by its **A**-spectrum since the strong regularity property of G was first determined. The number of edges of G of a d-regular graph is given by $\frac{1}{2}nd$, and the number of triangles of the strongly regular graph G is given by $\frac{1}{6}nd\lambda$, so they are both determined once the four parameters of the strongly regular graphs are revealed. The diameter of a connected strongly regular graph is equal to 2 (note that complete graphs are excluded from the family of strongly regular graphs). Finally, the girth of the strongly regular graph G is determined as follows [34]:

1. If $\lambda > 0$, then the girth of G is equal to 3;

2. If $\lambda = 0$ and $\mu \geq 2$, then the girth of G is equal to 4;

3. If $\lambda = 0$ and $\mu = 1$, then the girth of G is equal to 5.

$\square$

Remark 2.4.44. A strongly regular graph is connected if and only if $\mu > 0$.

By [50, Proposition 2], no pair of **A**-cospectral graphs exists where one graph is regular, and the other is not. The following result extends this observation to strong regularity.

Corollary 2.4.45. *There are no two* **A***-cospectral connected graphs where one is strongly regular and the other is not.*

Proof. For a connected strongly regular graph, the strong regularity is determined by the **A**-spectrum. $\square$

Another corollary that follows from Theorem 2.4.43 applies to strongly regular DS graphs.

Corollary 2.4.46. *Let* G *be a connected strongly regular graph such that there is no other nonisomorphic strongly regular graph with an identical vector of parameters* (n, d, λ, μ)*. Then,* G *is a DS graph.*

Corollary 2.4.46 naturally raises the following question.

Question 2.4.47. *Which connected strongly regular graphs are determined by their vector of parameters* (n, d, λ, μ)*?*

A partial answer to Question 2.4.47 is provided below.

By Corollary 2.4.46, for connected strongly regular graphs, there exists an equivalence between their spectral determination (due to their regularity and in light of Theorem 2.3.17, based on the spectrum of their adjacency, Laplacian, or signless Laplacian matrices) and the uniqueness of these graphs for the given parameter vector (n, d, λ, μ).

The study of the number of nonisomorphic strongly regular graphs corresponding to a given set of parameters has been extensively explored. For example, by Theorem 2.4.34, there is a unique (up to isomorphism) strongly regular graph of the form $\mathsf{srg}(q^2, 2(q-1), q-2, 2)$ for any given $q \geq 2$ with $q \neq 4$. Specifically, this implies the uniqueness of $\mathsf{srg}(36, 10, 4, 2)$ (setting $q = 6$). On the other hand, a computer search by McKay and Spence established that there are $32,548$ strongly regular graphs of the form $\mathsf{srg}(36, 15, 6, 6)$, so none of them is DS (by Corollary 2.4.46).

Further results on the uniqueness or non-uniqueness of strongly regular graphs with a given parameter vector (n, d, λ, μ) can be found in [33] and the references therein. Infinite families of strongly regular DS graphs are presented in Theorem 2.4.42. Some known sporadic strongly regular DS graphs are listed in [50, Table 2], with an update in [51, Table 1]). The uniqueness of further strongly regular graphs with given parameter vectors, making them therefore DS graphs, was established, e.g., in [19, 21, 44, 107, 136].

A combination of [51, Table 1] and Corollary 2.4.32 implies that, apart from complete graphs on fewer than three vertices and all complete bipartite regular graphs (which are known to be DS, as stated in Item 2 of Theorem 2.3.10), also all the seven currently known triangle-free strongly regular graphs (see [137]) are DS. These include:

- The Pentagon graph C_5 that is $\mathsf{srg}(5, 2, 0, 1)$ (by Item 2 of Theorem 2.3.10, and see [22, Section 10.1]),
- The Petersen graph $\mathsf{srg}(10, 3, 0, 1)$ (by Corollary 2.4.32, and see [22, Section 10.3]),
- Clebsch graph $\mathsf{srg}(16, 5, 0, 2)$ (see [22, Section 10.7]),
- Hoffman-Singleton srg $\mathsf{srg}(50, 7, 0, 1)$ (see [22, Section 10.19]),
- Gewirtz graph $\mathsf{srg}(56, 10, 0, 2)$ (see [22, Section 10.20]),
- Mesner (M_{22}) graph $\mathsf{srg}(77, 16, 0, 4)$ (see [22, Section 10.27] and [19]),
- Higman-Sims graph $\mathsf{srg}(100, 22, 0, 6)$ (see [22, Section 10.31] and [107]).

An up-to-date list of strongly regular DS graphs — strongly regular graphs that are uniquely determined by their parameter vectors — as well as the number of strongly regular NICS graphs for given parameter vectors, is available on Brouwer's website [18]. An exclamation mark placed to the left of a parameter vector (n, d, λ, μ), without a preceding number, indicates a strongly regular DS graph. In contrast, an exclamation mark preceded by a natural number greater than 1 specifies the number of strongly regular NICS graphs with the corresponding parameter vector. For example, as shown in [18], strongly regular graphs with the parameter vectors $(13, 6, 2, 3)$, $(15, 6, 1, 3)$, $(17, 8, 3, 4)$, and $(21, 10, 3, 6)$, among others, are DS graphs. On the other hand, according to [18], there are 15 strongly regular NICS graphs with the parameter vector $(25, 12, 5, 6)$, 10 strongly regular NICS graphs with the parameter vector $(26, 10, 3, 4)$, and so forth.

To conclude, as strongly regular NICS graphs are not DS, $\mathbf{L}$-DS, or $\mathbf{Q}$-DS, we were recently informed of ongoing research by Cioaba *et al.* [39], which investigates the spectral properties of higher-order Laplacian matrices associated with these graphs. This research demonstrates that the spectra of these new matrices can distinguish some of the strongly regular NICS graphs. However, in other cases, strongly regular NICS graphs remain indistinguishable even with the spectra of these higher-order Laplacian matrices.

2.5 Graph operations for the construction of cospectral graphs

This section presents such graph operations, focusing on unitary and binary transformations that enable the systematic construction of cospectral graphs. These operations are designed to preserve the spectral properties of the original graph while potentially altering its structure, thereby producing non-isomorphic graphs with identical eigenvalues. By employing these techniques, one can generate diverse examples of cospectral graphs, offering valuable tools for investigating the limitations of spectral characterization and exploring the boundaries between graphs that are or are not determined by their spectrum, which then relates the scope of the present section to Section 2.4 that deals with graphs or graph families that are determined by their spectrum.

2.5.1 Coalescence

A construction of cospectral trees has been offered in [129], implying that almost all trees are not DS.

Definition 2.5.1. Let $\mathsf{G}_1 = (V_1, E_1), \mathsf{G}_2 = (V_2, E_2)$ be two graphs with n_1, n_2 vertices, respectively. Let $v_1 \in V_1$ and $v_2 \in V_2$ be an arbitrary choice of vertices in both graphs. The *coalescence of* G_1 *and* G_2 *with respect to* v_1 *and* v_2 is the graph with $n_1 + n_2 - 1$ vertices, obtained by the union of G_1 and G_2 where v_1 and v_2 are identified as the same vertex in the united graph.

Theorem 2.5.2. *Let* $\mathsf{G}_1 = (V_1, E_1), \mathsf{G}_2 = (V_2, E_2)$ *be two cospectral graphs, and let* $v_1 \in V_1$ *and* $v_2 \in V_2$ *be an arbitrary choice of vertices in both graphs. Let* H_1 *and* H_2 *be the subgraphs of* G_1 *and* G_2 *that are induced by* $V_1 \setminus \{v_1\}$ *and* $V_2 \setminus \{v_2\}$*, respectively. Let* Γ *be a graph and* $u \in V(\Gamma)$*. If* H_1 *and* H_2 *are cospectral, then the coalescence of* G_1 *and* Γ *with respect to* v_1 *and* u *is cospectral to the coalescence of* G_2 *and* Γ *with respect to* v_2 *and* u*.*

Combinatorial arguments that rely on the coalescence operation on graphs lead to a striking asymptotic result in [129], stating that the fraction of n-vertex trees with cospectral and nonisomorphic mates, which are also trees, approaches one as n tends to infinity. Consequently, the fraction of the n-vertex nonisomorphic trees that are determined by their spectrum (DS) approaches zero as n tends to infinity. In other words, this means that almost all trees are not DS (with respect to their adjacency matrix) [129].

2.5.2 Seidel switching

Seidel switching is one of the well-known methods for the construction of cospectral graphs.

Definition 2.5.3. Let G be a graph, and let $\mathcal{U} \subseteq \mathsf{V}(\mathsf{G})$. Constructing a graph $\mathsf{G}_{\mathcal{U}}$ by preserving all the edges in G between vertices within $\mathcal{U}$, as well as all edges in G between vertices within the complement set $\mathcal{U}^{\mathrm{c}} = \mathsf{V}(\mathsf{G}) \setminus \mathcal{U}$, while modifying adjacency and nonadjacency between any two vertices where one is in $\mathcal{U}$ and the other is in $\mathcal{U}^{\mathrm{c}}$, is referred to (up to isomorphism) as *Seidel switching* of G with respect to $\mathcal{U}$.

By Definition 2.5.3, the Seidel switching of G with respect to $\mathcal{U}$ is equivalent to its Seidel switching with respect to $\mathcal{U}^{\mathrm{c}}$. Let $\mathbf{A}(\mathsf{G})$ and $\mathbf{A}(\mathsf{G}_{\mathcal{U}})$ be the adjacency matrices of a graph G and its Seidel switching $\mathsf{G}_{\mathcal{U}}$, and let $\mathbf{A}_{\mathcal{U}}$ and $\mathbf{A}_{\mathcal{U}^{\mathrm{c}}}$ be the submatrices of $\mathbf{A}(\mathsf{G})$ that, respectively, refer to the adjacency matrices of the subgraphs of G induced by $\mathcal{U}$ and $\mathcal{U}^{\mathrm{c}}$. Then, for some $\mathbf{B} \in \{0, 1\}^{|\mathcal{U}^{\mathrm{c}}| \times |\mathcal{U}|}$, we get

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{A}_{\mathcal{U}} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{A}_{\mathcal{U}^{\mathrm{c}}} \end{pmatrix}, \tag{2.5.1}$$

and by Definition 2.5.3,

$$\mathbf{A}(\mathsf{G}_{\mathcal{U}}) = \begin{pmatrix} \mathbf{A}_{\mathcal{U}} & \overline{\mathbf{B}}^{\mathrm{T}} \\ \overline{\mathbf{B}} & \mathbf{A}_{\mathcal{U}^{\mathrm{c}}} \end{pmatrix}, \tag{2.5.2}$$

where $\overline{\mathbf{B}}$ is obtained from $\mathbf{B}$ by interchanging zeros and ones. If G is a regular graph, the following necessary and sufficient condition for $\mathsf{G}_{\mathcal{U}}$ to be a regular graph of the same degree of its vertices.

Theorem 2.5.4. *[49, Proposition 1.1.7] Let G be a d-regular graph on n vertices. Then, $\mathsf{G}_{\mathcal{U}}$ is also d-regular if and only if $\mathcal{U}$ induces a regular subgraph of degree k, where $|\mathcal{U}| = n - 2(d - k)$.*

The next result shows the relevance of Seidel switching for the construction of regular and cospectral graphs.

Theorem 2.5.5. *[49, Proposition 1.1.8] Let G be a d-regular graph, $\mathcal{U} \subseteq \mathsf{V}(\mathsf{G})$, and let $\mathsf{G}_{\mathcal{U}}$ be obtained from G by Seidel switching. If $\mathsf{G}_{\mathcal{U}}$ is also a d-regular graph, then G and $\mathsf{G}_{\mathcal{U}}$ are cospectral (and due to their regularity, they are $\mathcal{X}$-cospectral for every $\mathcal{X} \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$).*

Remark 2.5.6. Theorem 2.5.5 provides a method for finding cospectral regular graphs. These graphs may be, however, also isomorphic. If the graphs are nonisomorphic, then it gives a pair of NICS graphs.

Remark 2.5.7. A regular graph G on n vertices cannot be switched into another regular graph if n is odd (see [49, Corollary 4.1.10]), which means that the conditions in Theorem 2.5.5 cannot be satisfied for any regular graph of an odd order.

Remark 2.5.8. Seidel switching determines an equivalence relation on graphs. This follows from the fact that switching with respect to a subset $\mathcal{U} \in \mathsf{V}(\mathsf{G})$, and then with respect to a subset $\mathcal{V} \in \mathsf{V}(\mathsf{G})$, is the same as switching with respect to $(\mathcal{V} \setminus \mathcal{U}) \cup (\mathcal{U} \setminus \mathcal{V})$ (see [49, p. 18]).

Example 2.5.9. The Shrikhande graph can be obtained through Seidel switching applied to the line graph $\ell(\mathsf{K}_{4,4})$ with respect to four independent vertices of the latter (see [49, Example 1.2.4]). Both are 6-regular graphs (hence, they are cospectral graphs by Theorem 2.5.5). Moreover, the former graph is a strongly regular graph $\mathsf{srg}(16, 6, 2, 2)$, whereas the line graph $\ell(\mathsf{K}_{4,4})$ is not. Consequently, these are nonisomorphic and cospectral (NICS) 6-regular graphs on 16 vertices.

2.5.3 The Godsil and McKay method

Another construction of cospectral pairs of graphs was offered by Godsil and McKay in [65].

Theorem 2.5.10. *Let* G *be a graph with an adjacency matrix of the form*

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{B} & \mathbf{N} \\ \mathbf{N}^{\mathrm{T}} & \mathbf{C} \end{pmatrix} \tag{2.5.3}$$

where the sum of each column in $\mathbf{N} \in \{0,1\}^{b \times c}$ *is either* $0, b$ *or* $\frac{b}{2}$. *Let* $\widehat{\mathbf{N}}$ *be the matrix obtained by replacing each column* $\underline{c}$ *in* $\mathbf{N}$ *whose sum of elements is* $\frac{b}{2}$ *with its complement* $\mathbf{1}_n - \underline{c}$. *Then, the modified graph* $\widehat{\mathsf{G}}$ *whose adjacency matrix is given by*

$$\mathbf{A}(\widehat{\mathsf{G}}) = \begin{pmatrix} \mathbf{B} & \widehat{\mathbf{N}} \\ \widehat{\mathbf{N}}^{\mathrm{T}} & \mathbf{C} \end{pmatrix} \tag{2.5.4}$$

is cospectral with G.

Two examples of pairs of NICS graphs are presented in Section 1.8.3 of [20].

2.5.4 Graphs resulting from the duplication and corona graphs

Definition 2.5.11. [123] Let G be a graph with a vertex set $\mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$, and consider a copy of G with a vertex set $\mathsf{G} = \{u_1, \ldots, u_n\}$, where u_i is a duplicate of the vertex v_i. For each $i \in [n]$, connect the vertex u_i to all the neighbors of v_i in G, and then delete all edges in G. Similarly, for each $i \in [n]$, connect the vertex v_i to all the neighbors of u_i in the copied graph, and then delete all edges in the copied graph. The resulting graph, which has $2n$ vertices is called the *duplication graph* of G, and is denoted by $\mathrm{Du}(\mathsf{G})$ (see Figure 2.4).

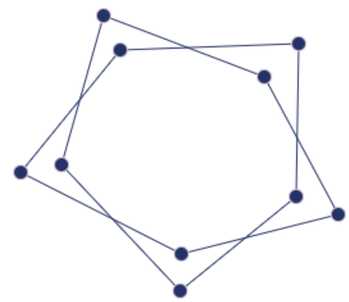


Figure 2.4: The duplication graph $\mathrm{Du}(\mathsf{C}_5)$ (see Definition 2.5.11).

Definition 2.5.12. [64] Let G_1 and G_2 be graphs on disjoint vertex sets of n_1 and n_2 vertices, and with m_1 and m_2 edges, respectively. The *corona* of G_1 and G_2, denoted by $\mathsf{G}_1 \circ \mathsf{G}_2$, is a graph on $n_1 + n_1 n_2$ vertices obtained by taking one copy of G_1 and n_1 copies of G_2, and then connecting, for each $i \in [n_1]$, the i-th vertex of G_1 to each vertex in the i-th copy of G_2 (see Figure 2.5).

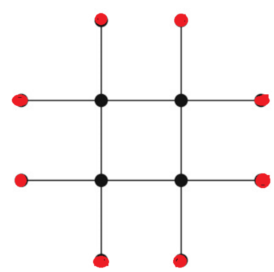


Figure 2.5: The corona graph $\mathsf{C}_4 \circ (2\,\mathsf{K}_1)$ (see Definition 2.5.12) consists of a single copy of C_4 (represented by the black vertices) and four copies of $2\,\mathsf{K}_1$ (represented by the red vertices).

Definition 2.5.13. [77] The *edge corona* of G_1 and G_2, denoted by $\mathsf{G}_1 \diamond \mathsf{G}_2$, is defined as the graph obtained by taking one copy of G_1 and $m_1 = \left|\mathsf{E}(\mathsf{G}_1)\right|$ copies of G_2, and then connecting, for each $j \in [m_1]$, the two end-vertices of the j-th edge of G_1 to every vertex in the j-th copy of G_2 (see Figure 2.6).

Definition 2.5.14. Let G_1 and G_2 be graphs with disjoint vertex sets of n_1 and n_2 vertices, respectively. Let $Du(\mathsf{G}_1)$ be the duplication graph of G_1 with vertex set $\mathsf{V}(\mathsf{G}_1) \cup \mathrm{U}(\mathsf{G}_1)$, where $\mathsf{V}(\mathsf{G}_1) = \{v_1, \ldots, v_{n_1}\}$ and with the duplicated vertex set $\mathrm{U}(\mathsf{G}_1) = \{u_1, \ldots, u_{n_1}\}$ (see Definition 2.5.11). The *duplication corona graph*, denoted by $\mathsf{G}_1 \boxminus \mathsf{G}_2$, is the graph obtained from $\mathrm{Du}(\mathsf{G}_1)$ and n_1 copies of G_2 by connecting, for each $i \in [n_1]$, the vertex $v_i \in \mathsf{V}(\mathsf{G}_1)$ of the graph $\mathrm{Du}(\mathsf{G}_1)$ to every vertex in the i-th copy of G_2 (see Figure 2.7).

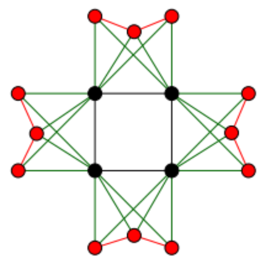


Figure 2.6: The edge-corona graph $C_4 \diamond P_3$ (see Definition 2.5.13.

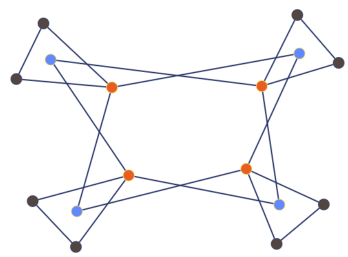


Figure 2.7: The duplication corona graph $C_4 \boxminus K_2$ (see Definition 2.5.14).

Definition 2.5.15. Let G_1 and G_2 be graphs with disjoint vertex sets of n_1 and n_2 vertices, respectively. Let $\mathrm{Du}(G_1)$ be the duplication graph of G_1 with the vertex set $V(G_1) \cup U(G_1)$, where $V(G_1) = \{v_1, \ldots, v_{n_1}\}$ and the duplicated vertex set $U(G_1) = \{u_1, \ldots, u_{n_1}\}$ (see Definition 2.5.11). The *duplication neighborhood corona*, denoted by $G_1 \boxplus G_2$, is the graph obtained from $\mathrm{Du}(G_1)$ and n_1 copies of G_2 by connecting the neighbors of the vertex $v_i \in V(G_1)$ of $\mathrm{Du}(G_1)$ to every vertex in the i-th copy of G_2 for $i \in [n_1]$ (see Figure 2.8).

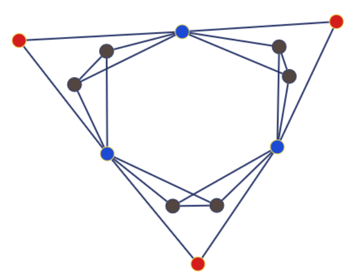


Figure 2.8: The duplication neighborhood corona $K_3 \boxplus K_2$ (see Definition 2.5.15).

Definition 2.5.16. Let G_1 and G_2 be graphs with disjoint vertex sets of n_1 and n_2 vertices, respectively. Let $\mathrm{Du}(\mathsf{G}_1)$ be the duplication graph of G_1 with vertex set $\mathsf{V}(\mathsf{G}_1) \cup \mathrm{U}(\mathsf{G}_1)$, where $\mathsf{V}(\mathsf{G}_1) = \{v_1, \ldots, v_{n_1}\}$ is the vertex set of G_1 and $\mathrm{U}(\mathsf{G}_1) = \{u_1, \ldots, u_{n_1}\}$ is the duplicated vertex set. The *duplication edge corona*, denoted by $\mathsf{G}_1 \boxplus \mathsf{G}_2$, is the graph obtained from $\mathrm{Du}(\mathsf{G}_1)$ and $\left|\mathsf{E}(\mathsf{G}_1)\right|$ copies of G_2 by connecting each of the two vertices $v_i, v_j \in \mathsf{V}(\mathsf{G}_1)$ of $\mathrm{Du}(\mathsf{G}_1)$ to every vertex in the k-th copy of G_2 whenever $\{v_i, v_j\} = e_k \in \mathsf{E}(\mathsf{G}_1)$ (see Figure 2.9).

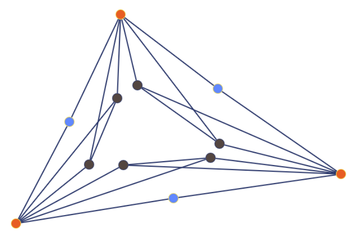


Figure 2.9: The duplication edge corona $\mathsf{K}_3 \boxplus \mathsf{K}_2$ (see Definition 2.5.16).

Definition 2.5.17. Consider two graphs G_1 and G_2 with n_1 and n_2 vertices and, respectively. The *closed neighborhood corona* of G_1 and G_2, denoted by $\mathsf{G}_1 \underline{\boxtimes} \mathsf{G}_2$, is a new graph obtained by creating n_1 copies of G_2. Each vertex of the i^{th} copy of G_2 is then connected to the i^{th} vertex and neighborhood of the i^{th} vertex of G_1 (see Figure 2.10).

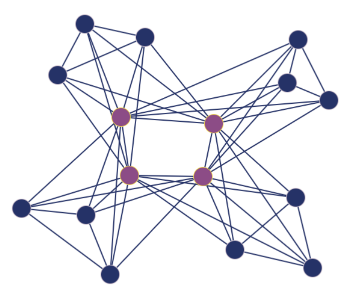


Figure 2.10: The closed neighborhood corona of the 4-length cycle C_4 and the triangle K_3, denoted by $\mathsf{C}_4 \underline{\boxtimes} \mathsf{K}_3$ (see Definition 2.5.17).

Theorem 2.5.18. *[4] Let $\mathsf{G}_1, \mathsf{H}_1$ be r_1-regular, cospectral graphs, and let G_2 and H_2 be r_2-regular, cospectral, and nonisomorphic (NICS) graphs. Then, the following holds:*

- *The duplication corona graphs $\mathsf{G}_1 \boxminus \mathsf{G}_2$ and $\mathsf{H}_1 \boxminus \mathsf{H}_2$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*

- *The duplication neighborhood corona $\mathsf{G}_1 \boxbar \mathsf{G}_2$ and $\mathsf{H}_1 \boxbar H_2$ are $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$-NICS irregular graphs.*

- *The duplication edge corona* $\mathsf{G}_1 \boxplus \mathsf{G}_2$ *and* $\mathsf{H}_1 \boxplus \mathsf{H}_2$ *are* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$*-NICS irregular graphs.*

Question 2.5.19. *Are the irregular graphs in Theorem 2.5.18 also cospectral with respect to the normalized Laplacian matrix?*

Theorem 2.5.20. *[135] Let* G_1 *and* G_2 *be cospectral regular graphs, and let* H *be an arbitrary graph. Then, the following holds:*

- *The closed neighborhood corona* $\mathsf{G}_1 \underline{\boxtimes} \mathsf{H}$ *and* $\mathsf{G}_2 \underline{\boxtimes} \mathsf{H}$ *are* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$*-NICS irregular graphs.*

- *The closed neighborhood corona* $\mathsf{H} \underline{\boxtimes} \mathsf{G}_1$ *and* $\mathsf{H} \underline{\boxtimes} \mathsf{G}_2$ *are* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}\}$*-NICS irregular graphs.*

Question 2.5.21. *Are the irregular graphs in Theorem 2.5.20 also cospectral with respect to the normalized Laplacian matrix?*

2.5.5 Graphs constructions based on the subdivision and bipartite incidence graphs

Definition 2.5.22. [49] Let G be a graph. The *subdivision graph* of G, denoted by $\mathrm{S}(\mathsf{G})$, is obtained from G by inserting a new vertex into every edge of G. Subdivision is the process of adding a new vertex along an edge, effectively splitting the edge into two edges connected in series through the new vertex (see Figure 2.11).

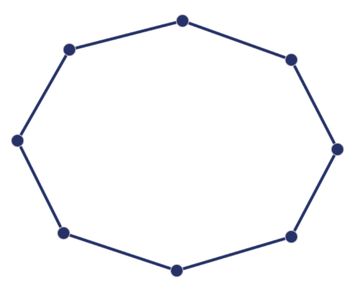


Figure 2.11: The subdivision graph of a 4-length cycle, denoted by $\mathrm{S}(\mathsf{C}_4)$, which is an 8-length cycle C_8 (see Definition 2.5.22).

Definition 2.5.23. [117] Let G be a graph. The *bipartite incidence graph* of G, denoted by $\mathrm{B}(\mathsf{G})$, is a bipartite graph constructed as follows: For each edge $e \in \mathsf{G}$, a new vertex u_e is added to the vertex set of G. The vertex u_e is then made adjacent to both endpoints of e in G (see Figure 2.12).

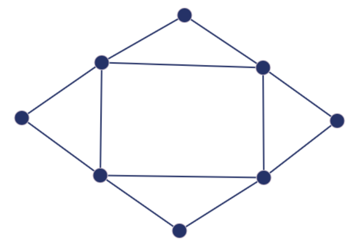


Figure 2.12: The bipartite incidence graph of a length-4 cycle C_4, denoted by $\mathrm{B}(\mathsf{C}_4)$ (see Definition 2.5.23).

Let G_1 and G_2 be two vertex-disjoint graphs with n_1 and n_2 vertices, and m_1 and m_2 edges, respectively. Four binary operations on these graphs are defined as follows.

Definition 2.5.24. The *subdivision-vertex-bipartite-vertex join* of graphs G_1 and G_2, denoted $\mathrm{S}(\mathsf{G}_1) \ddot{\vee} \mathrm{B}(\mathsf{G}_2)$, is the graph obtained from $\mathrm{S}(\mathsf{G}_1)$ and $\mathrm{B}(\mathsf{G}_2)$ by connecting each vertex of $\mathsf{V}(\mathsf{G}_1)$ (that is the subset of the original vertices in the vertex set of $\mathrm{S}(\mathsf{G}_1)$) with every vertex of $\mathsf{V}(\mathsf{G}_2)$ (that is the subset of the original vertices in the vertex set of $\mathrm{B}(\mathsf{G}_2)$).

By Definitions 2.5.22, 2.5.23, and 2.5.24, it follows that the graph $\mathrm{S}(\mathsf{G}_1) \ddot{\vee} \mathrm{B}(\mathsf{G}_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $n_1 n_2 + 2m_1 + 3m_2$ edges. Figure 2.13 displays the graph $\mathrm{S}(\mathsf{C}_4) \ddot{\vee} \mathrm{B}(\mathsf{P}_3)$.

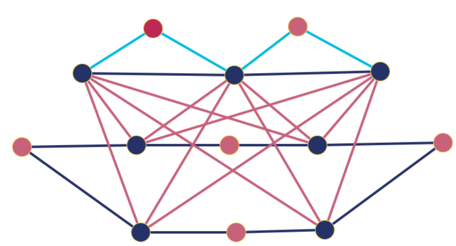


Figure 2.13: The graph $\mathrm{S}(\mathsf{C}_4) \ddot{\vee} \mathrm{B}(\mathsf{P}_3)$ (see Definition 2.5.24). The black vertices represent the vertices of the length-4 cycle C_4 and the vertices of the path P_3. The additional vertices in the subdivision graph $\mathrm{S}(\mathsf{C}_4)$ are the four red vertices located at the bottom of this figure (as shown in Figure 2.11). Similarly, the additional vertices in the bipartite incidence graph of the path P_3 are the two red vertices located at the top of this figure. The edges of the graph include the following: edges connecting each black vertex of C_4 to each black vertex of P_3, edges between the black and red vertices at the bottom of the figure (that correspond to the subdivision of C_4), and the four (light blue) edges connecting the two top red vertices to the top black vertices of the figure (that correspond to the bipartite incidence graph of P_3).

Definition 2.5.25. The *subdivision-edge-bipartite-edge join* of G_1 and G_2, denoted by $\mathrm{S}(\mathsf{G}_1) \overset{=}{\vee} \mathrm{B}(\mathsf{G}_2)$, is the graph obtained from $\mathrm{S}(\mathsf{G}_1)$ and $\mathrm{B}(\mathsf{G}_2)$ by connecting each vertex of $\mathsf{V}(\mathrm{S}(\mathsf{G}_1)) \setminus \mathsf{V}(\mathsf{G}_1)$ (that is the subset of the added vertices in the vertex set of $\mathrm{S}(\mathsf{G}_1)$) with every vertex of $\mathsf{V}(\mathrm{B}(\mathsf{G}_2)) \setminus \mathsf{V}(\mathsf{G}_2)$ (that is the subset of the added vertices in the vertex set of $\mathrm{B}(\mathsf{G}_2)$).

By Definitions 2.5.22, 2.5.23, and 2.5.25, it follows that the graph $\mathrm{S}(\mathsf{G}_1) \overset{=}{\vee} \mathrm{B}(\mathsf{G}_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1 m_2 + 2m_1 + 3m_2$ edges. Figure 2.14 displays the graph $\mathrm{S}(\mathsf{C}_4) \overset{=}{\vee} \mathrm{B}(\mathsf{P}_3)$.

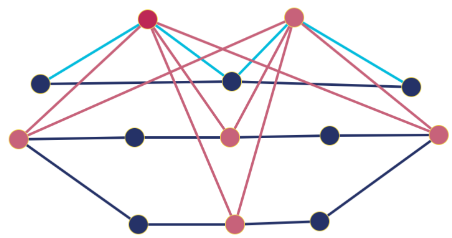


Figure 2.14: The graph $\mathrm{S}(\mathsf{C}_4) \overset{=}{\vee} \mathrm{B}(\mathsf{P}_3)$ (see Definition 2.5.25). In comparison to Figure 2.13, edges connecting each black vertex of C_4 to each black vertex of P_3 are deleted and replaced in this figure by edges connecting each red vertex of C_4 to each red vertex of P_3.

Definition 2.5.26. The *subdivision-edge-bipartite-vertex join* of G_1 and G_2, denoted by $\mathrm{S}(\mathsf{G}_1) \overline{\dot{\vee}} \mathrm{B}(\mathsf{G}_2)$, is the graph obtained from $\mathrm{S}(\mathsf{G}_1)$ and $\mathrm{B}(\mathsf{G}_2)$ by connecting each vertex of $\mathsf{V}(\mathrm{S}(\mathsf{G}_1)) \setminus \mathsf{V}(\mathsf{G}_1)$ (that is the subset of the added vertices in the vertex set of $\mathrm{S}(\mathsf{G}_1)$) with every vertex of $\mathsf{V}(\mathsf{G}_2)$ (that is the subset of the original vertices in the vertex set of $\mathrm{B}(\mathsf{G}_2)$).

By Definitions 2.5.22, 2.5.23, and 2.5.26, it follows that the graph $\mathrm{S}(\mathsf{G}_1) \overline{\dot{\vee}} \mathrm{B}(\mathsf{G}_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $m_1 n_2 + 2m_1 + 3m_2$ edges. Figure 2.15 displays the graph $\mathrm{S}(\mathsf{C}_4) \overline{\dot{\vee}} \mathrm{B}(\mathsf{P}_3)$.

Definition 2.5.27. The *subdivision-vertex-bipartite-edge join* of G_1 and G_2, denoted by $\mathrm{S}(\mathsf{G}_1) \dot{\overline{\vee}} \mathrm{B}(\mathsf{G}_2)$, is the graph obtained from $\mathrm{S}(\mathsf{G}_1)$ and $\mathrm{B}(\mathsf{G}_2)$ by connecting each vertex of $\mathsf{V}(\mathsf{G}_1)$ (that is the subset of the original vertices in the vertex set of $\mathrm{S}(\mathsf{G}_1)$) with every vertex of $\mathsf{V}(\mathrm{B}(\mathsf{G}_2)) \setminus \mathsf{V}(\mathsf{G}_2)$ (that is the subset of the added vertices in the vertex set of $\mathrm{B}(\mathsf{G}_2)$).

By Definitions 2.5.22, 2.5.23, and 2.5.27, it follows that the graph $\mathrm{S}(\mathsf{G}_1) \dot{\overline{\vee}} \mathrm{B}(\mathsf{G}_2)$ has $n_1 + n_2 + m_1 + m_2$ vertices and $n_1 m_2 + 2m_1 + 3m_2$ edges. Figure 2.16 displays the graph $\mathrm{S}(\mathsf{C}_4) \dot{\overline{\vee}} \mathrm{B}(\mathsf{P}_3)$.

We next present the main result of this subsection, which motivates the four binary graph operations introduced in Definitions 2.5.24–2.5.27.

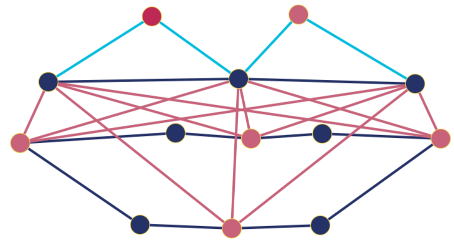


Figure 2.15: The graph $\mathsf{S}(\mathsf{C}_4)\,\overline{\dot{\vee}}\,\mathsf{B}(\mathsf{P}_3)$ (see Definition 2.5.26). In comparison to Figure 2.13, edges connecting each black vertex of C_4 to each black vertex of P_3 are deleted and replaced in this figure by edges connecting each red vertex of C_4 to each black vertex of P_3.

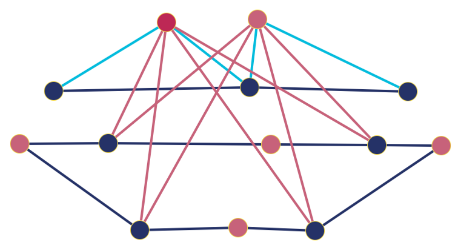


Figure 2.16: The graph $\mathsf{S}(\mathsf{C}_4)\,\dot{\overline{\vee}}\,\mathsf{B}(\mathsf{P}_3)$ (see Definition 2.5.27). In comparison to Figure 2.13, edges connecting each black vertex of C_4 to each black vertex of P_3 are deleted and replaced in this figure by edges connecting each black vertex of C_4 to each red vertex of P_3.

Theorem 2.5.28. *[54] Let G_i , H_i, where $i \in \{1,2\}$, be regular graphs, where G_1 need not be different from H_1. If G_1 and H_1 are* **A***-cospectral, and G_2 and H_2 are* **A***-cospectral, then*

- $\mathsf{S}(\mathsf{G}_1)\,\ddot{\vee}\,\mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1)\,\ddot{\vee}\,\mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*
- $\mathsf{S}(\mathsf{G}_1)\,\overline{\overline{\vee}}\,\mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1)\,\overline{\overline{\vee}}\,\mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*
- $\mathsf{S}(\mathsf{G}_1)\,\overline{\dot{\vee}}\,\mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1)\,\overline{\dot{\vee}}\,\mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*
- $\mathsf{S}(\mathsf{G}_1)\,\dot{\overline{\vee}}\,\mathsf{B}(\mathsf{G}_2)$ *and* $\mathsf{S}(\mathsf{H}_1)\,\dot{\overline{\vee}}\,\mathsf{B}(\mathsf{H}_2)$ *are irregular* $\{\mathbf{A}, \mathbf{L}, \mathcal{L}\}$*-NICS graphs.*

In light of Theorem 2.5.28, the following questions naturally arises.

Question 2.5.29. *Are the graphs in Theorem 2.5.28 also cospectral with respect to the signless Laplacian matrix (i.e.,* **Q***-cospectral)?*

2.5.6 Connected irregular NICS graphs

The (joint) cospectrality of regular graphs with respect to their adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices can be asserted by verifying their cospectrality with respect to only one of these matrices. In other words, regular graphs are X-cospectral for some $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$ if and only if they are cospectral with respect to all these matrices (see Proposition 2.3.15). For irregular graphs, this does not hold in general. Following [25], it is natural to ask the following question:

Question 2.5.30. *Are there pairs of irregular graphs that are* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$*-NICS, i.e., X-NICS with respect to every* $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$*?*

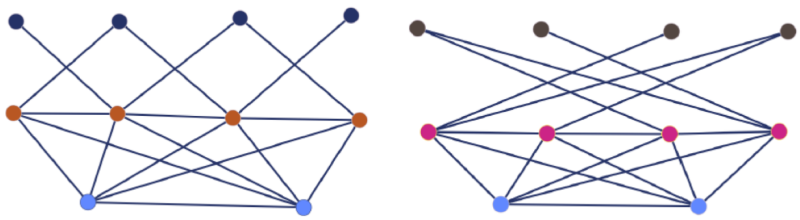
This question remained open until Hamud and Berman recently proposed a method for constructing pairs of irregular graphs that are X-cospectral with respect to every $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$, providing explicit constructions [74]. Building on that work, Sason demonstrated in [124] that for every even integer $n \geq 14$, there exist two connected, irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs on n vertices with identical independence, clique, and chromatic numbers, yet distinct Lovász ϑ-functions. We now present the preliminary definitions required to outline the relevant results in [74] and [124], and the construction of such cospectral irregular X-NICS graphs for all $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$.

Definition 2.5.31 (Neighbors splitting join of graphs)**.** [101] Let G and H be graphs with disjoint vertex sets, and let $\mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$. The *neighbors splitting (NS) join* of G and H is obtained by adding vertices $v'_1, \ldots, v'_n$ to the vertex set of $\mathsf{G} \vee \mathsf{H}$ and connecting v'_i to v_j if and only if $\{v_i, v_j\} \in \mathsf{E}(\mathsf{G})$. The NS join of G and H is denoted by $\mathsf{G} \veebar \mathsf{H}$.

Definition 2.5.32 (Nonneighbors splitting join of graphs [73, 74])**.** Let G and H be graphs with disjoint vertex sets, and let $\mathsf{V}(\mathsf{G}) = \{v_1, \ldots, v_n\}$. The *nonneighbors splitting (NNS) join* of G and H is obtained by adding vertices $v'_1, \ldots, v'_n$ to the vertex set of $\mathsf{G} \vee \mathsf{H}$ and connecting v'_i to v_j, with $i \neq j$, if and only if $\{v_i, v_j\} \notin \mathsf{E}(\mathsf{G})$. The NNS join of G and H is denoted by $\mathsf{G} \underline{\veebar} \mathsf{H}$.

Remark 2.5.33. In general, $\mathsf{G} \veebar \mathsf{H} \ncong \mathsf{H} \veebar \mathsf{G}$ and $\mathsf{G} \underline{\veebar} \mathsf{H} \ncong \mathsf{H} \underline{\veebar} \mathsf{G}$ (unless $\mathsf{G} \cong \mathsf{H}$).

Example 2.5.34 (NS and NNS join of graphs [74])**.** Figure 2.17 shows the NS and NNS joins of the path graphs P_4 and P_2, denoted by $\mathsf{P}_4 \underline{\vee} \mathsf{P}_2$ and $\mathsf{P}_4 \underline{\underline{\vee}} \mathsf{P}_2$, respectively.



(a) $P_4 \underline{\vee} P_2$ (b) $P_4 \underline{\underline{\vee}} P_2$

Figure 2.17: The neighbors splitting (NS) and nonneighbors splitting (NNS) joins of the path graphs P_4 and P_2 are depicted, respectively, on the left and right-hand sides of this figure. The NS and NNS joins of graphs are, respectively, denoted by $\mathsf{P}_4 \underline{\vee} \mathsf{P}_2$ and $\mathsf{P}_4 \underline{\underline{\vee}} \mathsf{P}_2$ [74].

Theorem 2.5.35 (Irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs)**.** *Let G_1 and H_1 be regular and cospectral graphs, and let G_2 and H_2 be regular, nonisomorphic, and cospectral (NICS) graphs. Then, the following statements hold:*

(1) The NS-join graphs $\mathsf{G}_1 \underline{\vee} \mathsf{G}_2$ and $\mathsf{H}_1 \underline{\vee} \mathsf{H}_2$ are irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs.

(2) The NNS join graphs $\mathsf{G}_1 \underline{\underline{\vee}} \mathsf{G}_2$ and $\mathsf{H}_1 \underline{\underline{\vee}} \mathsf{H}_2$ are irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs.

The proof of Theorem 2.5.35 is provided in [73, 74], and it relies heavily on the notion of the Schur complement (see Theorem 2.2.2). The interested reader is referred to the recently published paper [74] for further details.

Relying on Theorem 2.5.35, the following result is stated and proved in [124].

Theorem 2.5.36 (On irregular NICS graphs)**.** *For every even integer $n \geq 14$, there exist two connected, irregular $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$-NICS graphs on n vertices with identical independence, clique, and chromatic numbers, yet their Lovász ϑ-functions are distinct.*

That result is in fact strengthened in Section 4.2 of [124], and the interested reader is referred to that paper for further details. The proof of Theorem 2.5.36 is also constructive, providing explicit such graphs.

2.6 Open questions and outlook

We conclude this chapter by highlighting some of the most significant open questions in the fascinating area of research related to cospectral nonisomorphic graphs and graphs determined by their spectrum, leaving them as topics for further future study.

2.6.1 Haemers' conjecture

Haemers' conjecture [69, 70], a prominent topic in spectral graph theory, posits that almost all graphs are uniquely determined by their adjacency spectrum. This conjecture suggests that for large graphs, the probability of having two non-isomorphic graphs, on a large number of vertices, sharing the same adjacency spectrum is negligible. The conjecture has inspired extensive research, including studies on specific graph families, cospectrality, and algebraic graph invariants, contributing to deeper insights into the relationship between graph structure and eigenvalues. Haemers' conjecture is stated formally as follows.

Definition 2.6.1. For $n \in \mathbb{N}$, let $I(n)$ to be the numbers of distinct graphs on n vertices, up to isomorphism. For $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$, let $\alpha_{\mathcal{X}}(n)$ be the number of $\mathcal{X}$-DS graphs on n vertices, up to isomorphism, and (for the sake of simplicity of notation) let $\alpha(n) \triangleq \alpha_{\{A\}}(n)$

Conjecture 2.6.2. *[69] For sufficiently large n, almost all of the graphs are DS, i.e.,*

$$\lim_{n\to\infty} \frac{\alpha(n)}{I(n)} = 1. \tag{2.6.1}$$

Several results lend support to this conjecture [69, 89, 145], but a complete proof in its full generality remains elusive. By [66], the number of graphs with n vertices up to isomorphism is

$$I(n) = (1 - \epsilon(n)) \frac{2^{n(n-1)/2}}{n!}, \tag{2.6.2}$$

where $\lim_{n\to\infty} \epsilon(n) = 0$. By Stirling's approximation for $n!$ and straightforward algebra, $I(n)$ can be verified to be given by

$$I(n) = 2^{\frac{n(n-1)}{2} \left(1-\epsilon(n)\right)}. \tag{2.6.3}$$

It is shown in [89] that the number of DS graphs on n vertices is at least e^{cn} for some constant $c > 0$ (see also the discussion in Section 2.4.5). In light of Remark 2.3.13, Conjecture 2.6.2 leads to the following question.

Question 2.6.3. *For what minimal subset* $\mathcal{X} \subset \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$ *(if any) does the limit*

$$\lim_{n\to\infty} \frac{\alpha_{\mathcal{X}}(n)}{I(n)} = 1 \tag{2.6.4}$$

hold?

Some computer results somewhat support Haemers' Conjecture. Approximately 80% of the graphs with 12 vertices are DS [23]. A new class of graphs is defined in [145], offering an algorithmic method for finding all the $\{\mathbf{A}, \overline{\mathbf{A}}\}$-cospectral mates of a graph G in this class (i.e., graphs that are $\{\mathbf{A}, \overline{\mathbf{A}}\}$-cospectral with G). Using this algorithm, they found that out of $10,000$ graphs with 50 vertices, chosen uniformly at random from this class, at least 99.5% of them were $\{\mathbf{A}, \overline{\mathbf{A}}\}$-DS.

2.6.2 DS properties of structured graphs

This chapter explores various structures of graph families determined by their spectrum with respect to one or more of their associated matrices, as well as the related problem of constructing pairs of nonisomorphic graphs that are cospectral with respect to some or all of these matrices. Several questions are interspersed throughout the chapter (see Questions 2.4.47, 2.5.19, 2.5.21, 2.5.29, and 2.5.30), which remain open for further study.

In addition to serving as a survey on spectral graph determination, this chapter suggests a new alternative proof to Theorem 3.3 in [104], asserting that all Turán graphs are determined by their adjacency spectrum (see Theorem 2.4.21). This proof is based on a derivation of the adjacency spectrum of the family of Turán graphs (see Theorem 2.4.18). Since these graphs are generally bi-regular multipartite graphs (i.e., their vertices have only two possible degrees, which in this case are consecutive integers), it does not necessarily imply that Turán graphs are determined by the spectrum of some other associated matrices, such as the Laplacian, signless Laplacian, or normalized Laplacian matrices. Determining whether this holds is an open question.

The distance matrix of a connected graph is the symmetric matrix whose columns and rows are indexed by the graph vertices, and its entries are equal to the pairwise distances between the corresponding vertices. The distance spectrum is the multiset of eigenvalues of the distance matrix, and its characterization has been a subject of fruitful research (see [7,78] for comprehensive surveys on the distance spectra of graphs, and [10,75] on the spectra of graphs with respect to variants of the distance matrix). Nonisomorphic graphs may share an identical adjacency spectrum, as well as an identical distance spectrum, and therefore be graphs that are not determined by either their adjacency or distance spectrum. This holds, e.g., for the Shrikhande graph and the line graph of the complete bipartite graph $\mathsf{K}_{4,4}$, which are nonisomorphic strongly regular graphs with the identical parameters $(16, 6, 2, 2)$, sharing an identical $\mathbf{A}$-spectrum given by $\{6, [2]^6, [-2]^6\}$ and an identical distance spectrum given by $\{24, [0]^9, [-4]^6\}$. There exist, however, graphs that are determined by their distance spectrum but are not determined by their adjacency spectrum. In this context, [80] proves that complete multipartite graphs are uniquely determined by their distance spectra (this result confirms a conjecture proposed in [94] and extends the special case established there for complete bipartite graphs). A Turán graph is, in particular, determined by its distance spectrum [80], and by its adjacency spectra (see Theorem 2.4.21 here). On the other hand, while complete multipartite graphs are not generally determined by their adjacency spectra (e.g., for complete bipartite graphs, see Theorem 2.4.7), they are necessarily determined by their distance spectra [80]. Another family of graphs that are determined by their distance spectrum but are not determined by their adjacency spectrum, named d-cube graphs, is provided in [87, 88]. These graphs have their vertices represented by binary n-length tuples, where any two vertices are adjacent if and only if their corresponding binary n-tuples differ in one coordinate; it is also shown that these graphs have exactly three distinct distance eigenvalues. The question of which multipartite graphs, or graphs in general, are determined by their distance spectra remains open.

Another newly obtained proof presented in this chapter refers to a necessary and sufficient condition in [104] for complete bipartite graphs to be **A**-DS (see Theorem 2.4.7 and Remark 2.4.8 here). These graphs are also bi-regular, and the two possible vertex degrees can differ by more than one. Both of these newly obtained proofs, discussed in Section 2.4, provide insights into the broader question of which (structured) multipartite graphs are determined by their adjacency spectrum or, more generally, by the spectra of some of their associated matrices.

Even if Haemers' conjecture is eventually proved in its full generality, it remains surprising when a new family of structured graphs is shown to be DS (or X-DS, more generally). This is because for certain structured graphs, such as strongly regular graphs and trees, their spectra often fail to uniquely determine them [50,51]. This stark contrast between the fact that almost all random graphs of high order are likely to be DS and the existence of interesting structured graphs that are not DS has significant implications.

In addition to the questions posed earlier in this chapter, we raise the following additional concrete question:

Question 2.6.4. By Theorem 2.4.40, the family of generalized friendship graphs $F_{p,q}$ is $\mathcal{L}$-DS. Are these graphs also DS with respect to their other associated matrices?

We speculate that the DS property of a graph correlates, to some extent, with the number of symmetries that the graph possesses, and we hypothesize that the size of the automorphism group of a graph can partially indicate whether it is DS.

A justification for this claim is that the automorphism group of a graph reflects its symmetries, which can influence the eigenvalues of its adjacency matrix. Highly symmetric graphs (i.e., those with large automorphism groups) often exhibit eigenvalue multiplicities and patterns that are shared by other nonisomorphic graphs, making such graphs less likely to be DS. Conversely, graphs with trivial automorphism groups are typically less symmetric and may have eigenvalues that uniquely determine their structure, increasing the likelihood that they are DS. As noted in [66], almost all graphs have trivial automorphism groups. This observation aligns with the conjecture that most graphs are DS, as the absence of symmetry reduces the likelihood of two nonisomorphic graphs sharing the same spectrum.

It is noted, however, that the DS property of graphs is not solely dictated by their automorphism groups. Specifically, a graph with a large automorphism group can still be DS if its eigenvalues uniquely encode its structure (see Section 2.4.7). In contrast to these DS graphs, other graphs with trivial automorphism groups are not guaranteed to be DS; in such cases, the spectrum might not capture enough structural information to uniquely determine the graph. Typically, graphs with a small number of distinct eigenvalues seem to be, in general, the hardest graphs to distinguish by their spectrum. As noted in [52], it seems that most graphs with few eigenvalues (e.g., most of the strongly regular graphs) are not determined by their spectrum, which served as one of the motivations of the work in [52] in studying graphs whose normalized Laplacian has three eigenvalues.

To conclude, the size of the automorphism group of a graph can provide some indication of whether it is DS, but it is not a definitive criterion. While large automorphism groups often correlate with the graph not being DS due to shared eigenvalues among nonisomorphic graphs, this is not an absolute rule. Therefore, the claim should be understood as a general observation that requires qualification to account for exceptions.

Chapter 3

The graphs of pyramids

Chapter Overview

For natural numbers $k < n$ we study the graphs $T_{n,k} := K_k \vee \overline{K_{n-k}}$. For $k = 1$, $T_{n,1}$ is the star S_{n-1}. For $k > 1$ we refer to $T_{n,k}$ as a *graph of pyramids*. We prove that the graphs of pyramids are determined by their spectrum, and that a star S_n is determined by its spectrum iff n is prime. We also show that the graphs $T_{n,k}$ are completely positive iff $k \leq 2$.

3.1 Introduction

In the second year of my undergraduate studies, I took an advanced course in matrix theory. This chapter is one of the outcomes of this course. Two of the topics studied in the course were completely positive (CP) matrices and graphs and cospectrality of graphs. In one of the problems in the course we used the Schur complement to show that the graph T_n, the graph of $n-2$ triangles with a common base, is CP and DS.

In this chapter, we generalize T_n to $\mathsf{T}_{n,k}$ - graphs that consist of $n-k$ pyramids K_{k+1} with K_k as a common base and use the Schur complement to compute their spectrum. For $k > 1$ we refer to $\mathsf{T}_{n,k}$ as graphs of pyramids and show that they are determined by their spectrum (DS). For $k = 1$, $\mathsf{T}_{n,1}$ is a star S_n, and it is DS if and only if $n-1$ is prime. Since the motivation for this chapter was the proof that $\mathsf{T}_n = \mathsf{T}_{n,2}$ is completely positive, we start the chapter with a short survey of complete positivity. We summarize the chapter with a table that classifies the graphs $\mathsf{T}_{n,k}$ according to their being DS/CP.

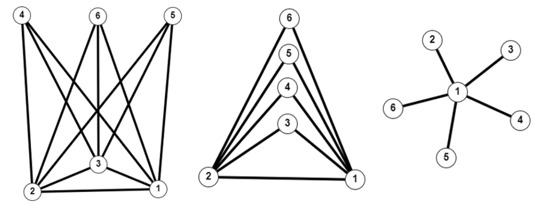


Figure 3.1: the graphs $\mathsf{T}_{6,3}$, $\mathsf{T}_{6,2}$ and $\mathsf{T}_{6,1}$

3.2 Completely positive matrices and graphs

A matrix $\mathbf{A}$ is *completely positive* if there exists a nonnegative matrix $\mathbf{B}$ s.t. $\mathbf{A} = \mathbf{B}\mathbf{B}^T$ (see Definition 2.2.4). The readers are referred to [16] and [130] for the properties and the many applications of CP matrices. A necessary condition for a matrix to be completely positive is that it is *doubly nonnegative* i.e. nonnegative and positive semidefinite. This condition is not sufficient.

Example 3.2.1. The matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix} \tag{3.2.1}$$

is doubly nonnegative but not CP.

When is the necessary condition sufficient ?

Definition 3.2.2. 1. The graph of a matrix $\mathbf{A} \in \mathrm{Sym}_n$, denoted $\mathsf{G}(\mathbf{A})$, is a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ where

$$\mathsf{V} = \langle n \rangle\ ,\ \mathsf{E} = \left\{ (i, j)\ \mid\ i \neq j\ ,\ a_{i,j} \neq 0 \right\}. \tag{3.2.2}$$

2. If $\mathsf{G}(\mathbf{A}) = \mathsf{G}$, we say that $\mathbf{A}$ is a *matrix realization* of G.

Example 3.2.3. The graph of the matrix in Example 3.2.1 is a pentagon.

Definition 3.2.4. A graph G is *completely positive* (*CP graph*) if every doubly nonnegative matrix realization of G is completely positive.

Examples of CP graphs are :

- Graphs with less than 5 vertices [105].
- Bipartite graphs [13].
- T_n [86].

Definition 3.2.5. A *long odd cycle* is a cycle of odd length greater than 3.

Graphs that contain a long odd cycle are not completely positive [14]. Examples of graphs that contain a long odd cycle are B_n [15], a cycle C_n with a chord between vertices 1 and 3 (Figure 3.2), for every $n \geq 5$.

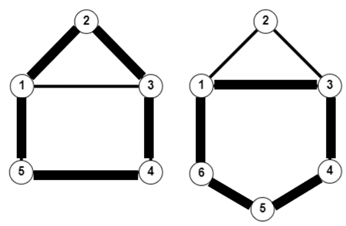


Figure 3.2: The graphs B_5 and B_6.

A characterization of completely positive graphs is :

Theorem 3.2.6. *[86] The following properties of a graph* G *are equivalent :*

1. G *is CP.*
2. G *does not contain a long odd cycle*
3. *The line graph of* G *is perfect.*

The equivalence between 2 and 3 was proved in [139].

3.3 The graphs T_n

Recall that the graph T_n consists of $n-2$ triangles with a common base.

Theorem 3.3.1. *The spectrum of* T_n *is*

$$\sigma(\mathsf{T}_n) = \left\{ \left(\frac{1 - \sqrt{8n-15}}{2} \right), (-1), (0)^{n-3}, \left(\frac{1 + \sqrt{8n-15}}{2} \right) \right\}. \tag{3.3.1}$$

Proof. The adjacency matrix of T_n is

$$\mathbf{A}(\mathsf{T}_n) = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & \cdots & \vdots \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.3.2}$$

Let $\mathbf{M} = \lambda\,\mathbf{I} - \mathbf{A}\,(\mathsf{T}_n)$ be the characteristic matrix of T_n. The characteristic polynomial of T_n is

$$\det(\mathbf{M}) = \det \begin{pmatrix} \lambda & -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & -1 & \cdots & -1 \\ -1 & -1 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \lambda & 0 & 0 \\ -1 & -1 & 0 & 0 & \ddots & 0 \\ -1 & -1 & 0 & 0 & \cdots & \lambda \end{pmatrix}. \tag{3.3.3}$$

Denote $\mathbf{A} = \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix}$, $\mathbf{B} = -\,\mathbf{J}_{2,n}$, $\mathbf{D} = \lambda \cdot \mathbf{I}_{n-2}$. Then

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{pmatrix}. \tag{3.3.4}$$

Using Theorem 2.2.2 , we get

$$\det(\mathbf{M}) = \det(\mathbf{D}) \cdot \det\left(\mathbf{M}/\mathbf{D}\right) = \det(\mathbf{D}) \cdot \det\left(\mathbf{A} - \mathbf{B}^T \mathbf{D}^{-1} \mathbf{B}\right) \tag{3.3.5}$$

$$= \det\left(\lambda \cdot \mathbf{I}_{n-2}\right) \cdot \det\left(\mathbf{A} - \mathbf{J}_{2,n} \cdot \frac{1}{\lambda}\, \mathbf{I}_{n-2} \cdot \mathbf{J}_{n,2}\right) \tag{3.3.6}$$

$$= (\lambda)^{n-2} \cdot \det\left(\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} - \frac{n-2}{\lambda} \cdot \mathbf{J}_{2,2}\right) = (\lambda)^{n-2} \cdot \det\begin{pmatrix} \lambda - \frac{n-2}{\lambda} & \frac{n-2}{\lambda} \\ \frac{n-2}{\lambda} & \lambda - \frac{n-2}{\lambda} \end{pmatrix} \tag{3.3.7}$$

$$= (\lambda)^{n-2} \cdot \left[\left(\lambda - \frac{n-2}{\lambda}\right)^2 - \left(\frac{n-2}{\lambda}\right)^2\right] = \lambda \cdot (\lambda + 1) \cdot \left(\lambda^2 - \lambda + 2 \cdot (2 - n)\right). \tag{3.3.8}$$

Hence, the spectrum is:

$$\sigma\left(\mathsf{T}_n\right)=\left\{\left(\frac{1-\sqrt{8n-15}}{2}\right),(-1),(0)^{n-3},\left(\frac{1+\sqrt{8n-15}}{2}\right)\right\}. \tag{3.3.9}$$

□

Observe that for $n = 3$ we get the spectrum of a triangle $\sigma(\mathsf{T}_3) = \sigma(\mathsf{K}_3) = \sigma(\mathsf{C}_3) = \left\{(-1)^2, (2)\right\}$.

In the next section we study the graphs $\mathsf{T}_{n,k}$, and show that for $k > 1$ they are DS (Theorem 3.4.4). Since $\mathsf{T}_n = \mathsf{T}_{n,2}$, we get the following theorem as special case of Theorem 3.4.4.

Theorem 3.3.2. *The graph T_n is determined by its spectrum.*

In the previous section we showed that T_n is CP. For $k > 2$, $\mathsf{T}_{n,k}$ contains a long odd cycle, so they are not CP. Hence T_n are the only graphs of pyramids that are both DS and CP.

3.4 The graphs $\mathsf{T}_{n,k}$

Let $k < n$ be natural numbers. Define $\mathsf{T}_{n,k} = \mathsf{K}_k \vee \overline{\mathsf{K}_{n-k}}$ (not to be confused with Turan graph $T(n,k)$). For $k = 1$, $\mathsf{T}_{n,1} = \mathsf{S}_n$, and for $k = 2$, $\mathsf{T}_{n,2} = \mathsf{T}_n$.

Since the graph $\mathsf{T}_{n,k}$, $k > 1$ consists of $n - k$ pyramids (K_{k+1}) that intersect in K_k we refer to it as a *graph of pyramids*.

Theorem 3.4.1. *The spectrum of $\mathsf{T}_{n,k}$ is*

$$\sigma(\mathsf{T}_{n,k})=\left\{(\lambda_2),(-1)^{k-1},(0)^{n-k-1},(\lambda_1)\right\}, \tag{3.4.1}$$

where $\lambda_1, \lambda_2 = \frac{(k-1)\pm\sqrt{(k-1)^2+4k(n-k)}}{2}$.

Remark 3.4.2. We precede the proof with a sanity check. For $k = 1$, we get the spectrum of $\mathsf{S}_n = \mathsf{K}_{1,n-1}$ (Theorem 2.4.4). For $k = 2$, we get the spectrum of T_n (Theorem 3.3.1).

Proof. (of Theorem 3.4.1) Let $k \leq n$ be natural numbers and denote $r := n - k$. The adjacency matrix of $\mathsf{T}_{n,k}$ is

$$\mathbf{A}(\mathsf{T}_{n,k})=\begin{pmatrix}\mathbf{J}_k-\mathbf{I}_k & \mathbf{J}_{k,r}\\ \mathbf{J}_{r,k} & \mathbf{0}_r\end{pmatrix}. \tag{3.4.2}$$

Let $\mathbf{M} = \lambda\,\mathbf{I}_n - \mathbf{A}(\mathsf{T}_{n,k})$ be the characteristic matrix of $\mathsf{T}_{n,k}$. The characteristic polynomial of $\mathsf{T}_{n,k}$ is

$$\det(\mathbf{M})=\det\begin{pmatrix}(\lambda+1)\,\mathbf{I}_k-\mathbf{J}_k & -\mathbf{J}_{k,r}\\ -\mathbf{J}_{r,k} & \lambda\,\mathbf{I}_r\end{pmatrix}. \tag{3.4.3}$$

Denote $\mathbf{B} = (\lambda + 1)\,\mathbf{I}_k - \mathbf{J}_k$. Then

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} & -\mathbf{J}_{k,r} \\ -\mathbf{J}_{r,k} & \lambda\,\mathbf{I}_r \end{pmatrix}. \tag{3.4.4}$$

The Schur complement $\mathbf{M}/\lambda\,\mathbf{I}_r$ is

$$\mathbf{M}/\lambda\,\mathbf{I}_r = \mathbf{B} - \mathbf{J}_{k,r}\cdot(\lambda\,\mathbf{I})^{-1}\cdot\mathbf{J}_{r,k} = \mathbf{B} - \frac{1}{\lambda}\,\mathbf{J}_{k,r}\cdot\mathbf{J}_{r,k} = \mathbf{B} - \frac{r}{\lambda}\,\mathbf{J}_k \tag{3.4.5}$$

$$= (\lambda + 1)\,\mathbf{I}_k - \mathbf{J}_k - \frac{r}{\lambda}\,\mathbf{J}_k = (\lambda + 1)\,\mathbf{I}_k - \frac{r + \lambda}{\lambda}\,\mathbf{J}_k\,. \tag{3.4.6}$$

By the Schur complement Theorem (Theorem 2.2.2)

$$\det(\mathbf{M}) = \det(\lambda\,\mathbf{I}_r)\cdot\det(\mathbf{M}/\lambda\,\mathbf{I}_r) = \lambda^r\cdot\det(\mathbf{M}/\lambda\,\mathbf{I}_r)\,. \tag{3.4.7}$$

The determinant of the Schur complement is :

$$\det(\mathbf{M}/\lambda\,\mathbf{I}_r) = \det\left((\lambda + 1)\,\mathbf{I}_k - \frac{r + \lambda}{\lambda}\,\mathbf{J}_k\right) = \det\left(\frac{r + \lambda}{\lambda}\cdot\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\,\mathbf{I}_k - \mathbf{J}_k\right)\right) \tag{3.4.8}$$

$$= \left(\frac{r + \lambda}{\lambda}\right)^k\cdot\det\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\,\mathbf{I}_k - \mathbf{J}_k\right). \tag{3.4.9}$$

Since $\sigma(\mathbf{J}_k) = \left\{(0)^{k-1}, (k)\right\}$, the spectrum of $\frac{\lambda\cdot(\lambda+1)}{\lambda+r}\,\mathbf{I}_k - \mathbf{J}_k$ is

$$\sigma\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\,\mathbf{I}_k - \mathbf{J}_k\right) = \left\{\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r} - k\right), \left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\right)^{k-1}\right\}, \tag{3.4.10}$$

so

$$\det\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\,\mathbf{I}_k - \mathbf{J}_k\right) = \left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r} - k\right)\cdot\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\right)^{k-1} \tag{3.4.11}$$

$$\Longrightarrow \det(\mathbf{M}/\lambda\,\mathbf{I}_r) = \left(\frac{r + \lambda}{\lambda}\right)^k\cdot\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r} - k\right)\cdot\left(\frac{\lambda\cdot(\lambda + 1)}{\lambda + r}\right)^{k-1} \tag{3.4.12}$$

$$= \frac{(\lambda + 1)^{k-1}\cdot(\lambda\cdot(\lambda + 1) - k(\lambda + r))}{\lambda}. \tag{3.4.13}$$

Hence,

$$\det(\mathbf{M}) = \lambda^r \cdot \det\left(\mathbf{M}/\lambda\mathbf{I}_r\right) = \lambda^r \cdot \left(\frac{(\lambda+1)^{k-1}\cdot(\lambda\cdot(\lambda+1)-k(\lambda+r))}{\lambda}\right) \tag{3.4.14}$$

$$= \lambda^{r-1}\cdot(\lambda+1)^{k-1}\cdot(\lambda(\lambda+1)-k(\lambda+r)) \tag{3.4.15}$$

$$= \lambda^{r-1}\cdot(\lambda+1)^{k-1}\cdot\left(\lambda^2+(1-k)\lambda-rk\right). \tag{3.4.16}$$

Thus, the eigenvalues of $\mathsf{T}_{n,k}$ are 0 with multiplicity $n-k-1$, -1 with multiplicity $k-1$ and the two roots of $\lambda^2+(1-k)\lambda-(n-k)k$. □

Is $\mathsf{T}_{n,k}$ determined by its spectrum ? We consider two cases - the star graphs and the graphs of pyramids.

Claim 3.4.3. *The star graph $\mathsf{S}_n = \mathsf{T}_{n,1}$ is DS iff $n-1$ is prime.*

Proof. The star graph $\mathsf{G} \triangleq \mathsf{S}_n = \mathsf{K}_{1,n-1}$ is a complete bipartite graph. Thus, by Theorem 2.4.7 G is DS iff the pair $(n-1, 1)$ is an AM-minimizer, which is equivalent to $n-1$ being prime. □

Theorem 3.4.4. *The graphs of pyramids are DS.*

Before proving this theorem, we present a preliminary theorem that will be used in the proof.
The following theorem is a corollary of Theorem 2.2.3:

Theorem 3.4.5. *Let H be an induced subgraph of G. Let $\lambda_n \le ... \le \lambda_2 \le \lambda_1$ be the eigenvalues of G and $\mu_m \le ... \le \mu_2 \le \mu_1$ the eigenvalues of H. Then*

1. $\mu_1 \le \lambda_1$ *and* $\lambda_n \le \mu_m$.
2. *For every real number a, $\nu(\mathsf{G}, a) \ge \nu(\mathsf{H}, a)$ and $\mu(\mathsf{G}, a) \ge \mu(\mathsf{H}, a)$.*
3. $\operatorname{rank}(\mathbf{A}(\mathsf{H})) \le \operatorname{rank}(\mathbf{A}(\mathsf{G}))$.

Proof of Theorem 3.4.4. Let $1 < k \le n$ and let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph that is cospectral with $\mathsf{T}_{n,k}$. We have to show that G and $\mathsf{T}_{n,k}$ are isomorphic. We start by showing that

- K_{k+2} and P_4 are not induced subgraphs of G (Lemma 3.4.7).
- Every two non-empty induced subgraphs of G are connected by an edge (Lemma 3.4.8).
- If U is a maximum clique in G, then every edge in G has an end in U (Lemma 3.4.10).

Lemma 3.4.6. 1. G *has one positive eigenvalue.*

2. G *has k negative eigenvalues.*

3. G *has one eigenvalue strictly smaller than* -1.

Proof. Since G is cospectral with $\mathsf{T}_{n,k}$, its spectrum is

$$\sigma(\mathsf{G}) = \left\{(\lambda_2), (-1)^{k-1}, (0)^{n-k-1}, (\lambda_1)\right\} \tag{3.4.17}$$

where $\lambda_1, \lambda_2 = \frac{(k-1)\pm\sqrt{(k-1)^2+4k(n-k)}}{2}$. □

Lemma 3.4.7. *1.* G *does not contain a clique of size* $k+2$.

2. G *does not have* P_4 *as an induced subgraph.*

Proof. The spectrum of K_{k+2} is $\sigma(\mathsf{K}_{k+2}) = \left\{(-1)^{k+1}, (k+1)\right\}$, and the spectrum of P_4 is $\sigma(\mathsf{P}_4) = \left\{2 \cdot cos\left(\frac{\pi \cdot k}{5}\right) \mid k \in \langle 4\rangle\right\}$. K_{k+2} has $k+1$ negative eigenvalues and P_4 has 2 positive eigenvalues. By Lemma 3.4.6, G has k negative eigenvalues and 1 positive eigenvalue. Thus, by Theorem 3.4.5, P_4 and K_{k+2} are not induced subgraphs of G. □

Lemma 3.4.8. *Every two disjoint non-empty induced subgraphs in* G *are connected by an edge.*

Proof. Assume there exist two disjoint non-empty induced subgraphs $\mathsf{H}_1 = (\mathsf{V}_1, \mathsf{E}_1)$ and $\mathsf{H}_2 = (\mathsf{V}_2, \mathsf{E}_2)$. Let $\mathsf{H} = \mathsf{H}_1 \dot{\cup} \mathsf{H}_2$ be the subgraph induced by $\mathsf{V}_1 \cup \mathsf{V}_2$.

$$\mathbf{A}(\mathsf{H}) = \begin{pmatrix} \mathbf{A}(\mathsf{H}_1) & 0 \\ 0 & \mathbf{A}(\mathsf{H}_2) \end{pmatrix}. \tag{3.4.18}$$

The spectrum of H is given by $\sigma(\mathsf{H}) = \sigma(\mathsf{H}_1) + \sigma(\mathsf{H}_2)$. Both H_1 and H_2 have a positive eigenvalue, since they are non-empty. Thus, H has at least two positive eigenvalues, contradicting Theorem 3.4.5. □

For the next lemma we need the following definition.

Definition 3.4.9. Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph and let $U \subseteq \mathsf{V}$ be a set of vertices. U is said to be a *maximum clique* if it is a clique, and G does not have a clique of greater order.

Lemma 3.4.10. *Let* V_1 *be a maximum clique in G, and let* $\mathsf{H}_1 = (\mathsf{V}_1, \mathsf{E}_1)$. *Let* $\mathsf{H}_2 = (\mathsf{V}_2, \mathsf{E}_2)$ *be the subgraph induced by the vertices* $\mathsf{V}_2 := \mathsf{V} \setminus \mathsf{V}_1$. *Then* H_2 *is an empty graph.*

Proof. Without loss of generality we assume that $\mathsf{V}_1 = \langle m\rangle$. By Lemma 3.4.7 $m \leq k+1$. We have to show that H_2 is an empty graph. Suppose to the contrary that there is an edge $e = (v_1, v_2) \in \mathsf{E}_2$, for some $v_1, v_2 \in \mathsf{V}_2$. Since H_1 is induced by a maximum clique, there exists a vertex $u_1 \in \mathsf{V}_1$ s.t. $(v_1, u_1) \notin \mathsf{E}$ (otherwise, the subgraph obtained by $\mathsf{V} \dot{\cup} \{v_1\}$ is a complete graph of order greater than m, contradicting the maximality of H_1). Similarly, there exists a vertex $u_2 \in \mathsf{V}_1$ s.t. $(v_2, u_2) \notin \mathsf{E}$.

We can also assume that $u_1 \neq u_2$, since if both v_1, v_2 are connected to all the vertices except u_1, then $(V_1 \setminus \{u\}) \dot{\cup} \{v_1, v_2\}$ is a clique, contradicting the maximality of V_1.

We now show that the following cases are impossible :

1. $(v_1, u_2) \notin \mathsf{E}$ and $(v_2, u_1) \notin \mathsf{E}$
2. $(v_1, u_2) \in \mathsf{E}$ but $(v_2, u_1) \notin \mathsf{E}$
3. $(v_1, u_2) \notin \mathsf{E}$ but $(v_2, u_1) \in \mathsf{E}$
4. $(v_1, u_2) \in \mathsf{E}$ and $(v_2, u_1) \in \mathsf{E}$

Case 1 is impossible since in this case the subgraph obtained by the vertices $\{v_1, v_2, u_1, u_2\}$ is $\mathsf{K}_2 \dot{\cup} \mathsf{K}_2$. Thus, G has induced cliques that are not connected by an edge, contradicting Lemma 3.4.8. Case 2 is impossible since the subgraph induced by the vertices $\{v_1, v_2, u_1, u_2\}$ is P_4, contradicting Lemma 3.4.7. Similarly, Case 3 is impossible. The proof that Case 4 is impossible needs more work:

Since G is cospectral with $\mathsf{T}_{n,k}$ they have the same number of edges. Thus, there exists another edge in E_2. By Lemma 3.4.8, G does not have disjoint cliques. Thus, there is an edge (v_1, v_3) for some $v_3 \in \mathsf{V}_2$. By Lemma 3.4.7 there exists $u_3 \in \mathsf{V}_1$ s.t. $(u_3, v_3) \notin \mathsf{E}$. Up to isomorphism, there are two possibilities for the subgraph induced by vertices $\{v_1, v_2, v_3, u_1, u_2, u_3\}$ - $(v_2, v_3) \in \mathsf{E}_2$ or $(v_2, v_3) \notin \mathsf{E}_2$.

If $(v_2, v_3) \in \mathsf{E}_2$, then $\{v_1, v_2, v_3\}$ is a triangle in H_2. Since H_1 is a maximum clique, there exists a vertex $u_3 \in \mathsf{V}_1$ s.t. $(v_3, u_3) \notin \mathsf{E}$ (otherwise $\{v_1, v_3\}$ or $\{v_2, v_3\}$ satisfies one of the previous conditions). In this case, the subgraph induced by the vertices $\{v_1, v_2, v_3, u_1, u_2, u_3\}$ is H_3 in Figure 3.3. Its adjacency matrix is

$$\mathbf{A}(\mathsf{H}_3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \tag{3.4.19}$$

and its spectrum is

$$\sigma(\mathsf{H}_3) = \left\{(0)^3, (-2)^2, (4)\right\}. \tag{3.4.20}$$

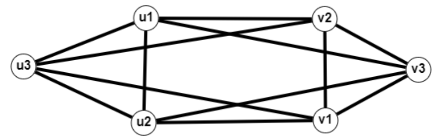


Figure 3.3: H_3

Hence, H_3 has 2 eigenvalues strictly smaller than -1, contradicting Theorem 3.4.5.

If $(v_2, v_3) \notin \mathsf{E}_2$, there exists a vertex $u_3 \in V_1$ s.t. $(v_3, u_3) \notin \mathsf{E}$. Assume that there exists $u_3 \notin \{u_1, u_2\}$ such that $(v_1, u_3) \in \mathsf{E}$ and $(v_2, u_3) \in \mathsf{E}$ (otherwise one of the previous conditions occur, with respect to v_3). In this case, the subgraph induced by the vertices $\{v_1, v_2, v_3, u_1, u_2, u_3\}$ is H_4 in Figure 3.4. Its adjacency matrix is

$$\mathbf{A}(\mathsf{H}_4) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{3.4.21}$$

and its spectrum is

$$\sigma(\mathsf{H}_4) = \left\{ \left(\frac{-\sqrt{5}-1}{2} \right), (-2.231), (-0.483), (0), \left(\frac{-\sqrt{5}-1}{2} \right), (3.714) \right\}. \tag{3.4.22}$$

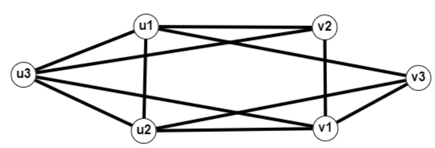


Figure 3.4: H_4

Hence, H_4 also has 2 eigenvalues strictly smaller than -1, contradicting Theorem 3.4.5. □

Now we can complete the proof of the theorem. The adjacency matrix of G is

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{A}(\mathsf{H}_1) & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0}_{n-m} \end{pmatrix} = \begin{pmatrix} \mathbf{J}_m - \mathbf{I}_m & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0}_{n-m} \end{pmatrix} \tag{3.4.23}$$

for some block matrix $\mathbf{B}$. Recall that

$$\mathbf{A}(\mathsf{T}_{n,k}) = \begin{pmatrix} \mathbf{J}_{k+1} - \mathbf{I}_{k+1} & \mathbf{J}_{k+1,n-k-1} \\ \mathbf{J}_{n-k-1,k+1} & \mathbf{0}_{n-k-1} \end{pmatrix}. \tag{3.4.24}$$

We want to show that $m = k+1$ and $\mathbf{B} = \mathbf{J}_{n\times(n-m)}$. Assume to the contrary $m \neq k+1$. By Lemma 3.4.7, $m < k+1$. Then clearly the number of edges of G is smaller then the number of edges of $\mathsf{T}_{n,k}$, contradicting the fact that the number of edges is determined by the spectrum, so two cospectral graphs must have the same number of edges (See 2.1 and the followed discussion). Hence, $m = k+1$ and for some $\mathbf{B} \in \mathbb{R}^{(k+1)\times(n-k-1)}$

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{J}_{k+1} - \mathbf{I}_{k+1} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0}_{n-k-1} \end{pmatrix}. \tag{3.4.25}$$

Again by considering the number of edges, all the entries of $\mathbf{B}$ are equal to 1. Thus

$$\mathbf{A}(\mathsf{G}) = \begin{pmatrix} \mathbf{J}_{k+1} - \mathbf{I}_{k+1} & \mathbf{J}_{k+1,n-k-1} \\ \mathbf{J}_{n-k-1,k+1} & \mathbf{0}_{n-k-1} \end{pmatrix} = \mathbf{A}(\mathsf{T}_{n,k}). \tag{3.4.26}$$

□

3.5 Summary

We showed that the graphs of pyramids $\mathsf{T}_{n,k}$ $(2 \leq k < n)$ are DS (Theorem 3.4.4), and that for $k = 1$, the graph $\mathsf{T}_{n,1} = \mathsf{S}_n$ is DS if and only if $n-1$ is prime. In the next table we list the graphs $\mathsf{T}_{n,k}$ according to their being DS/CP:

	CP	not CP
DS	$k = 2$ or $k = 1,\ n$ is prime	$3 \leq k$
not DS	$k = 1,\ n$ is composite	

Table 3.1: Classification of the graphs $\mathsf{T}_{n,k}$ according to their being DS/CP.

We have an empty field in the table, since the only graphs $\mathsf{T}_{n,k}$ that are not DS are stars, and stars are CP since they are bipartite. This suggests the following questions on general graphs (not only $\mathsf{T}_{n,k}$).

Question 3.5.1. *What is the smallest order v of a graph that is not CP neither DS ?*

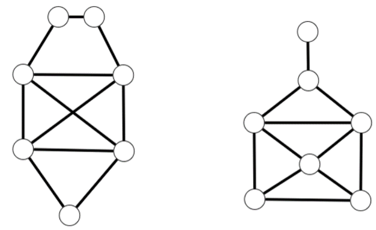


Figure 3.5: Two cospectral graphs with 7 vertices G_1 and G_2

solution. The two graphs in Figure 3.5 are not DS, and by Theorem 3.2.6 are not CP, so $\nu \leq 7$. On the other hand, $\nu \geq 6$ since all the graphs with 4 vertices are CP, and there is only one pair of non-isomorphic cospectral graphs with 5 vertices (Figure 2.1), and both are CP. Computer search shows that $\nu > 6$ [47], so $\nu = 7$. □

Chapter 4

On the transitivity of generalized-Hamming graphs and their complements

Chapter Overview

The generalized-Hamming graph $\mathcal{G}_{q,n,d}$, which arises naturally in graph theory and coding theory, is the regular graph on $\mathbb{F}_q^n$ in which two vertices are adjacent if their Hamming distance is less than d, and it is vertex-transitive. We classify all parameters (q, n, d) for which $\mathcal{G}_{q,n,d}$ is edge-transitive or distance-transitive, and separately classify all parameters for which its complement has these properties. We prove that $\mathcal{G}_{q,n,d}$ is edge-transitive if and only if it is distance-transitive, and that this occurs precisely when $d = 2$, $(q, d) = (2, 3)$, or $(q, d) = (2, n)$. For the complement graphs, we determine all parameters yielding edge- or distance-transitivity using spectral methods based on Krawtchouk polynomials and the structure of the Hamming association scheme. In contrast to the generalized-Hamming graphs, where the parameter sets corresponding to edge- and distance-transitivity coincide, we show that for their complements the set of parameters yielding distance-transitivity is strictly contained in the set yielding edge-transitivity. As an application, we compute the exact values of the Lovász ϑ-function of generalized-Hamming graphs, as well as of their complements, in all cases where either one of them is edge-transitive.

4.1 Introduction

Let $q \in \mathbb{N}$ be a prime power, and let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$. In this chapter, the *generalized-Hamming graph* $\mathcal{G}_{q,n,d}$ denotes the graph with vertex set $\mathbb{F}_q^n$ in which two distinct vertices are adjacent if their Hamming distance is less than d. In the literature, the term generalized-Hamming graph is used to refer to this graph or to closely related variants [5, 92, 119, 147]. The generalized-Hamming graph also arises in information theory in connection with both the classical Gilbert–Varshamov bound and the Shannon capacity of graphs (see [79, 106, 121, 126, 138]).

In [126], the generalized-Hamming graph $\mathcal{G}_{2,5,3}$ was used to show that Schrijver's ϑ-function need not be an upper bound on the Shannon capacity of graphs. Relying on [124], the selection of that graph as a potential example was justified by the fact that the graph and its complement are both vertex-transitive, the graph is edge-transitive, whereas its complement is not (had the graph and its complement both been vertex- and edge-transitive, it could not serve as an example [124]). This observation motivates the present paper, whose goal is to classify the parameters q, n, d for which the generalized-Hamming graph or its complement are edge-transitive. We show that in all cases where the generalized-Hamming graph is edge-transitive, it is even distance-transitive (recall that distance-transitivity implies edge-transitivity, but the opposite is not necessarily true). The paper also classifies the parameters q, n, d for which the complement of the generalized-Hamming graph, denoted by $\overline{\mathcal{G}_{q,n,d}}$, is edge-transitive, and those for which it is distance-transitive. The proof of the latter classification relies on spectral methods in graph theory and the theory of association schemes. To the best of our knowledge, a complete classification of the parameters q, n, d for which generalized-Hamming graphs or their complements are edge-transitive or distance-transitive has not previously appeared in the literature.

The main theorems in this paper are as follows.

Theorem 4.1.1. *Let $q \in \mathbb{N}$ be a prime power, let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$, and let $\mathsf{G} = \mathcal{G}_{q,n,d}$. Then, the following statements are equivalent:*

1. G *is edge-transitive.*
2. G *is distance-transitive.*
3. *Either $d = 2$, or $(q, d) = (2, 3)$, or $(q, d) = (2, n)$.*

Theorem 4.1.2. *Let $q \in \mathbb{N}$ be a prime power, let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$, and let $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$. Then,*

1. G *is edge-transitive if and only if either $(q, d) = (2, n-1)$ for an even n, or $d = n$.*
2. G *is distance-transitive if and only if one of the following two conditions holds:*

   (Ła) $(q, d) = (2, n-1)$ *for an even n,*

   (Łb) $d = n$ *and either $q = 2$ or $n = 2$.*

Corollary 4.1.3. *Let G be a generalized-Hamming graph such that G and $\overline{\mathsf{G}}$ are both edge-transitive. Then, either $(q, d) = (2, n)$ or $(q, d, n) = (2, 3, 4)$.*

The paper alternates between background material and new results. The even-numbered sections provide necessary background on graph transitivity, association schemes, Krawtchouk polynomials, and the Lovász ϑ-function, while the odd-numbered sections present the proofs of our results and their applications. An exception is Section 4.4.4, where we derive spectral properties of the complement of generalized-Hamming graphs. Since these results appear to be new, Section 4.4.4 contains a complete proof instead of referring to external sources.

In regard to the odd-numbered sections, Theorem 4.1.1 is proved in Section 4.3, and Theorem 4.1.2 is proved in Section 4.5. The results are then applied in Section 4.7 to compute the Lovász ϑ-function of edge-transitive generalized-Hamming graphs and their complements.

4.2 Graph transitivity

All graphs in this paper are finite, simple, undirected, and connected. We denote by G a graph with vertex set $\mathsf{V}(\mathsf{G})$ and edge set $\mathsf{E}(\mathsf{G})$. For every vertex $v \in \mathsf{V}(\mathsf{G})$, we denote by $\mathcal{N}(v)$ the neighborhood of v, i.e., the set of vertices adjacent to v. The *distance* between two vertices $u, v \in \mathsf{V}(\mathsf{G})$ is the length of the shortest path between them and is denoted by $d_{\mathsf{G}}(u, v)$. A *graph automorphism* is a permutation of the vertex set that preserves edges and non-edges. The set of all graph automorphisms of G forms an *automorphism group*, denoted by $\mathrm{Aut}(\mathsf{G})$. The following types of graph symmetries are considered:

1. G is *vertex-transitive* if, for every two vertices $u, v \in \mathsf{V}(\mathsf{G})$, there exists an automorphism of G that maps u to v.

2. G is *edge-transitive* if, for every pair of edges $\{u_1, v_1\}, \{u_2, v_2\} \in \mathsf{E}(\mathsf{G})$, there exists an automorphism of G that maps $\{u_1, v_1\}$ to $\{u_2, v_2\}$.

3. G is *distance-transitive* if, for every four vertices $u_1, u_2, v_1, v_2 \in \mathsf{V}(\mathsf{G})$ satisfying the equality $d_{\mathsf{G}}(v_1, v_2) = d_{\mathsf{G}}(u_1, u_2)$, there exists an automorphism of G that maps v_1 to u_1 and v_2 to u_2.

If a graph is distance-transitive, then it is also edge-transitive and vertex-transitive. A Cayley graph of an abelian group is vertex-transitive [67, Theorem 3.1.2], and the generalized-Hamming graph is a Cayley graph of the additive group $\mathbb{F}_q^n$. In particular,

$$\mathcal{G}_{q,n,d} = \mathrm{Cay}\left(\mathbb{F}_q^n, \mathcal{S}_{n,d}\right) \tag{4.2.1}$$

$$\mathcal{S}_{n,d} := \left\{\mathbf{x} \in \mathbb{F}_q^n \, : \, 1 \le w_H(\mathbf{x}) \le d-1\right\}, \tag{4.2.2}$$

where $w_H(\mathbf{x})$ is the Hamming weight of the vector $\mathbf{x}$, i.e., the number of nonzero coordinates of $\mathbf{x}$, and $\mathcal{S}_{n,d}$ is called a *connection set*. Similarly, the complement graph is also a Cayley graph:

$$\overline{\mathcal{G}_{q,n,d}} = \mathrm{Cay}\left(\mathbb{F}_q^n, \mathbb{F}_q^n \setminus (\mathcal{S}_{n,d} \cup \{\mathbf{0}\})\right). \tag{4.2.3}$$

Thus, both the generalized-Hamming graph and its complement are vertex-transitive, but they are not necessarily edge-transitive (hence, they need not be distance-transitive).

4.3 Transitive generalized-Hamming graphs

This section proves Theorem 4.1.1, which fully characterizes the parameters for which generalized-Hamming graphs are edge-transitive or distance-transitive, and establishes that a generalized-Hamming graph is edge-transitive if and only if it is distance-transitive.

The generalized-Hamming graph $\mathcal{G}_{q,n,2}$ is also called the Hamming graph, and it is known to be distance-transitive [42, Table 6.1]. A proof that $\mathcal{G}_{2,n,3}$ is distance-transitive is presented in [108]. The following lemma completes the proof of one direction of Theorem 4.1.1.

Lemma 4.3.1. *The generalized-Hamming graph $\mathcal{G}_{2,n,n}$ is distance-transitive.*

Proof. Every generalized-Hamming graph is vertex-transitive since it is a Cayley graph. Consequently, it is also distance-transitive if and only if for every two vertices $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n \setminus \{\mathbf{0}\}$, such that $\mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{u}) = \mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{v})$, there exists a graph automorphism that maps $\{\mathbf{0}, \mathbf{u}\}$ to $\{\mathbf{0}, \mathbf{v}\}$. Let $\mathsf{G} = \mathcal{G}_{2,n,n}$. For every vertex $\mathbf{v} \in \mathbb{F}_2^n$, the only vertex that is not adjacent to $\mathbf{v}$ is its complement $\overline{\mathbf{v}} \triangleq \mathbf{v} \oplus \mathbf{1}_n$, where $\oplus$ is the componentwise modulo-2 sum. Let $\mathbf{u}$ and $\mathbf{v}$ be two vertices such that $\mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{v}) = \mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{u})$. Consider the mapping $\varphi\colon \mathbb{F}_q^n \to \mathbb{F}_q^n$ given by

$$\varphi(\mathbf{x}) = \begin{cases} \mathbf{u} & \text{if } \mathbf{x} = \mathbf{v}, \\ \overline{\mathbf{u}} & \text{if } \mathbf{x} = \overline{\mathbf{v}}, \\ \mathbf{v} & \text{if } \mathbf{x} = \mathbf{u}, \\ \overline{\mathbf{v}} & \text{if } \mathbf{x} = \overline{\mathbf{u}}, \\ \mathbf{x} & \text{otherwise.} \end{cases} \tag{4.3.1}$$

If $\mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{v}) = \mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{u}) = 2$, then $\mathbf{v} = \mathbf{u} = \mathbf{1}_n$ and φ is the identity mapping. Otherwise, we have $\mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{v}) = \mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{u}) = 1$. In the latter case, φ permutes between $\mathbf{v}$ and $\mathbf{u}$, and also between their complements, while keeping all other vertices unchanged. Hence, φ is a graph automorphism that maps $\{\mathbf{0}, \mathbf{v}\}$ to $\{\mathbf{0}, \mathbf{u}\}$. □

The next lemma proves that every generalized-Hamming graph not listed in Theorem 4.1.1 is not edge-transitive. The proof identifies an invariant of edges in edge-transitive graphs and shows that this invariant takes different values for two edges of the graph.

Lemma 4.3.2. *Let $q, n, d \in \mathbb{N}$ be such that (q, n, d) does not satisfy any of the conditions in Item 3 of Theorem 4.1.1. Then, the generalized-Hamming graph $\mathcal{G}_{q,n,d}$ is not edge-transitive.*

Proof. Let $\mathsf{G} = \mathcal{G}_{q,n,d}$ be a generalized-Hamming graph such that (q, n, d) does not satisfy any of the conditions in Item 3 of Theorem 4.1.1, and suppose to the contrary that G is edge-transitive. Define $F\colon \mathcal{N}(\mathbf{0}) \to \mathbb{N}$ to be the number of neighbors of a vertex $\mathbf{v} \in \mathcal{N}(\mathbf{0})$ that are not neighbors of $\mathbf{0} \in \mathbb{F}_q^n$, i.e.,

$$F(\mathbf{v}) = |\mathcal{N}(\mathbf{v}) \setminus \mathcal{N}(\mathbf{0})|. \tag{4.3.2}$$

Under the assumption that G is edge-transitive, for every pair of vertices $\mathbf{u}, \mathbf{v} \in \mathcal{N}(\mathbf{0})$, there exists an automorphism φ of G such that $\varphi(\{\mathbf{0}, \mathbf{u}\}) = \{\mathbf{0}, \mathbf{v}\}$. Hence either $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi(\mathbf{u}) = \mathbf{v}$, or $\varphi(\mathbf{0}) = \mathbf{v}$ and $\varphi(\mathbf{u}) = \mathbf{0}$. In the latter case, since G is a Cayley graph with a symmetric connection set, both the translation $\tau_{-\mathbf{v}}\colon \mathbf{x} \mapsto \mathbf{x} - \mathbf{v}$ and the negation map $\iota\colon \mathbf{x} \mapsto -\mathbf{x}$ are automorphisms. Therefore, the composition

$$\psi := \iota \circ \tau_{-\mathbf{v}} \circ \varphi$$

is an automorphism of G satisfying $\psi(\mathbf{0}) = \mathbf{0}$ and $\psi(\mathbf{u}) = \mathbf{v}$. Consequently, for every pair $\mathbf{u}, \mathbf{v} \in \mathcal{N}(\mathbf{0})$, there exists an automorphism ψ fixing $\mathbf{0}$ and mapping $\mathbf{u}$ to $\mathbf{v}$. Graph automorphisms preserve adjacency, and hence

$$\begin{aligned}
F(\mathbf{u}) &= |\mathcal{N}(\mathbf{u}) \setminus \mathcal{N}(\mathbf{0})| \\
&= |\mathcal{N}(\psi(\mathbf{u})) \setminus \mathcal{N}(\psi(\mathbf{0}))| \\
&= |\mathcal{N}(\mathbf{v}) \setminus \mathcal{N}(\mathbf{0})| \\
&= F(\mathbf{v}).
\end{aligned}$$

Hence the value $F(\mathbf{v})$ is identical for all vertices $\mathbf{v} \in \mathcal{N}(\mathbf{0})$. In particular, consider the two vertices

$$\mathbf{u}_1 = (1, 0, 0, \ldots, 0), \qquad \mathbf{u}_2 = (1, 1, 0, \ldots, 0).$$

Since the third condition in Theorem 4.1.1 is not satisfied, it follows that $d \geq 3$. Hence, $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{N}(\mathbf{0})$, and therefore $F(\mathbf{u}_1) = F(\mathbf{u}_2)$.

We use combinatorial arguments to derive closed-form expressions for $F(\mathbf{u}_1)$ and $F(\mathbf{u}_2)$. We adopt the convention for binomial coefficients that $\binom{n}{k} = 0$ if $k > n$ or $k < 0$.

The neighbors of $\mathbf{u}_1$ that are not neighbors of $\mathbf{0}$ are precisely the vertices in $\mathbb{F}_q^n$ whose first coordinate equals 1 and whose Hamming weight is d. Hence, the number of such vertices is

$$F(\mathbf{u}_1) = \binom{n-1}{d-1}(q-1)^{d-1}. \tag{4.3.3}$$

The set of neighbors of $\mathbf{u}_2$ that are not neighbors of $\mathbf{0}$ is the disjoint union of the following sets:

- The neighbors of $\mathbf{u}_2$ that are not neighbors of $\mathbf{0}$ and have Hamming weight d. Such vertices cannot begin with the subsequences $(0,0)$, $(0,1)$, or $(1,0)$, since in each of these cases their Hamming weight would be less than d, and hence they would be adjacent to $\mathbf{0}$. Therefore, they must begin with $(1,1)$. The number of such vertices is $\binom{n-2}{d-2}(q-1)^{d-2}$.
- The neighbors of $\mathbf{u}_2$ that are not neighbors of $\mathbf{0}$ and have Hamming weight $d+1$. By the same reasoning, each of these vertices starts with $(1,1)$, and their number is $\binom{n-2}{d-1}(q-1)^{d-1}$.

Overall,

$$F(\mathbf{u}_2) = \binom{n-2}{d-2}(q-1)^{d-2} + \binom{n-2}{d-1}(q-1)^{d-1}. \tag{4.3.4}$$

Consequently, by (4.3.3) and (4.3.4), the equality $F(\mathbf{u}_2) - F(\mathbf{u}_1) = 0$ yields

$$\binom{n-2}{d-2}(q-1)^{d-2} + \binom{n-2}{d-1}(q-1)^{d-1} - \binom{n-1}{d-1}(q-1)^{d-1} = 0, \tag{4.3.5}$$

$$\iff \quad \binom{n-2}{d-2} + \binom{n-2}{d-1}(q-1) - \binom{n-1}{d-1}(q-1) = 0, \tag{4.3.6}$$

$$\iff \quad \frac{1}{n-d} + \frac{q-1}{d-1} - \frac{(n-1)(q-1)}{(n-d)(d-1)} = 0, \tag{4.3.7}$$

$$\iff \quad (d-1) + (n-d)(q-1) - (n-1)(q-1) = 0, \tag{4.3.8}$$

$$\iff \quad (q-2)(1-d) = 0. \tag{4.3.9}$$

By (4.3.9), if $q \neq 2$, we obtain a contradiction to the assumption that G is edge-transitive (recall that $d \geq 3$).

Otherwise, if none of the conditions in Item 3 of Theorem 4.1.1 are satisfied, then $q = 2$ and $4 \leq d \leq n-1$. In the latter case,

$$\mathbf{u}_3 \triangleq (1,1,1,0,\dots,0) \in \mathcal{N}(\mathbf{0}).$$

Let $q = 2$ and $4 \leq d \leq n-1$. We calculate $F(\mathbf{u}_3)$ and compare it to $F(\mathbf{u}_1)$, since, by the above, these two values are identical as $\mathbf{u}_1, \mathbf{u}_3 \in \mathcal{N}(\mathbf{0})$. The set of neighbors of $\mathbf{u}_3$ that are not neighbors of $\mathbf{0}$ is the disjoint union of the following sets:

- The neighbors of $\mathbf{u}_3$ with Hamming weight d. These vertices can differ from $\mathbf{u}_3$ in at most one of the first three coordinates; hence, the Hamming weight of their first three coordinates must be 2 or 3. We consider these two cases separately.

  1. If this Hamming weight is 2, then the first three coordinates of these vertices are either $(0, 1, 1)$, $(1, 0, 1)$, or $(1, 1, 0)$, and in each of these three cases, there are $\binom{n-3}{d-2}$ possibilities for the remaining coordinates.

  2. If the first three coordinates are $(1, 1, 1)$, then there are $\binom{n-3}{d-1}$ possibilities for the remaining coordinates.

  Therefore, the total number of neighbors of $\mathbf{u}_3$ with Hamming weight d is

$$3\binom{n-3}{d-2} + \binom{n-3}{d-3}.$$

- The neighbors of $\mathbf{u}_3$ with Hamming weight $d+1$. Such vertices must have their first three coordinates equal to $(1, 1, 1)$, and there are $\binom{n-3}{d-2}$ such vertices.

- The neighbors of $\mathbf{u}_3$ with Hamming weight $d+2$. Again, the first three coordinates must be $(1, 1, 1)$, and there are $\binom{n-3}{d-1}$ such vertices.

Overall,

$$F(\mathbf{u}_3) = 4\binom{n-3}{d-2} + \binom{n-3}{d-3} + \binom{n-3}{d-1}. \tag{4.3.10}$$

By invoking Pascal's identity

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}, \quad a, b \in \mathbb{N},$$

three times, equality (4.3.10) can be rewritten as

$$\begin{aligned} F(\mathbf{u}_3) &= 3\binom{n-3}{d-2} + \left[\binom{n-3}{d-2} + \binom{n-3}{d-3}\right] + \binom{n-3}{d-1} \\ &= 3\binom{n-3}{d-2} + \binom{n-2}{d-2} + \binom{n-3}{d-1} \\ &= 2\binom{n-3}{d-2} + \left[\binom{n-3}{d-2} + \binom{n-3}{d-1}\right] + \binom{n-2}{d-2} \\ &= 2\binom{n-3}{d-2} + \binom{n-2}{d-1} + \binom{n-2}{d-2} \\ &= 2\binom{n-3}{d-2} + \binom{n-1}{d-1}. \end{aligned} \tag{4.3.11}$$

Consequently, by (4.3.3) (for $q = 2$) and (4.3.11), the equality $F(\mathbf{u}_3) - F(\mathbf{u}_1) = 0$ yields

$$F(\mathbf{u}_3) - F(\mathbf{u}_1) = 2\binom{n-3}{d-2} = 0. \tag{4.3.12}$$

Since $4 \leq d \leq n-1$ in the considered case, equality (4.3.12) holds if and only if $d-2 > n-3$, i.e., $d > n-1$. This leads to a contradiction. Therefore, in this case as well, we obtain a contradiction to the assumption that G is edge-transitive. □

4.4 Association Schemes

The proof of Theorem 4.1.2, which appears in the next section (Section 4.5), relies on the theory of association schemes. This section begins by briefly recalling the necessary definitions and properties of association schemes.

4.4.1 Association schemes and their matrix representation

Definition 4.4.1 (association scheme)**.** Let $\mathcal{X}$ be a finite set and let $m \in \mathbb{N}$. An $(m+1)$-*class association scheme* on $\mathcal{X}$ is a partition $\mathcal{R} = \{R_0, R_1, \dots, R_m\}$ of $\mathcal{X} \times \mathcal{X}$ (i.e., $\bigcup_{i=0}^{m} R_i = \mathcal{X} \times \mathcal{X}$ and $R_i \cap R_j = \varnothing$ for all $i \neq j$) that satisfies the following properties:

- $R_0 = \{(x, x) : x \in \mathcal{X}\}$ is the identity relation.
- For every $0 \leq i \leq m$, the relation $R_i^{\star} \triangleq \{(y, x) : (x, y) \in R_i\}$ also belongs to $\mathcal{R}$ (i.e., $R_i^{\star} = R_j$ for some j). In the special case where $R_i^{\star} = R_i$ for every $0 \leq i \leq m$, the association scheme is called *symmetric*.

- For every $0 \le i, j, k \le m$, there exist integers p_{ij}^k such that for every $(x, y) \in R_k$,

$$\left| \{z \in \mathcal{X} : (x, z) \in R_i \text{ and } (z, y) \in R_j\} \right| = p_{ij}^k,$$

  i.e., the above quantity depends only on i, j, k and not on the particular pair $(x, y) \in R_k$. The integers $\{p_{ij}^k\}$ are called the *intersection numbers* of the association scheme.

For every $i \in \{0, 1, \ldots, m\}$, the *adjacency matrix representation* of the relation R_i is the matrix $\mathbf{A}_i \in \{0, 1\}^{|\mathcal{X}| \times |\mathcal{X}|}$, where

$$\mathbf{A}_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.1}$$

One can represent the relations of an association scheme by adjacency matrices. Accordingly, an association scheme $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ can be represented by the set of matrices $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m\}$. In this representation, the conditions of Definition 4.4.1 are rewritten as follows:

- $\sum_{i=0}^m \mathbf{A}_i = \mathbf{J}$, where $\mathbf{J}$ denotes the all-ones matrix of size $|\mathcal{X}| \times |\mathcal{X}|$.
- $\mathbf{A}_0 = \mathbf{I}$ is the identity matrix of size $|\mathcal{X}| \times |\mathcal{X}|$.
- For every $0 \le i \le m$, the equality $\mathbf{A}_i^T = \mathbf{A}_j$ holds for some $0 \le j \le m$. If the association scheme is symmetric then $\mathbf{A}_i^T = \mathbf{A}_i$ for every $0 \le i \le m$.
- For every $0 \le i, j \le m$,

$$\mathbf{A}_i \, \mathbf{A}_j = \sum_{k=0}^m p_{ij}^k \, \mathbf{A}_k = \mathbf{A}_j \, \mathbf{A}_i \, . \tag{4.4.2}$$

Since the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute pairwise, they admit a common basis of eigenvectors and can therefore be simultaneously diagonalized. The matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ span a commutative algebra called the *Bose–Mesner algebra*.

Definition 4.4.2 (Bose–Mesner algebra)**.** Let $\mathcal{R}$ be an association scheme on a finite set $\mathcal{X}$ with adjacency matrices $\mathbf{A}_0, \ldots, \mathbf{A}_m$. The *Bose–Mesner algebra* of $\mathcal{R}$, denoted by $\mathcal{B}(\mathcal{R})$, is the subalgebra of $\mathbb{R}^{|\mathcal{X}| \times |\mathcal{X}|}$ consisting of all real linear combinations of these matrices:

$$\mathcal{B}(\mathcal{R}) = \operatorname{span}_{\mathbb{R}}\{\mathbf{A}_0, \ldots, \mathbf{A}_m\}.$$

Since the products $\mathbf{A}_i \, \mathbf{A}_j$ are again linear combinations of $\mathbf{A}_0, \ldots, \mathbf{A}_m$ (see (4.4.2)), this span is closed under matrix multiplication.

4.4.2 Dual basis and eigenmatrices

Apart from the basis formed by the adjacency matrices, the Bose–Mesner algebra has another distinguished basis called the *dual basis*, consisting of orthogonal projections (the primitive idempotents). The following theorem states this fact.

Theorem 4.4.3. *[55, Theorem 2.2] Let $\mathcal{R}$ be an association scheme on a set $\mathcal{X}$ with adjacency matrices $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m\}$. Then there exists a basis of $\mathcal{B}(\mathcal{R})$ consisting of orthogonal projections $\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_m$. In particular, there exist scalars $q_k(i)$ and $p_i(k)$ for $0 \le i, k \le m$ such that*

$$\mathbf{E}_k = \frac{1}{|\mathcal{X}|} \sum_{i=0}^{m} q_k(i)\, \mathbf{A}_i, \qquad \mathbf{A}_i = \sum_{k=0}^{m} p_i(k) \mathbf{E}_k. \tag{4.4.3}$$

Moreover, for all $0 \le i, j \le m$,

$$\mathbf{E}_i \mathbf{E}_j = \delta_{i,j} \mathbf{E}_i, \text{ where } \delta_{i,j} \triangleq \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.4}$$

Define the matrices $\mathbf{P} \triangleq [p_i(k)]_{i,k=0}^{m}$ and $\mathbf{Q} \triangleq [q_k(i)]_{k,i=0}^{m}$. The matrices $\mathbf{P}$ and $\mathbf{Q}$ are called the *first eigenmatrix* and the *second eigenmatrix* of the association scheme, respectively, and they satisfy

$$\mathbf{PQ} = \mathbf{QP} = |\mathcal{X}|\, \mathbf{I}. \tag{4.4.5}$$

Association schemes may be viewed as a natural generalization of distance-regular graphs. In particular, every distance-regular graph naturally gives rise to an association scheme, called its *distance scheme*. The Hamming scheme arises in this way from the Hamming graph $\mathrm{H}(n, q) = \mathcal{G}_{q,n,2}$. Introductions to association schemes can be found, e.g., in [55,56] and in [102, Chapter 21].

4.4.3 The Hamming scheme and Krawtchouk polynomials

Definition 4.4.4 (The Hamming scheme)**.** The Hamming association scheme $\mathcal{H}$ is an $(n+1)$-class symmetric association scheme on the set $\mathbb{F}_q^n$, with adjacency matrices $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$. For $0 \le i \le n$, the matrix $\mathbf{A}_i$ is the *distance-i matrix*, defined by

$$(\mathbf{A}_i)_{\mathbf{u},\mathbf{v}} = \begin{cases} 1 & \text{if } \mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n \text{ differ in exactly } i \text{ coordinates,} \\ 0 & \text{otherwise.} \end{cases}$$

The dual basis of the Hamming scheme is given in terms of the Krawtchouk polynomials.

Definition 4.4.5 (Krawtchouk polynomials)**.** [56, Eq. (16)] Let $q, n \in \mathbb{N}$, $q \geq 2$, and $0 \leq i, \ell \leq n$. The *q-ary Krawtchouk polynomial* $K_\ell(i; n, q)$ is defined by

$$K_\ell(i; n, q) = \sum_{r=0}^{\ell} \binom{i}{r} \binom{n-i}{\ell-r} (-1)^r \, (q-1)^{\ell-r}. \tag{4.4.6}$$

Note that $K_\ell(i; n, q)$ is a polynomial of degree ℓ in i, since for every $r \in \{0, 1, \ldots, \ell\}$ the product $\binom{i}{r}\binom{n-i}{\ell-r}$ is itself a polynomial of degree ℓ in i.

Theorem 4.4.6. *[56, Example 1]. The entries of the first and second eigenmatrices* $\mathbf{P} \triangleq [p_i(k)]_{i,k=0}^m$ *and* $\mathbf{Q} \triangleq [q_k(i)]_{k,i=0}^m$ *of the Hamming scheme are, respectively, given by*

$$p_i(k) = K_i(k; n, q), \quad q_k(i) = K_k(i; n, q). \tag{4.4.7}$$

In particular, we have

$$\mathbf{A}_i = \sum_{k=0}^{n} K_i(k; n, q) \, \mathbf{E}_k, \tag{4.4.8}$$

$$\mathbf{E}_k = \frac{1}{q^n} \sum_{i=0}^{n} K_k(i; n, q) \, \mathbf{A}_i \, . \tag{4.4.9}$$

Theorem 4.4.7 (Properties of Krawtchouk polynomials)**.** *Let* $n \in \mathbb{N}$, $q \geq 2$, *and let* $K_l(i; n, q)$ *denote the q-ary Krawtchouk polynomials for* $0 \leq l, i \leq n$. *Then, the following properties hold:*

1. *[43, Proposition 1.2]: The Krawtchouk polynomials satisfy the generating function identity*

$$\sum_{k=0}^{n} K_k(i; n, q) \, z^k = (1 + (q-1)z)^{n-i} (1-z)^i. \tag{4.4.10}$$

2. *[43, Theorem 2.1]: For every* $i, d \in [n]$, *the following summation formula holds*

$$\sum_{k=0}^{d} K_k(i; n, q) = K_d(i-1; n-1, q). \tag{4.4.11}$$

Theorem 4.4.8 (Character-theoretic representation of Krawtchouk polynomials)**.** *Let p be a prime, let* $q = p^m$, *and let* $0 \leq i, k \leq n$. *For* $\mathbf{x} \in \mathbb{F}_q^n$ *with* $w_H(\mathbf{x}) = i$, *the following holds:*

1. *[118, Eq. (29)]: The Krawtchouk polynomial admits the representation*

$$K_k(i; n, q) = \sum_{\substack{\mathbf{y} \in \mathbb{F}_q^n \\ w_H(\mathbf{y}) = k}} \chi_{\mathbf{y}}(\mathbf{x}), \tag{4.4.12}$$

where $\chi_{\mathbf{y}}$ *is the additive character of* $\mathbb{F}_q^n$ *indexed by* $\mathbf{y}$ *and defined by (see [93, Theorem 5.7])*

$$\chi_{\mathbf{y}}(\mathbf{x}) = \exp\left(\frac{2\pi i}{p} \cdot \mathrm{Tr}\left(\langle \mathbf{x}, \mathbf{y} \rangle\right)\right),$$

$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n x_j y_j$ *denotes the standard inner product over* $\mathbb{F}_q$*, and* $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ *is the trace function given by*

$$\mathrm{Tr}(a) = \sum_{j=0}^{m-1} a^{p^j}, \qquad a \in \mathbb{F}_q.$$

2. *The trace map* $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ *is* $\mathbb{F}_p$*-linear and surjective [93, Theorem 2.23].*

Remark 4.4.9. For $q = 2$, the Krawtchouk polynomials defined in (4.4.6) reduce to the binary Krawtchouk polynomials

$$K_j(i) \triangleq K_j(i; n, 2) = \sum_{r=0}^{j} (-1)^r \binom{i}{r} \binom{n-i}{j-r}, \qquad j \in \{0, 1, \ldots, n\}. \tag{4.4.13}$$

4.4.4 Spectral properties of the complement of the generalized-Hamming graph

The theory of association schemes enables a characterization of the spectral properties of the complement of the generalized-Hamming graph. The next result, stated in Theorem 4.4.10, provides such a characterization. Since this result appears to be new, we include a proof for completeness.

Theorem 4.4.10. *Let* $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$ *be the complement of the generalized-Hamming graph.*

1. *The adjacency matrix of* G *admits the decomposition*

$$\mathbf{A}(\mathsf{G}) = \sum_{j=0}^{n} \gamma_j \mathbf{E}_j, \tag{4.4.14}$$

where the scalars γ_j *are the eigenvalues of* $\mathbf{A}(\mathsf{G})$*, given by*

$$\gamma_0 = \Delta(\mathsf{G}), \quad \gamma_j = -K_{d-1}(j-1; n-1, q), \ \forall 1 \le j \le n, \tag{4.4.15}$$

and $\Delta(\mathsf{G}) = \sum_{i=d}^n \binom{n}{i}(q-1)^i$ *denotes the degree of the regular graph* G.

2. *$\gamma_1 < \gamma_m$ for every m with $2 \leq m \leq n-1$.*

3. *$\gamma_1 = \gamma_n$ if and only if $q = 2$ and d is odd.*

4. *For every polynomial p,*

$$p(\mathbf{A}(\mathsf{G})) = \sum_{j=0}^{n} p(\gamma_j)\,\mathbf{E}_j. \tag{4.4.16}$$

Proof. Let $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$ be the complement of the generalized-Hamming graph, $1 < d < n$.

1. The adjacency matrix of G can be expressed as follows:

$$\mathbf{A}(\mathsf{G}) = \sum_{i=d}^{n} \mathbf{A}_i = \sum_{i=d}^{n}\left(\sum_{j=0}^{n} K_i(j;n,q)\mathbf{E}_j\right) = \sum_{j=0}^{n}\left(\sum_{i=d}^{n} K_i(j;n,q)\right)\mathbf{E}_j \triangleq \sum_{j=0}^{n}\gamma_j\mathbf{E}_j, \tag{4.4.17}$$

where $\{\mathbf{A}_i\}_{i=0}^{n}$ are the distance i matrices of the Hamming scheme, and $\{\mathbf{E}_j\}_{j=0}^{n}$ is the dual basis of its Bose-Mesner algebra. The values $\{\gamma_j\}_{j=0}^{n}$ are given by

$$\gamma_j \triangleq \sum_{i=d}^{n} K_i(j;n,q) = \sum_{i=0}^{n} K_i(j;n,q) - \sum_{i=0}^{d-1} K_i(j;n,q), \tag{4.4.18}$$

and form the spectrum of G with respect to its adjacency matrix. The largest eigenvalue γ_0 is the degree of regularity of G, i.e.

$$\gamma_0 = \Delta(\mathsf{G}) = \sum_{i=d}^{n}\binom{n}{i}(q-1)^i. \tag{4.4.19}$$

By evaluating the generating function at $z = 1$ from Theorem 4.4.7, we can conclude that for every $0 < j \leq n$,

$$\sum_{i=0}^{n} K_i(j;n,q) = 0. \tag{4.4.20}$$

By the Summation formula in Theorem 4.4.7, for every $1 \leq j \leq n$,

$$\gamma_j = -\sum_{i=0}^{d-1} K_i(j;n,q) = -K_{d-1}(j-1;n-1,q). \tag{4.4.21}$$

Together, equalities (4.4.19) and (4.4.21) conclude the proof.

2. We compare γ_1 and γ_m for $2 \leq m \leq n-1$. By (4.4.21),

$$\gamma_1 - \gamma_m = K_{d-1}(m-1; n-1, q) - K_{d-1}(0; n-1, q) \tag{4.4.22}$$
$$= K_{d-1}(m-1; n-1, q) - (q-1)^{d-1}\binom{n-1}{d-1}. \tag{4.4.23}$$

Let $\mathbf{x} \in \mathbb{F}_q^{n-1}$ be a vector with Hamming weight $w_H(\mathbf{x}) = m-1$. By the character-theoretic representation of Krawtchouk polynomials (Theorem 4.4.8),

$$K_{d-1}(m-1; n-1, q) = \sum_{\mathbf{y}\in\mathbb{F}_q^{n-1} : w_H(\mathbf{y})=d-1} \omega^{\mathrm{Tr}(\langle \mathbf{x}, \mathbf{y}\rangle)}, \tag{4.4.24}$$

where $\omega = e^{2\pi i/p}$. Since $|\omega^{\mathrm{Tr}(\langle \mathbf{x}, \mathbf{y}\rangle)}| = 1$, we have

$$K_{d-1}(m-1; n-1, q) \leq \sum_{\mathbf{y}\in\mathbb{F}_q^{n-1} : w_H(\mathbf{y})=d-1} 1 = (q-1)^{d-1}\binom{n-1}{d-1}. \tag{4.4.25}$$

We claim that the inequality is strict. Indeed, equality would require that

$$\omega^{\mathrm{Tr}(\langle \mathbf{x}, \mathbf{y}\rangle)} = 1 \quad \text{for all } \mathbf{y} \text{ with } w_H(\mathbf{y}) = d-1,$$

i.e., the additive character $\mathbf{y} \mapsto \omega^{\mathrm{Tr}(\langle \mathbf{x}, \mathbf{y}\rangle)}$ is identically equal to 1 on this set. However, when $\mathbf{x} \neq 0$, this character is nontrivial (see, e.g., [93, Ch. 5] or [102, Ch. 5]), and therefore it cannot be identically equal to 1 on all vectors of a given weight. Since $w_H(\mathbf{x}) = m-1 \geq 1$, we conclude that the inequality in (4.4.25) is strict. Hence,

$$\gamma_1 - \gamma_m < 0, \qquad \forall\, 2 \leq m \leq n-1. \tag{4.4.26}$$

3. In the cases where $j = 1$ or $j = n$, the eigenvalues γ_1 and γ_n can be expressed explicitly as follows:

$$\gamma_1 = -K_{d-1}(0; n-1, q) = -(q-1)^{d-1}\binom{n-1}{d-1} \tag{4.4.27}$$
$$\gamma_n = -K_{d-1}(n-1; n-1, q) = (-1)^{d}\binom{n-1}{d-1}. \tag{4.4.28}$$

Hence,

$$\gamma_1 = \gamma_n \iff q = 2 \text{ and } d \text{ is odd}. \tag{4.4.29}$$

4. By Equations (4.4.4) and (4.4.14), every polynomial $p(\mathbf{A}(\mathsf{G})) = \sum_{k=0}^{r} \alpha_k\, \mathbf{A}(\mathsf{G})^k$ of the adjacency matrix can be expressed in terms of the orthogonal projections as follows:

$$p(\mathbf{A}(\mathsf{G})) = \sum_{k=0}^{r} \alpha_k\, \mathbf{A}(\mathsf{G})^k = \sum_{k=0}^{r} \alpha_k \left(\sum_{j=0}^{n} \gamma_j \mathbf{E}_j \right)^k = \sum_{k=0}^{r} \sum_{j=0}^{n} \alpha_k \gamma_j^k \mathbf{E}_j = \sum_{j=0}^{n} p(\gamma_j) \mathbf{E}_j. \quad (4.4.30)$$

□

Remark 4.4.11. The eigenvalues of the generalized-Hamming graph $\mathsf{H} \triangleq \mathcal{G}_{q,n,d}$ are given in [148]:

$$(\Delta(\mathsf{H}), K_{d-1}(0; n-1, q) - 1, \ldots, K_{d-1}(n-1; n-1, q) - 1). \quad (4.4.31)$$

Since $\mathcal{G}_{q,n,d}$ is regular with degree $\Delta(\mathsf{H})$, By [20, Section 1.3.2] the eigenvalues of $\overline{\mathcal{G}_{q,n,d}}$ are given as a function of $|\,\mathsf{V}(\mathsf{H})| = q^n$ and the eigenvalues of H as follows:

$$(q^n - 1 - \Delta(\mathsf{H}), -K_{d-1}(0; n-1, q), \ldots, -K_{d-1}(n-1; n-1, q)). \quad (4.4.32)$$

This expression is consistent with the spectrum of $\overline{\mathcal{G}_{q,n,d}}$ given in Theorem 4.4.10. Parts 2, 3, and 4 of Theorem 4.4.10 will be used in the proof of Theorem 4.1.2, specifically in the proof of Lemmata 4.5.1 and 4.5.2.

4.5 Proof of Theorem 4.1.2

Let $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$ be the complement of the generalized-Hamming graph. Since G is a Cayley graph, it is vertex-transitive, so it is edge-transitive if and only if for every two neighbors of the zero vector, $\mathbf{u}, \mathbf{v} \in \mathcal{N}(\mathbf{0})$, there exists an automorphism $\varphi \in \mathrm{Aut}(\mathsf{G})$ that maps the edge $\{\mathbf{0}, \mathbf{u}\}$ to $\{\mathbf{0}, \mathbf{v}\}$. The proof of Theorem 4.1.2 is divided into several lemmas. We first show that G is not edge-transitive under certain parameter choices. Afterwards, we prove that in the remaining cases G is indeed edge-transitive.

Lemma 4.5.1. *Let $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$. If $q \neq 2$ and $1 < d < n$, or if $q = 2$, $1 < d < n$ and d is even, then G is not edge-transitive.*

Proof. By the connectivity of the graph G, the largest eigenvalue γ_0 of its adjacency matrix is simple (see, e.g., [41, Proposition 12.1.1]); in particular, $\gamma_1 < \gamma_0$. By Part 2 of Theorem 4.4.10, it also holds that $\gamma_1 < \gamma_m$ for all $m \in \{2, \ldots, n-1\}$. Furthermore, under the hypotheses of the lemma that $q \neq 2$ or d is even if $q = 2$, Part 3 of Theorem 4.4.10 ensures that $\gamma_1 \neq \gamma_n$. Hence, the polynomial $p \colon \mathbb{R} \to \mathbb{R}$ given by

$$p(t) \triangleq \prod_{m \neq 1} \frac{t - \gamma_m}{\gamma_1 - \gamma_m}, \qquad t \in \mathbb{R}, \tag{4.5.1}$$

is well defined under the hypotheses of the lemma. By construction, it satisfies $p(\gamma_1) = 1$ and $p(\gamma_m) = 0$ for every $m \neq 1$ (i.e., $m \in \{0, 2, \dots, n\}$). Consequently, by Theorem 4.4.10 (see (4.4.16)), we obtain

$$p(\mathbf{A}(\mathsf{G})) = \mathbf{E}_1, \tag{4.5.2}$$

whose explicit expression is given by

$$\mathbf{E}_1 = \frac{1}{q^n} \sum_{i=0}^{n} K_1(i; n, q)\, \mathbf{A}_i\,. \tag{4.5.3}$$

Therefore, for every two vertices $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$ with Hamming distance $w_H(\mathbf{u}, \mathbf{v}) = i$, the entry that corresponds to $(\mathbf{u}, \mathbf{v})$ in the matrix $\mathbf{E}_1$ is given by

$$(\mathbf{E}_1)_{\mathbf{u},\mathbf{v}} = \frac{K_1(i; n, q)}{q^n} = \frac{(q-1)(n-i) - i}{q^n}. \tag{4.5.4}$$

Suppose, to the contrary, that G is edge-transitive. Let $\mathbf{u}, \mathbf{v} \in \mathcal{N}(\mathbf{0})$ be two vertices such that $w_H(\mathbf{u}) = n$ and $w_H(\mathbf{v}) = n - 1$. (Indeed, since G is the complement of $\mathcal{G}_{q,n,d}$ with $d < n$, it follows from the assumptions of the lemma that both $\mathbf{u}$ and $\mathbf{v}$ are adjacent to the vertex corresponding to the all-zero vector.) By the edge transitivity assumption, there exists an automorphism $\varphi \in \mathrm{Aut}(\mathsf{G})$ that maps the edge $\{\mathbf{0}, \mathbf{u}\}$ to $\{\mathbf{0}, \mathbf{v}\}$. Let P_φ be the permutation matrix that corresponds to an automorphism φ in G. Then,

$$\mathbf{E}_1 = p(\mathbf{A}(\mathsf{G})) = P_\varphi p(\mathbf{A}(\mathsf{G})) P_\varphi^{-1} = P_\varphi \mathbf{E}_1 P_\varphi^{-1}, \tag{4.5.5}$$

which implies that the following relation is satisfied:

$$(\mathbf{E}_1)_{\mathbf{0},\mathbf{v}} - (\mathbf{E}_1)_{\mathbf{0},\mathbf{u}} = 0. \tag{4.5.6}$$

Setting $i = n$ and $i = n - 1$ in (4.5.4) (as the Hamming weights of $\mathbf{u}$ and $\mathbf{v}$, respectively) gives

$$(\mathbf{E}_1)_{\mathbf{0},\mathbf{u}} = -\frac{n}{q^n} \tag{4.5.7}$$

$$(\mathbf{E}_1)_{\mathbf{0},\mathbf{v}} = \frac{q-n}{q^n} \tag{4.5.8}$$

$$\Longrightarrow (\mathbf{E}_1)_{\mathbf{0},\mathbf{v}} - (\mathbf{E}_1)_{\mathbf{0},\mathbf{u}} = \frac{1}{q^{n-1}} \neq 0, \tag{4.5.9}$$

contradicting equation (4.5.6). Hence, G cannot be edge-transitive under the hypotheses of Lemma 4.5.1.

□

Lemma 4.5.2. *Let* $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$. *If* $q = 2$ *and* $d < n-1$ *is an odd number, then* G *is not edge-transitive.*

Proof. The proof is similar to that of Lemma 4.5.1. By Theorem 4.4.10, the polynomial

$$p(t) \triangleq \prod_{m \notin \{1,n\}} \frac{t-\gamma_m}{\gamma_1-\gamma_m} \tag{4.5.10}$$

is well defined. Evaluating the polynomial at γ_i for $0 \leq i \leq n$ gives $p(\gamma_1) = p(\gamma_n) = 1$ and $p(\gamma_m) = 0$ for every $m \notin \{1, n\}$. Thus, by equality (4.4.16),

$$p(\mathbf{A}(\mathsf{G})) = \mathbf{E}_1 + \mathbf{E}_n. \tag{4.5.11}$$

An explicit expression for $\mathbf{E}_n$ is given by

$$\begin{aligned}\mathbf{E}_n &= \frac{1}{2^n}\sum_{i=0}^{n} K_n(i)\,\mathbf{A}_i \\ &= \frac{1}{2^n}\sum_{i=0}^{n} (-1)^i\,\mathbf{A}_i\,. \end{aligned}\tag{4.5.12}$$

where equality (4.5.12) holds since, by (4.4.13),

$$K_n(i) := K_n(i; n, 2) = (-1)^i, \qquad i \in \{0, 1, \ldots, n\}.$$

Let $\mathbf{u}, \mathbf{v} \in \mathcal{N}(\mathbf{0})$ be two vertices such that $w_H(\mathbf{u}) = n$ and $w_H(\mathbf{v}) = n-2$ (note that, since $d < n-1$, both $\mathbf{u}$ and $\mathbf{v}$ are neighbors of the vertex corresponding to the all-zero vector in $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$). Suppose to the contrary that G is edge-transitive. Then, there exists an automorphism $\varphi \in \mathrm{Aut}(\mathsf{G})$ that maps the edge $\{\mathbf{0}, \mathbf{u}\}$ to the edge $\{\mathbf{0}, \mathbf{v}\}$. Let P_φ be the permutation matrix corresponding to the automorphism φ. Since φ is a graph automorphism, we get

$$\mathbf{E}_1 + \mathbf{E}_n = p(\mathbf{A}(\mathsf{G})) = P_\varphi p(\mathbf{A}(\mathsf{G}))P_\varphi^{-1} = P_\varphi(\mathbf{E}_1 + \mathbf{E}_n)P_\varphi^{-1}. \tag{4.5.13}$$

Hence, the following relation must hold:

$$(\mathbf{E}_1 + \mathbf{E}_n)_{\mathbf{0},\mathbf{v}} - (\mathbf{E}_1 + \mathbf{E}_n)_{\mathbf{0},\mathbf{u}} = 0. \tag{4.5.14}$$

However, by equations (4.5.4), (4.5.12), and (4.5.13), we obtain

$$(\mathbf{E}_1 + \mathbf{E}_n)_{\mathbf{0},\mathbf{u}} = \frac{(n-2n) + (-1)^n}{2^n} = \frac{-n + (-1)^n}{2^n}, \tag{4.5.15}$$

$$(\mathbf{E}_1 + \mathbf{E}_n)_{\mathbf{0},\mathbf{v}} = \frac{(n-2(n-2)) + (-1)^{n-2}}{2^n} = \frac{-n + 4 + (-1)^n}{2^n}, \tag{4.5.16}$$

$$\Longrightarrow (\mathbf{E}_1 + \mathbf{E}_n)_{\mathbf{0},\mathbf{v}} - (\mathbf{E}_1 + \mathbf{E}_n)_{\mathbf{0},\mathbf{u}} = 2^{-(n-2)}, \tag{4.5.17}$$

which contradicts equation (4.5.14). Hence, the graph G cannot be edge-transitive. □

Lemma 4.5.3. *Let $\mathsf{G} = \overline{\mathcal{G}_{q,n,n}}$ be the complement of the generalized-Hamming graph with $d = n$. Then,*

1. G *is edge-transitive.*
2. *It is distance transitive if and only if $q = 2$ or $n = 2$.*

Proof. We first start by proving that G is edge-transitive, followed by the proof that G is distance-transitive if $q = 2$ or $n = 2$. Then we show that if the latter condition is violated, then G is not distance-transitive.

1. If $q = 2$, the only neighbor of $\mathbf{0}$ is $\mathbf{1}_n$. Hence, G is trivially edge-transitive.

   If $q > 2$, let $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$ have all entries nonzero. Define $\varphi \colon \mathbb{F}_q^n \to \mathbb{F}_q^n$ by

   $$\varphi(x_1, \ldots, x_n) = (u_1^{-1}x_1v_1, \ldots, u_n^{-1}x_nv_n).$$

   Then φ is a bijection with inverse

   $$\varphi^{-1}(y_1, \ldots, y_n) = (u_1y_1v_1^{-1}, \ldots, u_ny_nv_n^{-1}).$$

   Each coordinate is multiplied by a nonzero field element. Hence, Hamming distances are preserved. Therefore, $\varphi \in \mathrm{Aut}(\mathsf{G})$ (i.e., the mapping φ is a graph automorphism of G). Moreover, $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi(\mathbf{u}) = \mathbf{v}$. Thus, $\{\mathbf{0}, \mathbf{u}\}$ maps to $\{\mathbf{0}, \mathbf{v}\}$. Hence, G is edge-transitive.

2. If $q = 2$, then G is a disjoint union of K_2's. Thus, it is distance-transitive.

   Assume $q \geq 3$, and consider the cases where $n = 2$ or $n \geq 3$ separately.

   Case $n = 2$**.** The vertices at distance 2 from $\mathbf{0}$ are those of weight 1. For any two such vertices $u, v \in \mathbb{F}_q^n$, there exists a coordinate permutation π such that u and $\pi(v)$ have the same support. Composing with φ gives distance-transitivity.

   Case $n > 2$**.** We show that G is not distance-transitive. Let

$$\mathbf{u} = (1, 0, 0, \dots, 0), \qquad \mathbf{v} = (0, 1, 1, \dots, 1).$$

   Both are at distance 2 from $\mathbf{0}$. Assume, for contradiction, that G is distance-transitive. Then, there exists $\varphi \in \mathrm{Aut}(\mathsf{G})$ fixing $\mathbf{0}$ and mapping $\mathbf{u}$ to $\mathbf{v}$. Define

$$C(\mathbf{x}, \mathbf{y}) = |\,\mathcal{N}(\mathbf{x}) \cap \mathcal{N}(\mathbf{y})|.$$

   In a distance-transitive graph, $C(\mathbf{x}, \mathbf{y})$ depends only on $d(\mathbf{x}, \mathbf{y})$. Hence,

$$C(\mathbf{0}, \mathbf{u}) = C(\mathbf{0}, \mathbf{v}).$$

   Computation of $C(\mathbf{0}, \mathbf{u})$**.** A common neighbor z satisfies $z_i \neq 0$ for all $i \in [n]$, and $z_1 \neq 1$. Thus,

$$z_1 \in \mathbb{F}_q \setminus \{0, 1\}, \quad z_i \in \mathbb{F}_q \setminus \{0\} \ (i \geq 2).$$

   Hence,

$$C(\mathbf{0}, \mathbf{u}) = (q-2)(q-1)^{n-1}.$$

   Computation of $C(\mathbf{0}, \mathbf{v})$**.** Now z satisfies $z_i \neq 0$ for all i, and $z_i \neq 1$ for $i \geq 2$. Thus,

$$z_1 \in \mathbb{F}_q \setminus \{0\}, \quad z_i \in \mathbb{F}_q \setminus \{0, 1\} \ (i \geq 2).$$

   Hence,

$$C(\mathbf{0}, \mathbf{v}) = (q-1)(q-2)^{n-1}.$$

   For $n > 2$, $C(\mathbf{0}, \mathbf{u}) \neq C(\mathbf{0}, \mathbf{v})$, which leads to a contradiction. Therefore, G is not distance-transitive.

□

Lemma 4.5.4. *If* $(q, d) = (2, n-1)$ *and* n *is even, then* $\mathsf{G} = \overline{\mathcal{G}_{2,n,n-1}}$ *is distance-transitive, and hence edge-transitive.*

Proof. The distance layers from $\mathbf{0}$ are determined by Hamming weights. In particular, the diameter of G is $\lceil n/2 \rceil$, and the distance-k layer consists of all vectors $\mathbf{v}$ with

$$w_H(\mathbf{v}) \in \begin{cases} \{n-k+1,\ n-k\}, & \text{if } k \text{ is odd,} \\ \{k-1,\ k\}, & \text{if } k \text{ is even.} \end{cases}$$

Since n is even, any two vectors in the same layer have weights that are either equal or differ by 1. Let $1 \leq k \leq \frac{n}{2}$, and let $\mathbf{u}, \mathbf{v} \in \mathsf{V}(\mathsf{G})$ satisfy

$$\mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{u}) = \mathrm{d}_{\mathsf{G}}(\mathbf{0}, \mathbf{v}) = k.$$

If $w_H(\mathbf{u}) = w_H(\mathbf{v})$, then there exists a coordinate permutation π that maps $\mathbf{u}$ to $\mathbf{v}$. Since π fixes $\mathbf{0}$, it is an automorphism that maps $\{\mathbf{u}, \mathbf{0}\}$ to $\{\mathbf{v}, \mathbf{0}\}$.

Assume now that $w_H(\mathbf{u}) \neq w_H(\mathbf{v})$. Since n is even, we may assume

$$w_H(\mathbf{u}) = 2i, \qquad w_H(\mathbf{v}) = 2i-1,$$

for some $1 \leq i \leq \frac{n}{2}$. Since coordinate permutations are automorphisms of G, it suffices to construct an automorphism mapping $(\mathbf{0}, \tilde{\mathbf{u}})$ to $(\mathbf{0}, \tilde{\mathbf{v}})$, where

$$\tilde{\mathbf{u}} = (\mathbf{J}_{2i-1}, \mathbf{0}_{n-2i}, 1), \tag{4.5.18}$$
$$\tilde{\mathbf{v}} = (\mathbf{J}_{2i-1}, \mathbf{0}_{n-2i}, 0). \tag{4.5.19}$$

Define $\varphi \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ by

$$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \mathrm{par}(\mathbf{x})), \tag{4.5.20}$$
$$\mathrm{par}(\mathbf{x}) = w_H(\mathbf{x}) \bmod 2. \tag{4.5.21}$$

Then φ is an involution, and hence bijective. Moreover, $\varphi(\mathbf{0}) = \mathbf{0}$. We show that φ is an automorphism of G. Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$.

- If $\mathrm{d}_H(\mathbf{x}, \mathbf{y}) = n$, then the first $n-1$ coordinates of $\varphi(\mathbf{x})$ and $\varphi(\mathbf{y})$ differ. Hence,

  $$\mathrm{d}_H(\varphi(\mathbf{x}), \varphi(\mathbf{y})) \geq n-1,$$

  so $\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\} \in \mathsf{E}(\mathsf{G})$.

- If $\mathrm{d}_H(\mathbf{x}, \mathbf{y}) = n-1$, then $x_k = y_k$ for a unique k. Since n is even,

  $$\mathrm{par}(\mathbf{x}) + \mathrm{par}(\mathbf{y}) = \mathrm{par}(\mathbf{x} + \mathbf{y}) = n - 1 \equiv 1 \pmod 2,$$

so $\mathrm{par}(\mathbf{x}) \neq \mathrm{par}(\mathbf{y})$. Hence,

$$(\varphi(\mathbf{x}))_i \neq (\varphi(\mathbf{y}))_i \quad \text{for all } i \neq k,$$

and therefore

$$\mathrm{d}_H(\varphi(\mathbf{x}), \varphi(\mathbf{y})) \geq n - 1.$$

Thus, $\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\} \in \mathsf{E}(\mathsf{G})$.

Since φ is bijective, it follows that $\varphi \in \mathrm{Aut}(\mathsf{G})$. Finally, since $w_H(\tilde{\mathbf{u}}) = 2i$ and $w_H(\tilde{\mathbf{v}}) = 2i - 1$, we have $\mathrm{par}(\tilde{\mathbf{u}}) = 0$ and $\mathrm{par}(\tilde{\mathbf{v}}) = 1$. Hence, $\varphi(\tilde{\mathbf{u}}) = \tilde{\mathbf{v}}$. This completes the proof of the lemma. $\square$

Completion of the proof of Theorem 4.1.2

Summarizing Lemmas 4.5.1–4.5.4, we obtain the following:

1. If $d = n$, then $\overline{\mathcal{G}_{q,n,d}}$ is edge-transitive. In this case, it is distance-transitive if and only if $q = 2$ or $n = 2$ (Lemma 4.5.3).
2. If $d = n - 1$ and n is even, then $\overline{\mathcal{G}_{q,n,d}}$ is distance-transitive, and hence edge-transitive (Lemma 4.5.4).
3. In all remaining cases, the graph is not edge-transitive. Specifically, this holds if $q \neq 2$ and $1 < d < n$, or if $q = 2$ and $1 < d < n - 1$ (Lemma 4.5.1), or if $d = n - 1$ and n is odd (Lemma 4.5.2). In these cases, the graph is also not distance-transitive, since distance-transitivity implies edge-transitivity.

This completes the proof of Theorem 4.1.2.

4.6 The Lovász ϑ function

The Lovász ϑ-function of a graph, denoted $\vartheta(\mathsf{G})$, is a graph invariant that exhibits deep connections to several fundamental graph invariants, including the independence number $\alpha(\mathsf{G})$, clique number $\omega(\mathsf{G})$, chromatic number $\chi(\mathsf{G})$, fractional chromatic number $\chi_{\mathrm{f}}(\mathsf{G})$, Shannon capacity $\Theta(\mathsf{G})$, and others (see, e.g., [9,114]). For a comprehensive overview, we refer to [124], and for a unified framework encompassing these invariants, see [61]. Owing to its central role, the Lovász ϑ-function has also been generalized in [120] for the study of graph homomorphisms. For the definition of the Lovász ϑ-function, see definitions 2.2.19, 2.2.20. and the followed discussion. The present section provides additional necessary background on the Lovász ϑ-function for Section 4.7.

Theorem 4.6.1 (Properties of the Lovász ϑ-function)**.** *Let* G *be a finite, simple, and undirected graph on N vertices. The following properties hold:*

1.

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) \geq N, \tag{4.6.1}$$

with an equality if G *is vertex-transitive or strongly regular.*

2. *The following inequalities hold:*

$$\alpha(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \chi_{\mathrm{f}}(\overline{\mathsf{G}}) \leq \chi(\overline{\mathsf{G}}) \tag{4.6.2}$$

$$\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi_{\mathrm{f}}(\mathsf{G}) \leq \chi(\mathsf{G}). \tag{4.6.3}$$

Definition 4.6.2 (Maximum cut and surplus)**.** Let $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ be a graph. A *maximum cut* of G is a partition of the vertex set $\mathsf{V}(\mathsf{G})$ into two disjoint subsets $\mathcal{S}$ and $\mathcal{T}$ that maximizes the number of edges with one endpoint in $\mathcal{S}$ and the other in $\mathcal{T}$.

1. The *max-cut* of G, denoted by $\mathrm{mc}(\mathsf{G})$, is the maximum number of such edges, taken over all partitions of $\mathsf{V}(\mathsf{G})$. Equivalently, $\mathrm{mc}(\mathsf{G})$ is the maximum number of edges in a bipartite spanning subgraph of G.

2. The *surplus* of G is defined as

$$\mathrm{sp}(\mathsf{G}) \triangleq \mathrm{mc}(\mathsf{G}) - \frac{1}{2}\,|\,\mathsf{E}(\mathsf{G})|.$$

Theorem 4.6.3 (Lower bounds on the surplus, [9])**.** *For every graph* G*, its max-cut satisfies*

$$\mathrm{sp}(\mathsf{G}) \geq \frac{1}{\pi} \cdot \frac{|\,\mathsf{E}(\mathsf{G})|}{\vartheta(\overline{\mathsf{G}}) - 1} \geq 0. \tag{4.6.4}$$

In particular, any upper bound on $\vartheta(\overline{\mathsf{G}})$ yields a corresponding (possibly weaker) lower bound on the surplus $\mathrm{sp}(\mathsf{G})$.

The next theorem bounds the Lovász ϑ-function in terms of the second largest eigenvalue of the adjacency matrix of the graph.

Theorem 4.6.4 (Bounds on the Lovász ϑ-function of regular graphs, [125])**.** *Let* G *be a* Δ*-regular graph of order* N*, which is neither complete nor empty. Then, the following bounds hold for the Lovász* ϑ*-function of* G *and its complement* $\overline{\mathsf{G}}$*:*

(1)

$$\frac{N - \Delta + \lambda_2(\mathsf{G})}{1 + \lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{N\lambda_N(\mathsf{G})}{\Delta - \lambda_N(\mathsf{G})}. \tag{4.6.5}$$

- *Equality holds in the leftmost inequality of* (4.6.5) *if* $\overline{\mathsf{G}}$ *is both vertex-transitive and edge-transitive, or if* G *is a strongly regular graph;*
- *Equality holds in the rightmost inequality of* (4.6.5) *if* G *is edge-transitive, or if* G *is a strongly regular graph.*

(2)

$$1 - \frac{\Delta}{\lambda_N(\mathsf{G})} \le \vartheta(\overline{\mathsf{G}}) \le \frac{N(1 + \lambda_2(\mathsf{G}))}{N - \Delta + \lambda_2(\mathsf{G})}. \tag{4.6.6}$$

- *Equality holds in the leftmost inequality of* (4.6.6) *if* G *is both vertex-transitive and edge-transitive, or if* G *is a strongly regular graph;*
- *Equality holds in the rightmost inequality of* (4.6.6) *if* $\overline{\mathsf{G}}$ *is edge-transitive, or if* G *is a strongly regular graph.*

Since the Lovász ϑ-function is computable in polynomial time via SDP, the sandwich inequalities in (4.6.2) and (4.6.3) and the lower bound on the surplus of a graph in Theorem 4.6.3 are particularly useful since they provide bounds on graph invariants whose computation is, in general, NP-hard.

4.7 Application

While the algorithm for solving the SDP problem in (2.2.6) is polynomial in $|\mathsf{V}(\mathsf{G})|$, for the generalized-Hamming graph $\mathcal{G}_{q,n,d}$ we have $|\mathsf{V}(\mathcal{G}_{q,n,d})| = q^n$. Hence, although the algorithm is polynomial in the number of vertices, its complexity is exponential in n. However, for any set of parameters (q, n, d) such that $\mathcal{G}_{q,n,d}$ or its complement is edge-transitive, it is possible to compute the exact value of the Lovász ϑ-function of both $\mathcal{G}_{q,n,d}$ and its complement $\overline{\mathcal{G}_{q,n,d}}$. All generalized-Hamming graphs are vertex-transitive, since they are Cayley graphs. Thus, by Theorem 4.6.1(1), for every choice of parameters (q, n, d), we have $\vartheta(\mathcal{G}_{q,n,d}) \cdot \vartheta(\overline{\mathcal{G}_{q,n,d}}) = q^n$.

Combining the equality conditions in Theorem 4.6.4 with the complete classification of edge-transitive generalized-Hamming graphs and their complements from Theorems 4.1.1 and 4.1.2, we obtain exact closed-form expressions for the Lovász ϑ-function in the following cases, summarized in the next theorem.

Theorem 4.7.1. *Let* $\mathsf{G} = \mathcal{G}_{q,n,d}$ *be a generalized-Hamming graph with regularity* $\Delta \triangleq \sum_{i=1}^{d-1} \binom{n}{i}(q-1)^i$. *Then*

1. *If either $d = 2$, or $(q, d) = (2, 3)$, or $(q, d) = (2, n)$, then*

$$\vartheta(\mathsf{G}) = -\frac{q^n \, \lambda_{min}(\mathsf{G})}{\Delta - \lambda_{min}(\mathsf{G})} \tag{4.7.1}$$

$$\vartheta(\overline{\mathsf{G}}) = 1 - \frac{\Delta}{\lambda_{min}(\mathsf{G})}, \tag{4.7.2}$$

   where $\lambda_{min}(\mathsf{G})$ is the smallest eigenvalue of the adjacency matrix of G.

2. *If either $(q, d) = (2, n - 1)$ for an even n, or $d = n$, then*

$$\vartheta(\mathsf{G}) = \frac{q^n - \Delta + \lambda_2(\mathsf{G})}{1 + \lambda_2(\mathsf{G})} \tag{4.7.3}$$

$$\vartheta(\overline{\mathsf{G}}) = \frac{q^n(1 + \lambda_2(\mathsf{G}))}{q^n - \Delta + \lambda_2(\mathsf{G})}, \tag{4.7.4}$$

   where $\lambda_2(\mathsf{G})$ is the second largest eigenvalue of the adjacency matrix of G.

The smallest and second largest eigenvalues λ_{min} and λ_2 are obtained from equalities (4.4.15). The complexity for computing these eigenvalues is $O(n^2)$, which is a significant improvement over the general SDP-based approach, whose complexity is exponential in n for this family of graphs.

Chapter 5

Summary and Outlook

This final chapter summarizes the main contributions of the thesis and outlines several directions for future research. We briefly review the two principal themes developed in this work: spectral graph determination and graph transitivity, while highlighting the main results, methods, and conceptual insights obtained in each direction. We then discuss open problems and possible extensions that arise naturally from the results of this thesis.

5.1 Summary of the thesis

This thesis investigates two central directions in algebraic graph theory: spectral graph determination and graph transitivity.

The first direction concerns spectral graph determination and is developed in Chapters 2 and 3. We study four matrices associated with a graph- the adjacency matrix, Laplacian, signless Laplacian, and normalized Laplacian- and analyze which structural properties can be recovered from their spectra. We survey classical and recent results on spectral graph determination, with particular emphasis on the adjacency matrix, and analyze their implications for various graph families.

In addition to this survey, we establish additional spectral properties of certain graph families, including strongly regular graphs, and develop new proof techniques for spectral characterization. In particular, we provide new proofs that complete bipartite graphs and Turán graphs are $\mathbf{A}$-DS by formulating their characterization as optimization problems constrained by spectral data. We also provide a systematic analysis of known constructions of non-isomorphic cospectral graphs.

Furthermore, we study the graphs of pyramids, and prove that they are $\mathbf{A}$-DS using tools from matrix analysis, including Cauchy interlacing and Schur complements.

The second direction of this thesis studies symmetry properties of generalized-Hamming graphs, and is developed in Chapter 4. We investigate the edge- and distance-transitivity of generalized-

Hamming graphs $\mathcal{G}_{q,n,d}$ and their complements, and characterize these properties in terms of the parameters q, n, d. Our analysis combines combinatorial arguments with algebraic techniques from group theory and the theory of association schemes. As an application, we derive closed-form expressions for the Lovász ϑ-function in cases where these graphs are edge-transitive.

These results highlight the effectiveness of spectral and algebraic methods in capturing both structural and symmetry properties of graphs. In particular, many graph invariants can be expressed or bounded in terms of eigenvalues, revealing deep connections between algebraic and combinatorial aspects of graph theory.

In summary, the main contributions of this thesis are as follows:

- A comprehensive study of spectral graph properties with respect to the matrices $\mathbf{A}, \mathbf{L}, \mathbf{Q}$, and $\mathcal{L}$, including new structural results and proof techniques.
- New characterizations of graphs determined by their spectrum, including alternative proofs of the spectral characterization of complete bipartite graphs and Turán graphs (Theorems 2.4.7 and 2.4.21), as well as the introduction of a new $\mathbf{A}$-DS family, the graphs of pyramids (Theorem 3.4.4).
- A characterization of the edge- and distance-transitivity of generalized-Hamming graphs and their complements, together with new spectral results and a closed-form expression for the Lovász ϑ-function of these graphs whenever either the graph or its complement is edge-transitive (Theorem 4.7.1).

This thesis also introduces several new proof techniques:

- Establishing spectral determination by characterizing a graph as the unique feasible solution of an optimization problem derived from spectral constraints.
- Deriving closed-form expressions for graph spectra using Schur complements.
- Constructing parity-based automorphisms of graphs.
- Proving that a graph G is not edge-transitive by constructing suitable polynomials in $\mathbf{A}(\mathsf{G})$ and showing that they fail to satisfy the required symmetry conditions. This technique may extend to other families of Cayley graphs.

5.2 Outlook

The results of this thesis suggest several promising directions for future research in spectral graph theory.

A central open problem in the area is Haemers' conjecture, which posits that most graphs are determined by their adjacency spectrum. While the results presented here provide further evidence supporting this conjecture, it remains widely open and likely requires fundamentally new ideas. A significant step toward resolving this conjecture would be to identify a class $\mathcal{X} \subseteq \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}, \overline{\mathbf{A}}, \overline{\mathbf{L}}, \overline{\mathbf{Q}}, \overline{\mathcal{L}}\}$ such that

$$\lim_{n\to\infty} \frac{\alpha_{\mathcal{X}}(n)}{I(n)} = 1.$$

Another natural direction is the identification of additional families of graphs that are determined by their spectrum, as well as the development of new general techniques for proving spectral determination. From an algorithmic perspective, it would also be of interest to design efficient methods for reconstructing graphs from spectral data.

The study of transitivity properties of generalized-Hamming graphs and their complements also opens several avenues for further research. One direction is to derive explicit descriptions of their automorphism groups. Techniques from the theory of association schemes may play a central role in this analysis. More broadly, the methods developed in this thesis could be applied to other graph families arising from association schemes, such as those related to the Johnson scheme.

Another promising direction is to investigate how transitivity properties influence other graph invariants, including the chromatic number, independence number, and Shannon capacity. This direction is particularly relevant in coding theory, where generalized-Hamming graphs are closely connected to error-correcting codes and the Gilbert-Varshamov bound.

Overall, these directions highlight the rich interplay between spectral methods, combinatorial structure, and algebraic techniques, and suggest that further advances in this area will continue to deepen our understanding of graph structure and symmetry.

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תרומות בתורת הגרפים האלגברית

חיבור על מחקר

הוגשה למילוי חלקי מהדרישות לקבלת התואר לשם מילוי חלקי של הדרישות לקבלת התואר
מוסמך למדעים במדעי המחשב

נעם קרופניק

הוגש לסנאט של הטכניון - מכון טכנולוגי לישראל
אב תשפ"ו חיפה יולי 2026

המחקר נעשה בפקולטה למדעי המחשב, בהנחיית פרופ' יגאל ששון ופרופ' אמריטוס אברהם ברמן.

מחבר חיבור זה מצהיר כי המחקר, כולל איסוף הנתונים, עיבודם והצגתם, התייחסות והשוואה למחקרים קודמים וכו', נעשה כולו בצורה ישרה, כמצופה ממחקר מדעי המבוצע לפי אמות המידה האתיות של העולם האקדמי. כמו כן, הדיווח על המחקר ותוצאותיו בחיבור זה נעשה בצורה ישרה ומלאה, לפי אותן אמות מידה.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי

תודות

ברצוני להודות למנחיי, פרופסור יגאל ששון ופרופסור אמריטוס אברהם ברמן, על ההנחיה והליווי לאורך כל תקופת לימודיי לתואר השני. התמיכה, ההכוונה והרוח הגבית שקיבלתי מכם היו גורם מרכזי בהשלמת עבודה זו. הייתה לי הזכות לעבוד עמכם וללמוד מן הידע הרב ומן החשיבה היצירתית שלבם. בנוסף, ברצוני להודות לפרופסור רוני רוט ופרופסור רפאל יוסטר על קריאת התיזה באופן מעמיק, והשתתפותם בוועדת הבחינה שלי.

אני מבקש להודות גם לפרופסור איתן יעקובי ולגברת מלינדה מרגוליס על העזרה והתמיכה שהענקתם לי עוד מראשית לימודיי לתואר הראשון, ועל כך שהאמנתם בי גם כשאני פקפקתי בעצמי.

עבודת תזה זו מוקדשת לסבתי, ד"ר דפנה שאשא, ולזכרו של סבי, פרופסור שאול שאשא ז"ל, שהשפעתם על חיי בכלל ועל דרכי האקדמית בפרט הייתה עצומה. סבי היה עבורי מנטור לחיים, ויכולנו לדבר על כל נושא, לשתף רעיונות, סיפורים, תובנות ובדיחות, והוא האדם החכם ביותר שפגשתי. הדיונים המתמטיים שסבתי ואני חולקים מאז ילדותי ועד היום הם תמיד מקור לנחת והשראה.

תודה רבה לאחותי ענבל ולאחי כרמל. כרמל, תודה על אינספור הדיונים על פונקציות, התפלגות הדלתא של דיראק ופילוסופיה של המתמטיקה. ענבל, תודה שהקשבת לשיתופים בנוגע לעבודתי, אף שמתמטיקה אינה התחום האהוב עלייך.

לבסוף, אני רוצה להודות להוריי, חיה וחגי, שהיו חלק חיוני ובלתי נפרד ממסע זה. תמיד תמכתם בי בכל דרכי והענקתם לי אהבה שאינה תלויה בדבר – בהצלחות ובכישלונות, בטוב וברע. אני בר־מזל להיות בנכם.

תקציר

תורת הגרפים האלגברית עוסקת בחקר מבנים קומבינטוריים באמצעות כלים אלגבריים כגון מטריצות, פולינומים וחבורות. לכל גרף ניתן לשייך מגוון מבנים אלגבריים, ובהם מטריצת השכנויות, מטריצות לפלסיאן מסוגים שונים, חבורת האוטומורפיזמים ועוד. שימוש בכלים אלו מאפשר לחשוף תכונות מבניות עמוקות של גרפים, ולעיתים להוביל לתוצאות שאינן נגישות בגישה הקומבינטורית הקלאסית. בתזה זו נחקרים הקשרים בין תכונות מבניות של גרפים לבין המבנים האלגבריים המשויכים להם, תוך שילוב כלים מתורת הגרפים הספקטרלית, תורת החבורות, ותורת סכמות הייחוס.

תורת הגרפים הספקטרלית היא תת־תחום מרכזי בתורת הגרפים האלגברית, החוקר גרפים באמצעות הערכים העצמיים והווקטורים העצמיים של מטריצות המייצגות אותם. בין המטריצות המרכזיות הנחקרות נמנות מטריצת השכנויות, מטריצת הלפלסיאן, מטריצת הלפלסיאן חסר הסימן, ומטריצת הלפלסיאן המנורמל. הספקטרום של גרף, ביחס לכל אחת ממטריצות אלו, הוא רב־קבוצת הערכים העצמיים של המטריצה (כולל הריבויים האלגבריים). לתורת הגרפים הספקטרלית יישומים רבים גם באלגוריתמיקה: בשונה מתכונות כגון מספר הקליקה והמספר הכרומטי, הידועות כבעיות קשות חישובית, הערכים העצמיים והווקטורים העצמיים של גרף ניתנים לחישוב באופן יעיל (בזמן פולינומי).

קיים קשר הדוק בין הערכים העצמיים של גרף לבין תכונות מבניות שונות שלו. מספקטרום השכנויות ניתן להסיק, בין היתר, את מספר הקודקודים, הקשתות והמשולשים בגרף, וכן האם הגרף רגולרי או דו־צדדי. מספקטרום הלפלסיאן ניתן להסיק את מספר רכיבי הקשירות של הגרף ואת מספר העצים הפורסים בגרף. מספקטרום הלפלסיאן חסר הסימן והלפלסיאן המנורמל ניתן להסיק מידע על רכיבי הקשירות הדו־צדדיים של הגרף. התזה סוקרת מאפיינים מבניים רבים הנקבעים על ידי הספקטרום ביחס לארבע המטריצות הנדונות, ומציגה גם תוצאות חדשות, הקשורות בין היתר לגרפים רגולריים במובן החזק.

אחת השאלות המרכזיות בתורת הגרפים הספקטרלית היא עד כמה הספקטרום של גרף קובע את מבנהו. שאלה זו הוצגה לראשונה על ידי M. Kac במאמרו משנת 1591, שכותרתו "האם ניתן לשמוע את צורתו של תוף?". ידוע כי גרפים איזומורפיים הם בעלי ספקטרום זהה, אך הכיוון ההפוך אינו נכון באופן כללי; כלומר, קיימים גרפים לא־איזומורפיים בעלי אותו ספקטרום. גרף מוגדר כגרף הנקבע על־פי הספקטרום (נ"ס) אם כל גרף אחר בעל אותו ספקטרום איזומורפי לו. בעשורים האחרונים הפכה בניית זוגות של גרפים לא־איזומורפיים בעלי ספקטרום זהה, לצד זיהוי גרפים נ"ס, למוקד מחקר מרכזי בתחום.

חלקה הראשון של התזה מוקדש לחקר הקשרים בין תכונות מבניות של גרפים לבין הספקטרום שלהם ביחס לארבע המטריצות שתוארו לעיל, תוך התמקדות במטריצת השכנויות, והוא מבוסס על שני מאמרים שפורסמו בנושא. בפרט, נדונים גרפים נ"ס ובניות של גרפים בעלי ספקטרום זהה. חלק זה כולל סקירה שיטתית של תוצאות קלאסיות ועדכניות, לצד הוכחות חדשות למספר טענות קיימות בתחום. בהוכחות אלו מנוסחות בעיות אופטימיזציה בדידה, שבהן פונקציית המטרה ותנאי האילוץ מתוארים באמצעות ספקטרום השכנויות של הגרף. בין היתר מוצגת הוכחה חדשה לכך שגרפי טוראן הם נ"ס, וכן ניתן משפט אפיון לקיום תכונה זו בגרפים דו־צדדיים מלאים, כתלות בגדלי הצדדים. הוכחות אלו אינן רק חלופיות להוכחות הקיימות בספרות, אלא מציעות פרספקטיבה המדגישה את הקשר בין מבנה קומבינטורי, תכונות ספקטרליות, ובעיות קיצון.

בנוסף מוצגת בתזה משפחה חדשה של גרפים, הנקראת גרפי פירמידות. גרפים אלו מורכבים מאוסף פירמידות מממד $n+1$ בעלות בסיס משותף מממד n. מתקבל ביטוי סגור לספקטרום השכנויות של גרפים אלו, ומוכח כי הם נקבעים לפי ספקטרום השכנויות.

חלקה השני של התזה עוסק בחקר תכונות סימטריה של גרפי המינג המוכללים וגרפיהם המשלימים. גרפי המינג המוכללים מהווים הכללה טבעית של גרפי המינג, ומאופיינים על ידי שלושה פרמטרים q,n,d. קודקודי הגרף הם איברי המרחב הווקטורי $\mathbb{F}_q^n$, כאשר שני קודקודים סמוכים אם ורק אם מרחק ההמינג ביניהם קטן מ־d. גרפים אלו מופיעים בהקשרים שונים בתורת הקודים ובתורת האינפורמציה, ובפרט בהוכחת שיפור של חסם גילברט–וורשמוב, וכן בהוכחה כי פונקציית ϑ של שרייבר אינה מהווה בהכרח חסם עליון לקיבול שאנון של גרף.

בעבודה זו נחקרות שתי תכונות סימטריה מרכזיות: טרנזיטיביות בצלעות וטרנזיטיביות במרחקים. תכונות אלו מהוות מדד חשוב לעושרה של חבורת האוטומורפיזמים של הגרף. ניתן אפיון מלא של הפרמטרים q,n,d עבורם גרפי המינג המוכללים מקיימים כל אחת מתכונות סימטריה אלו, וכן אפיון מלא לגרפים המשלימים. בפרט ניתנים תנאים הכרחיים ומספיקים לטרנזיטיביות בצלעות ולטרנזיטיביות במרחקים, ומוכח כי עבור גרפי המינג המוכללים שתי התכונות שקולות, בעוד ששקילות זו אינה מתקיימת עבור הגרפים המשלימים.

ההוכחות בחלק זה נשענות על כלים אלגבריים. בפרט נעשה שימוש בעובדה שגרפי המינג המוכללים הם גרפי קיילי, ובטכניקות מתורת החבורות לצורך ניתוח חבורת האוטומורפיזמים שלהם. בנוסף נעשה שימוש בתורת סכמות הייחוס, המאפשרת פירוק של מטריצות השכנויות לצירוף ליניארי של אופרטורים אורתוגונליים. במסגרת זו מתקבל גם תיאור מפורש של הספקטרום של גרפי המינג המוכללים ושל משלימיהם באמצעות פולינומי קראוצ'וק, שלהם תפקיד מרכזי בתורת ההצגות ובתורת הקודים.

לתוצאות המוצגות בחלק זה מספר יישומים. אחד המרכזיים שבהם נוגע לפונקציית ϑ של לובאס. פונקציה זו מהווה כלי חשוב בתורת האופטימיזציה הקומבינטורית ובתורת האינפורמציה, והיא קשורה בקשר הדוק לאינווריאנטים נוספים של גרפים, כגון מספר הקליקה והמספר הכרומטי. התזה מציגה ביטוי סגור לפונקציית לובאס של גרף המינג המוכלל ושל הגרף המשלים במקרים שבהם לפחות אחד מהם הינו טרנזיטיבי בצלעות.

תוצאות אלו מדגישות את עוצמתן של שיטות ספקטרליות ואלגבריות בניתוח בעיות קומבינטוריות מורכבות, וממחישות את הקשרים העמוקים בין מבנה, סימטריה וספקטרום של גרפים.