On Spectral Graph Determination and Transitivity of Generalized-Hamming Graphs and their Complements

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Thesis Defense

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Outline

The talk presents some of the results of my MSc thesis, with partial proofs. For the full details, please see:

[1] N. Krupnik and A. Berman, “The graphs of pyramids are determined by their spectrum,” *Linear Algebra and its Applications*, 2024.

[2] I. Sason, N. Krupnik, S. Hamud, and A. Berman, “On Spectral Graph Determination,” *Mathematics*, 2025, **13**, 549.

[3] N. Krupnik, I. Sason, and A. Berman, “On the transitivity of generalized-Hamming graphs and their complements,” submitted April 2026. Available at: https://arxiv.org/abs/2603.21627

Introduction and notations

- A graph $G = (V(G), E(G))$ is a pair where
  - $V(G)$ is the set of vertices. Denote $n \triangleq |V(G)|$.
  - $E(G) \subseteq \binom{V(G)}{2}$ is the set of edges.

  In this talk, all graphs are undirected, finite and simple (i.e. no self loops or multiple edges).
- The *adjacency matrix* of G, $\mathbf{A}(G)$, is a $n \times n$ matrix defined by:

$$\mathbf{A}(G)_{uv} = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of a graph

Definition

- The *spectrum of a matrix* $\mathbf{A}$, denoted $\sigma(\mathbf{A})$, is the multiset of its eigenvalues.
- The *spectrum of a graph* G, denoted $\sigma(\mathrm{G}) \triangleq \sigma(\mathbf{A}(\mathrm{G}))$, is the spectrum of its adjacency matrix.
- We denote the eigenvalues by $\lambda_1(\mathrm{G}) \geq \lambda_2(\mathrm{G}) \geq \ldots \geq \lambda_n(\mathrm{G})$.

Example

- The spectrum of $\mathbf{J}_4$ is $\sigma(\mathbf{J}_4) = \left\{[0]^3, 4\right\}$.
- **Example:** the complete graph K_4 has adjacency matrix $\mathbf{A}(\mathrm{K}_4) = \mathbf{J}_4 - \mathbf{I}_4$, so

$$\sigma(\mathrm{K}_4) = \sigma(\mathbf{J}_4 - \mathbf{I}_4) = \left\{[-1]^3, 3\right\}$$

Properties of the spectrum

- The number of vertices is the the number of eigenvalues (including multiplicities).
- The number of closed walks of length k in G is given by

$$\mathrm{Tr}(\mathbf{A}^k(\mathsf{G})) = \sum_{i=1}^{n} \lambda_i^k(\mathsf{G})$$

- G is bipartite $\iff$ $\sigma(\mathsf{G})$ is symmetric with respect to the origin, i.e., for every eigenvalue $\lambda \in \sigma(\mathsf{G})$, we have $-\lambda \in \sigma(\mathsf{G})$.
  - Otherwise, G contains an odd-length cycle.

The largest eigenvalue

- The largest eigenvalue $\lambda_1(\mathrm{G})$ satisfies

$$\frac{2|\,\mathrm{E}(G)|}{n} = \frac{1}{n}\sum_{v\in\mathrm{V}} \deg(v) \leq \lambda_1(\mathrm{G}) \leq \max_{v\in\mathrm{V}} \deg(v).$$

  - The upper bound is obtained by Gershgorin circle theorem.
  - The lower bound follows from the Rayleigh quotient.

$$\lambda_1(\mathrm{G}) = \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{x^T\,\mathbf{A}(\mathrm{G})x}{x^Tx} \geq \frac{\mathbf{1}^T\,\mathbf{A}(\mathrm{G})\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2|\,\mathrm{E}(G)|}{n}.$$

- G is regular if and only if $2|\,\mathrm{E}(\mathrm{G})| = n\lambda_1(\mathrm{G})$. In that case it is $\lambda_1(\mathrm{G})$-regular.

Strongly regular graphs

A graph G is called *strongly regular* with parameters (n, d, λ, μ), if:

- G is a d-regular graph on n vertices.
- Each pair of adjacent vertices has λ common neighbors.
- Each pair of distinct non-adjacent vertices has μ common neighbors.

Theorem

[1] *Let G be a non-complete connected graph. Then G is strongly regular* $\iff$ G *is regular and has three distinct eigenvalues.*

[1] J. J. Seidel, Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3, Linear Algebra and its Applications, 1968

Theorem

Let G be a connected strongly regular graph. Then the following properties of G are determined by its spectrum:

- *The parameters n, d, λ, μ.*
- *The Lovász ϑ function $\vartheta(\mathsf{G})$.*
- *The number of edges and triangles in G.*
- *The girth of G.*
- *The diameter of G.*

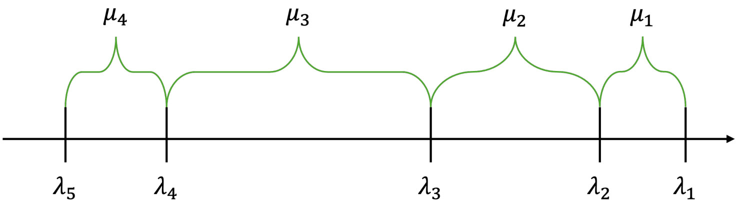
Cauchy interlacing theorem

Theorem (CIT)

Let $m, n \in \mathbb{N}$, $m < n$ and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of a symmetric matrix $\mathbf{N}$ *and let $\mu_1 \geq \ldots \geq \mu_m$ be the eigenvalues of a principal $m \times m$ submatrix* $\mathbf{M}$ *of* $\mathbf{N}$*. Then*

$$\lambda_{i+n-m} \leq \mu_i \leq \lambda_i$$

- Let H be an induced subgraph of G, then $\mathbf{A}(\mathrm{H})$ is a principal submatrix of $\mathbf{A}(\mathrm{G})$.

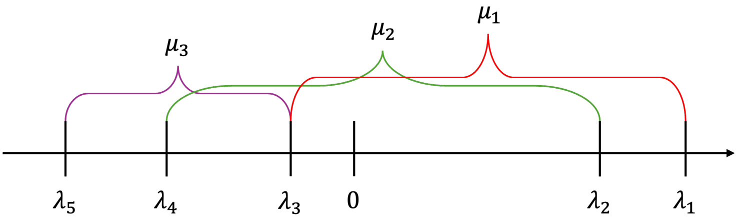


Example: $n = 5, m = 4$

Example

Let G be a graph and let H be an induced subgraph of G. Then

- The number of positive (negative) eigenvalues of H is at most the number of positive (negative) eigenvalues of G.
- $\mathrm{rank}(\mathbf{A}(\mathrm{H})) \leq \mathrm{rank}(\mathbf{A}(\mathrm{G}))$.
- $\lambda_1(\mathrm{H}) \leq \lambda_1(\mathrm{G})$ and $\lambda_m(\mathrm{H}) \geq \lambda_n(\mathrm{G})$.



Example: $n = 5, m = 3$

Spectral invariants

Matrix	# of edges	bipartite	# of components	# of bipartite components	# of closed walks
$\mathbf{A}$	Yes	Yes	No	No	Yes
$\mathbf{L}$	Yes	No	Yes	No	No
$\mathbf{Q}$	Yes	No	No	Yes	No
$\mathcal{L}$	No	Yes	Yes	Yes	No

Graph properties determined by the X-spectrum for $X \in \{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$.

Cospectral graphs

Definition

Two graphs G and H are said to be cospectral if $\sigma(\mathrm{G}) = \sigma(\mathrm{H})$.

Remark

- *The definition is not limited to the adjacency matrix. We can also define cospectrality with respect to the Laplacian, signless Laplacian or normalized Laplacian matrices.*
- *isomorphic graphs are cospectral.*
- *In the case where* G *and* H *are regular, cospectrality with respect to one matrix among* $\{\mathbf{A}, \mathbf{L}, \mathbf{Q}, \mathcal{L}\}$ *implies cospectrality with respect to the others.*

Non-isomorphic cospectral graphs

Example

Both of the graphs S_4 and $\mathrm{C}_4 \cup \mathrm{K}_1$ have spectrum $\sigma = \{-2, [0]^3, 2\}$.

Definition

A graph G is said to be *determined by its spectrum (DS)* if every graph that is cospectral to G is also isomorphic to G.

Example

The graphs $\mathrm{K}_n, \mathrm{C}_n, \mathrm{P}_n$ and all graphs with fewer than 5 vertices are DS.

Complete bipartite graphs

Definition

Let $\ell, r \in \mathbb{N}$. Define $\mathrm{K}_{\ell,r} = (L \cup R, \mathrm{E})$ to be the *complete bipartite graph* on $n \triangleq \ell + r$ vertices with parts of size $|L| = \ell$ and $|R| = r$. ($\mathrm{E} = \{\{u, v\} : u \in L,\ v \in R\}$).

The adjacency matrix of $\mathrm{K}_{\ell,r}$ is given by

$$\mathbf{A}(\mathrm{K}_{\ell,r}) = \begin{pmatrix} \mathbf{0}_{\ell} & \mathbf{J}_{\ell,r} \\ \mathbf{J}_{r,\ell} & \mathbf{0}_{r} \end{pmatrix}$$

Question

For which values of ℓ, r are the complete bipartite graphs $\mathrm{K}_{\ell,r}$ DS?

The spectrum of complete bipartite graphs

Claim

$$\sigma(\mathrm{K}_{\ell,r}) = \{-\sqrt{\ell r}, [0]^{n-2}, \sqrt{\ell r}\}$$

Proof.

- The adjacency matrix is $\mathbf{A}(\mathrm{K}_{\ell,r}) = \begin{pmatrix} \mathbf{0}_\ell & \mathbf{J}_{\ell,r} \\ \mathbf{J}_{r,\ell} & \mathbf{0}_r \end{pmatrix}$, and its rank is 2.
- Since $\mathrm{K}_{\ell,r}$ is bipartite, its spectrum is $\{-\lambda, [0]^{n-2}, \lambda\}$ for some $\lambda > 0$.
- The number of edges satisfies

$$\ell r = |\,\mathrm{E}(\mathrm{K}_{\ell,r})| = \frac{1}{2}\sum_{i=1}^{n} \lambda_i^2 = \lambda^2$$

Therefore, $\lambda = \sqrt{\ell r}$. $\square$

AM-minimizers

Notation

Let $\ell, r \in \mathbb{N}$*. Denote their arithmetic mean by* $\mathsf{AM}(\ell, r)$ *and their geometric mean by* $\mathsf{GM}(\ell, r)$*.*

Definition

The pair $(\ell, r) \in \mathbb{N}^2$ is said to be an AM-*minimizer* if it attains the minimum AM for fixed GM, i.e. if the following holds:

$$\mathsf{AM}(\ell, r) = \min\{\mathsf{AM}(x, y) : \mathsf{GM}(x, y) = \mathsf{GM}(\ell, r)\}$$

Example

For the geometric mean 9, $(9, 9)$ is an AM-minimizer, while $(3, 27)$ is not since

$$\mathsf{AM}(9, 9) = 9, \quad \mathsf{AM}(3, 27) = 15.$$

Spectral graph determination of complete bipartite graphs

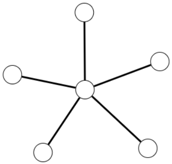
Theorem

The complete bipartite graph $\mathrm{K}_{\ell,r}$ *is DS* $\iff$ *the pair* (ℓ, r) *is an* AM-*minimizer.*

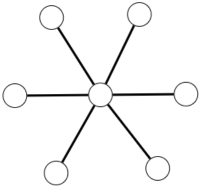
This theorem is equivalent to a result of Ma and Ren (2010). The proof is new.

Corollary

The star graph with n *edges,* S_n*, is DS* $\iff$ n *is prime.*



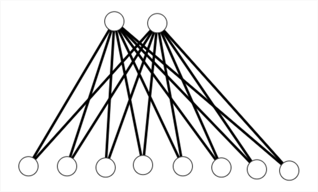
S_5 - DS



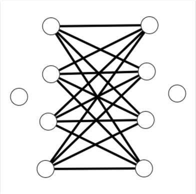
S_6 - Non-DS

Proof $\Rightarrow$.

- If (ℓ, r) is not an AM-minimizer, then there exists a pair (a, b) such that $\mathrm{GM}(a, b) = \mathrm{GM}(\ell, r)$ and $\mathrm{AM}(a, b) < \mathrm{AM}(\ell, r)$. Consider the graph $\mathrm{K}_{a,b}$.
- Since $\mathrm{GM}(a, b) = \mathrm{GM}(\ell, r)$, it has the same non-zero eigenvalues as $\mathrm{K}_{\ell,r}$.
- Since $\mathrm{AM}(a, b) < \mathrm{AM}(\ell, r)$, the multiplicity of the eigenvalue 0 is smaller in $\mathrm{K}_{a,b}$ than in $\mathrm{K}_{\ell,r}$.
- By adding isolated vertices, we get a non-isomorphic cospectral graph to $\mathrm{K}_{\ell,r}$.



$K_{2,8}$



$K_{4,4} \cup \overline{K_2}$

$\square$

Proof $\Leftarrow$.

Suppose (ℓ, r) is an AM-minimizer. Let G be a graph that is cospectral to $K_{\ell,r}$. Then,

- G is bipartite.
- There is a single non-trivial connected component H. (Otherwise, G would have at least two positive eigenvalues).
- H is complete bipartite (otherwise, it would have an induced P_4, contradicting CIT).
- $E(G) = GM^2(\ell, r) = \ell r$, and $V(G) = 2AM = \ell + r$.
- Therefore, G must be isomorphic to $K_{\ell,r}$.

□

The Turán graphs

Theorem

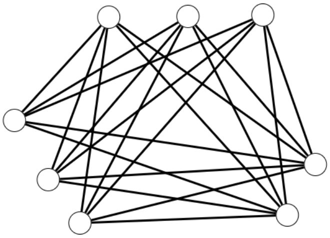
The k-partite Turán graph on n vertices $T(n, k)$ is DS.

Lemmas in the proof.

Let G be a graph cospectral to $T(n, k)$.
Then,

- G is connected.
- G is a complete multipartite graph.
- G has k parts.
- The sizes of the parts are the same as those in $T(n, k)$.

□



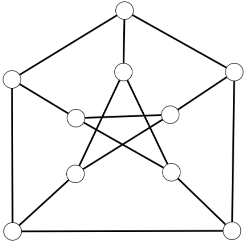
$T(8, 3)$

The Petersen graph

Theorem

The Petersen graph is DS.

The proof relies on the fact that the Petersen graph is isomorphic to $\overline{\ell(\mathsf{K}_5)}$.



The Petersen graph

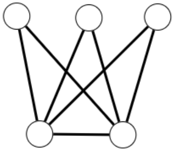
The graphs of pyramids

Definition

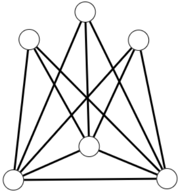
Let $1 < k < n \in \mathbb{N}$. Define the *graph of pyramids* $\mathsf{T}_{n,k}$ as $(n-k)$ pyramids K_{k+1} with a common base K_k.

Theorem

The graphs of pyramids are DS.



$\mathsf{T}_{5,2}$



$\mathsf{T}_{6,3}$

The Haemers' conjecture

Conjecture

Almost all graphs are determined by their spectrum.
i.e. if $\alpha(n)$ is the number of DS graphs on n vertices, and $\mathcal{I}(n)$ is the number of isomorphism classes of graphs on n vertices, then

$$\lim_{n\to\infty} \frac{\alpha(n)}{\mathcal{I}(n)} = 1.$$

Remark

The number of isomorphism classes of graphs on n vertices is given by

$$\mathcal{I}(n) = 2^{\frac{n(n-1)}{2}(1-\epsilon(n))}$$

where $\epsilon(n) \to 0$ as $n \to \infty$.

Results for and against the Haemers' conjecture

against:

- Almost all trees, and almost all strongly regular graphs, are not DS.
- There are many known constructions of pairs of cospectral mates, such as the Godsil-Mckay switching.

for:

- There are many well known families of DS graphs, some of which we discussed in the talk.
- A new construction of a family of DS graphs with e^{cn} graphs of size n.
- Computer results shows the $\sim 80\%$ of the graphs on 12 vertices are DS.

Definition (The Hamming metric)

Let q be a power of a prime number. Let us denote:

- $\mathbb{F}_q$ the finite field with q elements.
- $\mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$.
- For every $\mathbf{x} \in \mathbb{F}_q^n$, $w_H(\mathbf{x})$ denotes the *Hamming weight of* $\mathbf{x}$, i.e. the number of non-zero coordinates in $\mathbf{x}$.
- For every $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$, $\mathrm{d}_H(\mathbf{x}, \mathbf{y}) \triangleq w_H(\mathbf{x} - \mathbf{y})$ denotes the *Hamming distance between* $\mathbf{x}$ *and* $\mathbf{y}$.

The generalized-Hamming graph

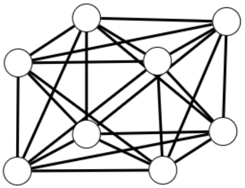
Definition

Let q be a power of a prime number, and let d, n be integers. The *generalized-Hamming graph* $\mathcal{G}_{q,n,d} = (\mathrm{V}, \mathrm{E})$ is defined as follows:

- The vertex set is $\mathrm{V} = \mathbb{F}_q^n$.
- Two vertices $\mathbf{x}, \mathbf{y} \in \mathrm{V}$ are adjacent if and only if $\mathrm{d}_H(\mathbf{x}, \mathbf{y}) < d$.

Remark

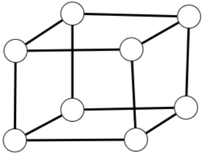
The number of vertices in $\mathcal{G}_{q,n,d}$ *is* $|\mathrm{V}(\mathcal{G}_{q,n,d})| = |\mathbb{F}_q^n| = q^n$.



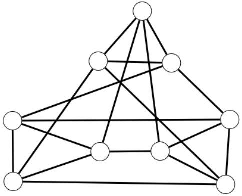
$\mathcal{G}_{2,3,3}$

Example

The Hamming graph is the generalized-Hamming graph $\mathcal{G}_{q,n,2}$. The Hypercube graph is the generalized-Hamming graph $\mathcal{G}_{2,n,2}$.



$\mathcal{G}_{2,3,2}$



$\mathcal{G}_{3,2,2}$

Related Works in Information and Coding Theory

- This graph was used in proving an improvement of the *Gilbert-Varshamov bound*, on the maximal size $A_q(n, d)$ of a code (subset of vectors) over $\mathbb{F}_q^n$ with a given minimum distance d. [2]
  - $A_q(n, d)$ is the independence number of the generalized-Hamming graph $\mathcal{G}_{q,n,d}$:

$$A_q(n, d) = \alpha(\mathcal{G}_{q,n,d})$$

- The graph $\mathcal{G}_{2,5,3}$ was used to show that the Schrijver ϑ-function need not be an upper bound on the Shannon capacity of the graph. [3]
  - If a graph G is vertex-transitive, and both G and its complement $\overline{\mathrm{G}}$ are edge-transitive, then the Schrijver ϑ-function is an upper bound on the Shannon capacity of the graph.

[2] Jiang, T. and Vardy, A., 2004. Asymptotic improvement of the Gilbert-Varshamov bound on the size of binary codes. IEEE Transactions on Information Theory, 50(8), pp.1655-1664

[3] Sason, I., 2025. An example showing that Schrijver's ϑ-function need not upper bound the Shannon capacity of a graph. AIMS Mathematics, 10(7), p.15294.

Reminder

Definitions (Transitivities of a graph)

- The *automorphism group* of a graph G, denoted $\mathrm{Aut}(\mathsf{G})$, is the group of all graph isomorphisms from G to itself.
- A graph is said to be *vertex-transitive* if for any two vertices x, y there exists an automorphism mapping x to y.
- A graph is said to be *edge-transitive* if for any two edges $\{x, y\}, \{u, v\} \in \mathsf{E}$ there exists an automorphism mapping $\{x, y\}$ to $\{u, v\}$.
- A graph G is said to be *distance-transitive* if for any two pairs of vertices (x, y) and (u, v), s.t. $\mathrm{d}_{\mathsf{G}}(x, y) = \mathrm{d}_{\mathsf{G}}(u, v)$, there exists an automorphism mapping x to u and y to v.

Remark

Distance-transitivity $\Rightarrow$ *edge- and vertex-transitivity.*

Generalized-Hamming graphs as Cayley graphs

- The generalized-Hamming graph $\mathcal{G}_{q,n,d}$ is a Cayley graph of the group $\mathbb{F}_q^n$:

$$\mathcal{G}_{q,n,d} = \mathrm{Cay}\left(\mathbb{F}_q^n, S\right)$$

  Where the generating set is an Hamming ball : $S = \{\mathbf{x} \in \mathbb{F}_q^n : w_H(\mathbf{x}) < d\}$.
- Similarly, $\overline{\mathcal{G}_{\mathrm{q,n,d}}} = \mathrm{Cay}\left(\mathbb{F}_q^n, \mathbb{F}_q^n \setminus S\right)$.
- Since S is a symmetric set, $\mathcal{G}_{q,n,d}, \overline{\mathcal{G}_{\mathrm{q,n,d}}}$ are vertex-transitive.

Question

- *Is* G *edge-transitive?*
- *Is* G *distance-transitive?*
- *What about its complement* $\overline{\mathrm{G}}$*?*

Transitivities of the generalized-Hamming graph

Theorem

Let $q \in \mathbb{N}$ be a prime power, let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$. Then the following statements are equivalent:

1. $\mathcal{G}_{q,n,d}$ *is edge-transitive.*
2. $\mathcal{G}_{q,n,d}$ *is distance-transitive.*
3. *Either* $d = 2$, *or* $(q, d) = (2, 3)$, *or* $(q, d) = (2, n)$.

Outline of the proof.

- If $d = 2$ then $\mathcal{G}_{q,n,2}$ is the Hamming graph, known to be distance-transitive.
- If $(q, d) = (2, 3)$ or $(q, d) = (2, n)$, we construct an explicit automorphism.
- In the other cases, define an automorphism invariant on the edges of $\mathcal{G}_{q,n,d}$, and show that it takes different values on different edges.

$\square$

Transitivities of the complement of the generalized-Hamming graph

Theorem

Let $q \in \mathbb{N}$ be a prime power, let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$. Then,

- $\overline{\mathcal{G}_{\mathsf{q,n,d}}}$ *is edge-transitive* $\iff$ *either* $(q, d) = (2, n-1)$ *for an even n, or* $d = n$.
- $\overline{\mathcal{G}_{\mathsf{q,n,d}}}$ *is distance-transitive* $\iff$ *one of the following two conditions holds:*
  1. $(q, d) = (2, n-1)$ *for an even n,*
  2. $d = n$ *and either* $q = 2$ *or* $n = 2$.

Remark

In contrast to the generalized-Hamming graph, the distance-transitivity of the complement graph $\overline{\mathcal{G}_{\mathsf{q,n,d}}}$ *is more restrictive then edge-transitivity.*

The proof uses the theory of association schemes, Krawtchouk polynomials, and Bose-Mesner algebras.

Outline of the proof

- in the case of edge transitivity and distance transitivity, the proof is constructive.
- In the other cases, decompose $\mathbf{A} \triangleq \mathbf{A}(\overline{\mathcal{G}_{q,n,d}})$ as

$$\mathbf{A} = \sum_{j=1}^{n} \gamma_j \mathbf{E}_j$$

  where $\mathbf{E}_j$ are orthogonal projections.
- Find a closed form expression for the γ_j's and $\mathbf{E}_j$'s.
- Construct a polynomial p s.t. $p(\mathbf{A}) = \mathbf{E}_1$.
- choose $\mathbf{v}, \mathbf{u} \in \mathcal{N}(\mathbf{0})$, and show that $(\mathbf{E}_1)_{\mathbf{0},\mathbf{u}} \neq (\mathbf{E}_1)_{\mathbf{0},\mathbf{v}}$.
- reach a contradiction with the edge transitivity.

Decomposing $\mathbf{A}$

- We use the theory of association schemes.
- Let $\{\mathbf{A}_i\}_{i=0}^n$ be the distance i matrices of the Hamming metric, defined as follows:

$$(\mathbf{A}_i)_{u,v} = \mathbb{1}(d_H(u,v) = i)$$

- The commutative algebra generated by the $\mathbf{A}_i$'s is known as the Bose-Mesner algebra of the Hamming scheme.
- Every Bose-Mesner algebra has a unique basis of orthogonal projections $\{\mathbf{E}_j\}_{j=0}^n$.
- $\mathbf{A}$ can be expressed in terms of the dual basis. There exists $\{\gamma_j\}_{j=0}^n$ such that

$$\mathbf{A} = \sum_{i=d}^n \mathbf{A}_i = \sum_{j=0}^n \gamma_j \mathbf{E}_j$$

The spectrum of the generalized-Hamming graph

- As a part of the proof we find the spectrum of $\mathcal{G}_{q,n,d}, \overline{\mathcal{G}_{q,n,d}}$.
- The eigenvalues are given in terms of the Krawtchouk polynomials, defined as:

$$K_\ell(i;n,q) = \sum_{r=0}^{\ell} \binom{i}{r}\binom{n-i}{\ell-r}(-1)^r (q-1)^{\ell-r}.$$

- There are at most $n+1$ distinct eigenvalues of $\mathcal{G}_{q,n,d}$, given in terms of the Krawtchouk polynomials:

$$\gamma_0 = \sum_{i=d}^{n} \binom{n}{i}(q-1)^i, \quad \forall 1 \le j \le n,\ \gamma_j = -K_{d-1}(j-1;n-1,q)$$

The Lovász ϑ function

The Lovász ϑ function is a well-studied graph invariant, with intriguing connections to many other invariants, such as $\alpha(\mathsf{G}), \omega(\mathsf{G}), \chi(\mathsf{G})$, and $\Theta(\mathsf{G})$.

Definition

The *Lovász ϑ function* of a graph G is defined as the solution of the following SDP problem:

$$\boxed{\begin{array}{l} \text{maximize } \ \mathrm{Tr}(\mathbf{B}\,\mathbf{J}_N) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \succeq 0, \\ \mathrm{Tr}(\mathbf{B}) = 1, \\ (\mathbf{A}(\mathsf{G}))_{i,j} = 1 \ \Rightarrow \ \mathbf{B}_{i,j} = 0, \quad i,j \in [N]. \end{cases} \end{array}}$$

Properties of the Lovász ϑ function

- Unlike other graph invariants, such as $\alpha(\mathsf{G}), \omega(\mathsf{G}), \chi(\mathsf{G})$ which are NP-hard to compute, the Lovász ϑ function can be approximated numerically, with a precision of r decimal digits, and complexity of $\mathrm{poly}(r|V|)$.
- Bounds on known graph invariants:
  1. $$\alpha(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}).$$
  2. $$\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G}).$$
  3. $$\vartheta(\mathsf{G}) \cdot \vartheta(\overline{\mathsf{G}}) \geq N,$$
     equality holds if G is vertex-transitive.

Theorem (Spectral bounds on the Lovász ϑ function [4])

Let G *be a* Δ*-regular graph on* N *vertices, which is neither complete nor empty. Then:*

1. $$\frac{N - \Delta + \lambda_2(\mathsf{G})}{1 + \lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{N\lambda_{min}(\mathsf{G})}{\Delta - \lambda_{min}(\mathsf{G})}.$$

    - *Equality holds in the leftmost inequality if* $\overline{\mathsf{G}}$ *is both vertex-transitive and edge-transitive, or if* G *is a strongly regular graph;*
    - *Equality holds in the rightmost inequality if* G *is edge-transitive, or if* G *is a strongly regular graph.*

[4] I. Sason, "Observations on Lovász ϑ-function, graph capacity, eigenvalues, and strong products," *Entropy*, 2023.

Theorem (continued)

2.

$$1 - \frac{\Delta}{\lambda_{min}(\mathsf{G})} \le \vartheta(\overline{\mathsf{G}}) \le \frac{N\big(1 + \lambda_2(\mathsf{G})\big)}{N - \Delta + \lambda_2(\mathsf{G})}.$$

- *Equality holds in the leftmost inequality if* G *is both vertex-transitive and edge-transitive, or if* G *is a strongly regular graph;*
- *Equality holds in the rightmost inequality if* $\overline{\mathsf{G}}$ *is edge-transitive, or if* G *is a strongly regular graph.*

By applying this theorem and the characterization of the edge-transitivity of generalized-Hamming graphs, we obtain the following:

Theorem

Let $\mathrm{G} = \mathcal{G}_{q,n,d}$ *be a generalized-Hamming graph with regularity* $\Delta \triangleq \sum_{i=1}^{d-1} \binom{n}{i}(q-1)^i$. *Then*

1. *If either* $d = 2$, *or* $(q, d) = (2, 3)$, *or* $(q, d) = (2, n)$ *(so* G *is edge-transitive), then*

$$\vartheta(\mathrm{G}) = -\frac{q^n\,\lambda_{min}(\mathrm{G})}{\Delta - \lambda_{min}(\mathrm{G})}, \quad \vartheta(\overline{\mathrm{G}}) = 1 - \frac{\Delta}{\lambda_{min}(\mathrm{G})},$$

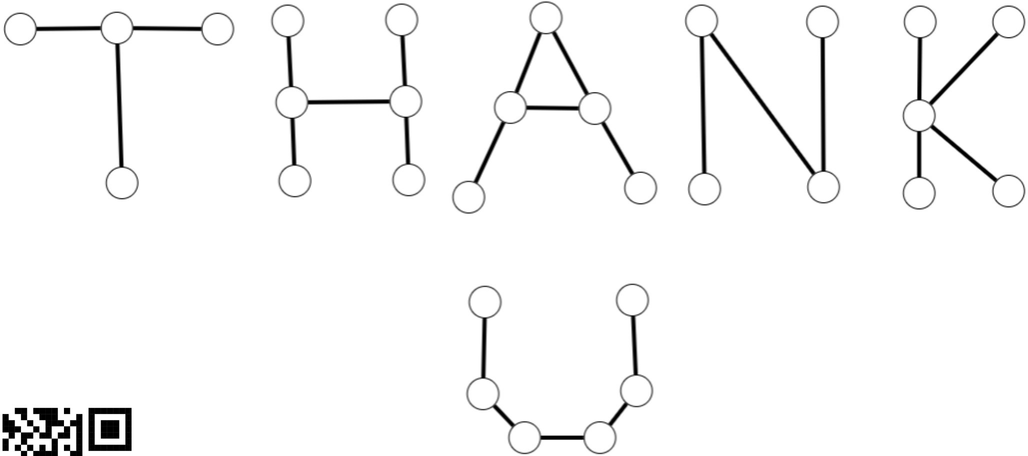
2. *If either* $(q, d) = (2, n-1)$ *for an even* n, *or* $d = n$ *(so* $\overline{\mathrm{G}}$ *is edge-transitive), then*

$$\vartheta(\mathrm{G}) = \frac{q^n - \Delta + \lambda_2(\mathrm{G})}{1 + \lambda_2(\mathrm{G})}, \quad \vartheta(\overline{\mathrm{G}}) = \frac{q^n(1 + \lambda_2(\mathrm{G}))}{q^n - \Delta + \lambda_2(\mathrm{G})}.$$

$\lambda_2(\mathrm{G}), \lambda_{min}(\mathrm{G})$ *are obtained by ordering the* γ_j*'s.*

Summary

- We reviewed some properties of the adjacency spectrum of a graph.
- We introduced spectral graph determination
- We showed that the complete bipartite graph $\mathsf{K}_{\ell,r}$ is DS if and only if the pair (ℓ, r) is an AM-minimizer. and introduced DS graphs such as the Turán graph, the Petersen graph, and the graphs of pyramids.
- We introduced the generalized-Hamming graph $\mathcal{G}_{q,n,d}$ and characterized the edge- and distance-transitivity of $\mathcal{G}_{q,n,d}$ and $\overline{\mathcal{G}_{\mathrm{q,n,d}}}$.
- We applied the characterization of the edge-transitivity of $\mathcal{G}_{q,n,d}$ and $\overline{\mathcal{G}_{\mathrm{q,n,d}}}$ to obtain closed form expressions for $\vartheta(\mathcal{G}_{q,n,d})$.



All components, except K, are DS.

End of presentation

Questions?

Operations on graphs

In our work we explore spectral properties of unitary and binary operations. Here are two examples

Theorem (The Gosdil & Mckay method)

Let G *be a graph, s.t.*

$$\mathbf{A}(\mathsf{G}) = \begin{bmatrix} \mathbf{B} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{C} \end{bmatrix}$$

where the column sum of each column of $\mathbf{N} \in \mathbb{R}^{b \times c}$ *is in* $\{0, b, \frac{b}{2}\}$. *Let* $\widehat{\mathbf{N}}$ *be the matrix obtained from* $\mathbf{N}$ *by replacing every column* $\mathbf{c}$ *whose sum is* $\frac{b}{2}$ *with the complement* $\mathbf{1} - \mathbf{c}$. *Then the graph realization of*

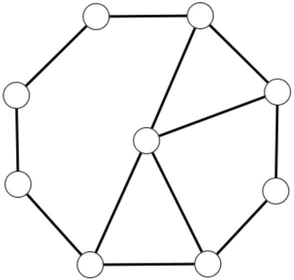
$$\begin{bmatrix} \mathbf{B} & \widehat{\mathbf{N}} \\ \widehat{\mathbf{N}}^T & \mathbf{C} \end{bmatrix}$$

Is cospectral to G.

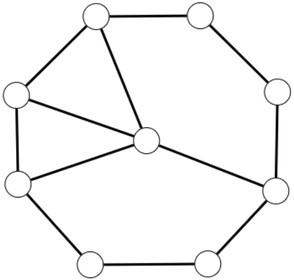
GM switching example

Example

The following two graphs are cospectral, by the GM switching method, with respect to the vertex in the middle.



Before GM switching

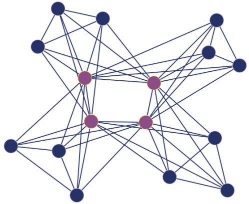


After GM switching

Closed neighborhood corona

Definition

The *closed neighborhood corona* of two graphs G_1 and G_2, denoted by $G_1 \underline{\boxtimes} G_2$, is a new graph obtained by creating $|V(G_1)|$ copies of G_2. Each vertex of the i^{th} copy of G_2 is then connected to the i^{th} vertex and neighborhood of the i^{th} vertex of G_1



$C_4 \underline{\boxtimes} K_3$

Constructing cospectral mates

Theorem

Let G_1 and G_2 be cospectral regular graphs, and let H be an arbitrary graph. Then, the following holds:

- *The closed neighborhood corona $\mathrm{G}_1 \underline{\boxtimes} \mathrm{H}$ and $\mathrm{G}_2 \underline{\boxtimes} \mathrm{H}$ are non isomorphic and cospectral graphs.*
- *The closed neighborhood corona $\mathrm{H} \underline{\boxtimes} \mathrm{G}_1$ and $\mathrm{H} \underline{\boxtimes} \mathrm{G}_2$ are non isomorphic and cospectral graphs.*

This theorem holds with respect to $\mathbf{A}, \mathbf{L}, \mathbf{Q}$.

Transitivities of the generalized-Hamming graph

Theorem

Let $q \in \mathbb{N}$ be a prime power, let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$. Then the following statements are equivalent:

1. *$\mathcal{G}_{q,n,d}$ is edge-transitive.*
2. *$\mathcal{G}_{q,n,d}$ is distance-transitive.*
3. *Either $d = 2$, or $(q, d) = (2, 3)$, or $(q, d) = (2, n)$.*

proof

The proof is constructed from the following Lemmata:

1. the Hamming graph $\mathcal{G}_{q,n,2}$ is distance-transitive.
2. The generalized-Hamming graph $\mathcal{G}_{2,n,3}$ is distance-transitive.
3. The generalized-Hamming graph $\mathcal{G}_{2,n,n}$ is distance-transitive.
4. If none of the previous conditions hold, then $\mathcal{G}_{q,n,d}$ is not edge-transitive.

Lemmata 2,3 are known results.

Proof of 3

- It is sufficient to show that for every $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$, s.t. $\mathrm{d}_{\mathsf{G}}(\mathbf{u}, \mathbf{0}) = \mathrm{d}_{\mathsf{G}}(\mathbf{v}, \mathbf{0})$, there exists an automorphism mapping $\mathbf{u}$ to $\mathbf{v}$ while fixing $\mathbf{0}$.
- Let $\mathbf{u}, \mathbf{v}$ be such pair of vertices. Define $\varphi \colon \mathbb{F}_q^n \to \mathbb{F}_q^n$ as follows:

$$\varphi(\mathbf{x}) = \begin{cases} \mathbf{u} & \text{if } \mathbf{x} = \mathbf{v}, \\ \overline{\mathbf{u}} & \text{if } \mathbf{x} = \overline{\mathbf{v}}, \\ \mathbf{v} & \text{if } \mathbf{x} = \mathbf{u}, \\ \overline{\mathbf{v}} & \text{if } \mathbf{x} = \overline{\mathbf{u}}, \\ \mathbf{x} & \text{otherwise.} \end{cases} \tag{1}$$

- φ is an automorphism, and $\varphi(\mathbf{0}) = \mathbf{0}$, $\varphi(\mathbf{v}) = \mathbf{u}$.

Proof of 4

- Consider the following automorphism invariant on the neighborhood of $\mathbf{0}$:

$$F\colon \mathcal{N}(\mathbf{0}), \quad F(\mathbf{x}) = |\mathcal{N}(\mathbf{x}) \setminus \mathcal{N}(\mathbf{0})| \tag{2}$$

- F is constant on the orbits of the action of $\mathrm{Aut}(\mathcal{G}_{q,n,d})$ on $\mathcal{N}(\mathbf{0})$.
- Set $\mathbf{u}_1 \triangleq (1, 0, 0, \ldots, 0)$, $\mathbf{u}_2 \triangleq (1, 1, 0, \ldots, 0)$, $\mathbf{u}_3 \triangleq (1, 1, 1, 0, \ldots, 0)$.
- show that F takes at least two different values on $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and thus they are in different orbits.

Orthogonal and orthonormal representations

Definition

Let G be a graph.

- Let $d \in \mathbb{N}$. An *orthogonal representation* of G in $\mathbb{R}^d$ is a mapping $f : \mathrm{V}(\mathrm{G}) \to \mathbb{R}^d$ such that for every $\{x, y\} \notin \mathrm{E}(\mathrm{G})$, $\langle f(x), f(y) \rangle = 0$.
- If in addition $||f(x)|| = 1$ for all $x \in \mathrm{V}(\mathrm{G})$, then f is called an *orthonormal representation*.

Strongly regular graphs

Theorem

Let G *be a connected strongly regular graph with parameters* (n, d, λ, μ). *Then* G *has 3 distinct eigenvalues given by:*

$$\left\{ d, \frac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right)\right\}$$

Corollary

For a strongly regular graph

$$(n, d, \lambda, \mu) \leftrightarrow \sigma(\mathsf{G})$$

cospectrality of strongly regular graphs

Question

Which strongly regular graphs are DS?

This is equivalent to asking which strongly regular graphs are determined by their parameters (n, d, λ, μ).

- For some parameters choices, such as $(n, n-1, n-2, \mu)$, the graph is determined by its parameters.
- for others this is far from the truth. For example, there are 32548 non-isomorphic strongly regular graphs with parameters $(36, 15, 6, 6)$.

Strongly regular DS graphs

Theorem

The following strongly regular graphs are DS:

- *Regular complete bipartite graphs.*
- *The line graph* $\ell(\mathsf{K}_{m,m})$ *for* $m \neq 4$.
- *the line graph* $\ell(\mathsf{K}_n)$ *for* $n \neq 8$.
- *The Petersen graph*

The proof is devided to the following lemmas: Let $\mathrm{G} = \overline{\mathcal{G}_{q,n,d}}$.

1. If $(q, d) = (2, n-1)$ for an even n then G is distance-transitive.
2. If $d = n$ then G is edge-transitive, and distance-transitive if and only if $q = 2$ or $n = 2$.
3. If none of the previous conditions hold, then G is not edge-transitive.

- The proof of $1, 2$ uses classical techniques.
- The proof of 3 is algebraic and relies on the theory of association schemes, Krawtchouk polynomials, and Bose-Mesner algebras.

The theory of association schemes

Definition (association scheme)

Let $\mathcal{X}$ be a finite set and let $m \in \mathbb{N}$. An $(m+1)$-*class symmetric association scheme* on $\mathcal{X}$ is a partition $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ of $\mathcal{X} \times \mathcal{X}$ that satisfies the following properties:

- $R_0 = \{(x, x) : x \in \mathcal{X}\}$ is the identity relation.
- for every $0 \leq i \leq m$, and $(x, y) \in R_i$, we have $(y, x) \in R_i$.
- For every $0 \leq i, j, k \leq m$, there exist integers p_{ij}^k such that for every $(x, y) \in R_k$,

$$\left| \left\{ z \in \mathcal{X} : (x, z) \in R_i \text{ and } (z, y) \in R_j \right\} \right| = p_{ij}^k,$$

  i.e., the above quantity depends only on i, j, k and not on the particular pair $(x, y) \in R_k$. The integers $\{p_{ij}^k\}$ are called the *intersection numbers* of the association scheme.

The adjacency matrix representation of an association scheme

Definition

Let $\mathcal{R}$ be an $(m+1)$-class symmetric association scheme on a set $\mathcal{X}$. For each relation $R_i \in \mathcal{R}$, define the adjacency matrix $\mathbf{A}_i \in \{0,1\}^{|\mathcal{X}| \times |\mathcal{X}|}$ as

$$(\mathbf{A}_i)_{x,y} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

The conditions on the R_i's are equivalent to the following conditions on the adjacency matrices $\mathbf{A}_i$:

- $\mathbf{A}_0 = \mathbf{I}$, the identity matrix.
- $\mathbf{A}_i \, \mathbf{A}_j = \sum_{k=0}^{m} p_{ij}^k \, \mathbf{A}_k$ for all $0 \leq i, j \leq m$.
- Symmetry: $\mathbf{A}_i^T = \mathbf{A}_i$ for all $0 \leq i \leq m$.

The Bose-Mesner algebra

Definition

Let $\mathcal{R}$ be an $(m+1)$-class symmetric association scheme on a finite set $\mathcal{X}$. The *Bose-Mesner algebra* $\mathcal{B}(\mathcal{R})$ is the commutative sub-algebra of the matrix algebra $\mathbb{R}^{|\mathcal{X}|\times|\mathcal{X}|}$ generated by the adjacency matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$.

$$\mathcal{B}(\mathcal{R}) = \mathrm{span}_{\mathbb{R}}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m\}$$

The dual basis

Theorem

1. *The Bose-Mesner algebra* $\mathcal{B}(\mathcal{R})$ *has a basis* $\mathbf{E}_0, \ldots, \mathbf{E}_m$ *of orthogonal idempotents, i.e.* $\mathbf{E}_i\mathbf{E}_j = \delta_{ij}\mathbf{E}_i$ *for all* $0 \leq i, j \leq m$.
2. *There exists scalars* $q_k(i), p_i(k)_{i,k}$ *such that*

$$\mathbf{E}_k = \frac{1}{|\mathcal{X}|}\sum_{i=0}^{m} q_k(i)\,\mathbf{A}_i, \quad \mathbf{A}_i = \sum_{k=0}^{m} p_i(k)\mathbf{E}_k.$$

The Hamming scheme

Definition

Let $q, n \in \mathbb{N}$ and $i, \ell \in \{0, 1, \dots, n\}$. The Krawtchouk polynomial is defined as

$$K_\ell(i; n, q) = \sum_{r=0}^{\ell} \binom{i}{r} \binom{n-i}{\ell-r} (-1)^r (q-1)^{\ell-r}.$$

Theorem

The dual basis $\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_n$ *of the Bose-Mesner algebra of the Hamming scheme* $\mathcal{H}(n, q)$ *is given by*

$$\mathbf{A}_i = \sum_{k=0}^{n} K_i(k; n, q)\, \mathbf{E}_k, \quad \mathbf{E}_k = \frac{1}{q^n} \sum_{i=0}^{n} K_k(i; n, q)\, \mathbf{A}_i\,.$$ [5]

[5] P. Delsarte, "An algebraic approach to the association schemes of coding theory," *Philips Res. Rep. Suppl.*, 1973.

Krawtchouk polynomials

In our work, we rely on the following properties of the Krawtchouk polynomials:

- **Generating function:**

$$\sum_{k=0}^{n} K_k(i; n, q)\, z^k = (1 + (q-1)z)^{n-i}(1-z)^i.$$

- **Character Theory definition:**

$$K_\ell(i; n, q) = \sum_{\mathbf{y} \in \mathbb{F}_q^n,\ w_H(\mathbf{y}) = \ell} \chi_{\mathbf{y}}(\mathbf{x}).$$

where $\mathbf{x} \in \mathbb{F}_q^n$ is an arbitrary vector such that $w_H(\mathbf{x}) = i$, and $\chi_{\mathbf{y}}(\mathbf{x})$ is the additive character of $\mathbb{F}_q^n$ associated with $\mathbf{y}$.

Decomposing $\mathbf{A}(\overline{\mathcal{G}_{q,n,d}})$

- The adjacency matrix of the $\mathrm{G} \triangleq \overline{\mathcal{G}_{q,n,d}}$ admits the decomposition

$$\mathbf{A}(\mathrm{G}) = \sum_{i=d}^{n} \mathbf{A}_i = \sum_{i=d}^{n} \left(\sum_{k=0}^{n} K_i(k; n, q) \mathbf{E}_k \right) = \sum_{k=0}^{n} \left(\underbrace{\sum_{i=d}^{n} K_i(k; n, q)}_{\gamma_k} \right) \mathbf{E}_k$$

- The coefficients γ_k are given by

$$\gamma_0 = \Delta(\mathrm{G}), \quad \gamma_j = -K_{d-1}(j-1; n-1, q), \, \forall 1 \leq j \leq n,$$

- For every polynomial p,

$$p\big(\mathbf{A}(\mathrm{G})\big) = \sum_{j=0}^{n} p(\gamma_j)\, \mathbf{E}_j.$$

Transitivities of the complement of the Gilbert graph

Lemma

Let $\mathsf{G} = \overline{\mathcal{G}_{q,n,d}}$*. If* $d \neq n$ *and* $(q, d) \neq (2, n-1)$*, then* $\mathcal{G}$ *is not edge-transitive.*

Outline of the proof

- define the polynomial
$$p(t) \triangleq \prod_{m \neq 1} \frac{t - \gamma_m}{\gamma_1 - \gamma_m},$$
and find for what (q, n, d) this polynomial is well defined (i.e., for what (q, n, d) do we have $\gamma_1 \neq \gamma_m$ for all $m \neq 1$).
- in the case where it is not defined, we construct a similar polynomial.
- $p(\mathbf{A}(\mathsf{G})) = \mathbf{E}_1$.

Outline of the proof (continued).

- find two edges $\{\mathbf{0}, \mathbf{v}\}, \{\mathbf{0}, \mathbf{w}\}$ such that $\mathbf{E}_1(\mathbf{0}, \mathbf{v}) \neq \mathbf{E}_1(\mathbf{0}, \mathbf{w})$.
- Under the assumption of edge-transitivity, there exists an automorphism φ mapping $\{\mathbf{0}, \mathbf{v}\}$ to $\{\mathbf{0}, \mathbf{w}\}$. Let $\mathbf{P}_\varphi$ be the corresponding permutation matrix, so

$$\mathbf{A}(\mathrm{G}) = \mathbf{P}_\varphi^{-1} \mathbf{A}(\mathrm{G}) \mathbf{P}_\varphi.$$

- Therefore

$$\mathbf{E}_1 = p(\mathbf{A}(\mathrm{G})) = p(\mathbf{P}_\varphi^{-1} \mathbf{A}(\mathrm{G}) \mathbf{P}_\varphi) = \mathbf{P}_\varphi^{-1} p(\mathbf{A}(\mathrm{G})) \mathbf{P}_\varphi = \mathbf{P}_\varphi \mathbf{E}_1 \mathbf{P}_\varphi^T.$$

$$\Rightarrow (\mathbf{E}_1)_{\mathbf{0},\mathbf{u}} = (\mathbf{E}_1)_{\mathbf{0},\mathbf{v}}$$

yielding a contradiction.

$\square$

The lovász ϑ function

Definition

Let G be a graph. The *Lovász ϑ function* of G is defined as

$$\vartheta(\mathrm{G}) = \min_{c,f} \max_{x \in \mathrm{V}(\mathrm{G})} \frac{1}{(c^T f(x))^2}$$

Where the minimum is taken over all unit vectors c and all orthonormal representations f of G.

Remark

The Lovász ϑ function is the solution of the following SDP problem:

$$\boxed{\begin{array}{l} \textit{maximize} \;\; \mathrm{Tr}(\mathbf{B}\,\mathbf{J}_N) \\ \textit{subject to} \\ \begin{cases} \mathbf{B} \succeq 0, \\ \mathrm{Tr}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i,j \in [N]. \end{cases} \end{array}}$$

Transitivities of the complement of the generalized-Hamming graph

Theorem

Let $q \in \mathbb{N}$ be a prime power, let $n, d \in \mathbb{N}$ with $2 \leq d \leq n$. Then,

- $\overline{\mathcal{G}_{\mathsf{q,n,d}}}$ *is edge-transitive* $\iff$ *either* $(q, d) = (2, n-1)$ *for an even* n*, or* $d = n$.
- $\overline{\mathcal{G}_{\mathsf{q,n,d}}}$ *is distance-transitive* $\iff$ *one of the following two conditions holds:*
    1. $(q, d) = (2, n-1)$ *for an even* n,
    2. $d = n$ *and either* $q = 2$ *or* $n = 2$.

The proof uses the theory of association schemes, Krawtchouk polynomials, and Bose-Mesner algebras.

The proof is divided into several Lemmas:

1. Let $G = \overline{\mathcal{G}_{q,n,d}}$. If $q \neq 2$ and $1 < d < n$, or if $q = 2$, $1 < d < n$ and d is even, then G is not edge-transitive.
2. Let $G = \overline{\mathcal{G}_{q,n,d}}$ where $q = 2$ and $d < n - 1$ is an odd number, then G is not edge-transitive.
3. Let $G = \overline{\mathcal{G}_{\mathrm{q,n,n}}}$ be the complement of the generalized-Hamming graph for $d = n$, then G is edge-transitive.
4. If $(q, d) = (2, n - 1)$ and n is even then $G = \overline{\mathcal{G}_{\mathrm{2,n,n-1}}}$ is edge transitive.

- The proof of 3,4 is constructive, and uses parity bijections.
- The proof of 1,2 is based on the theory of association schemes. Here we prove Lemma 1, after providing some background on association schemes.

Association schemes

Definition (association scheme)

Let $\mathcal{X}$ be a finite set and let $m \in \mathbb{N}$. A *symmetric* $(m+1)$-*class association scheme* on $\mathcal{X}$ is a partition $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ of $\mathcal{X} \times \mathcal{X}$ that satisfies the following properties:

- **Identity relation:** $R_0 = \{(x,x) : x \in \mathcal{X}\}$
- **Symmetry:** $\forall 0 \leq i \leq m$, $\forall x, y \in \mathcal{X}$, $(x,y) \in R_i \iff (y,x) \in R_i$.
- **Intersection pattern:** For every $0 \leq i, j, k \leq m$, there exist constants $p_{ij}^{(k)}$ such that for every $(x,y) \in R_k$,

  $$|\{z \in \mathcal{X} : (x,z) \in R_i, (z,y) \in R_j\}| = p_{ij}^{(k)}$$

  independently of (x,y). The integers $\{p_{ij}^{(k)}\}$ are the *intersection numbers* of the association scheme.

Adjacency matrix representation

Definition

Let $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ be a symmetric $(m+1)$-class association scheme on $\mathcal{X}$. Define the *adjacency matrix representation* to be $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m\}$ where $\mathbf{A}_i \in \{0,1\}^{|\mathcal{X}| \times |\mathcal{X}|}$, and

$$\mathbf{A}_i(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The following holds:

- $\sum_{i=0}^{m} \mathbf{A}_i = \mathbf{J}$
- $\mathbf{A}_0 = \mathbf{I}$
- $\mathbf{A}_i \mathbf{A}_j = \sum_{k=0}^{m} p_{ij}^{(k)} \mathbf{A}_k$, $\forall i,j \in \{0,1,\ldots,m\}$.

The Bose-Mesner Algebra

Definition

The *Bose-Mesner algebra* of an association scheme $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ on $\mathcal{X}$ is the commutative algebra generated by the adjacency matrix representation $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m\}$.

$$\mathcal{B}(\mathcal{R}) = \mathrm{span}_{\mathbb{R}}\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m\}$$

Theorem

The Bose-Mesner algebra $\mathcal{B}(\mathcal{R})$ has a basis of orthogonal projection matrices $\{\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_m\}$. i.e.

$$\mathbf{E}_i\mathbf{E}_j = \delta_{ij}\mathbf{E}_i, \quad \forall i, j \in \{0, 1, \ldots, m\}$$

This basis is called the dual basis of $\mathcal{B}(\mathcal{R})$.

The Hamming Scheme

Definition

The Hamming scheme $\mathcal{H}(n, q)$ is the symmetric association scheme defined on $\mathcal{X} = \mathbb{F}_q^n$ with the adjacency matrix representation $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ where $\mathbf{A}_i$ is the distance i matrix:

$$\mathbf{A}_i(x, y) = \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise.} \end{cases}$$

Definition

Let $q, n \in \mathbb{N}$, $q \geq 2$ and $0 \leq i, \ell \leq n$. The q-ary Krawtchouk polynomial $K_\ell(i; n, q)$ is defined by

$$K_\ell(i; n, q) = \sum_{r=0}^{\ell} \binom{i}{r} \binom{n-i}{\ell-r} (-1)^r (q-1)^{\ell-r}.$$

Theorem

Let $\mathcal{R} = \mathcal{H}(n, q)$ be the Hamming scheme. Let $\{\mathbf{A}_i\}_{i=0}^n$ be its adjacency matrix representation, and let $\{\mathbf{E}_i\}_{i=0}^n$ be its dual basis. Then, for every $0 \leq i, k \leq n$, the closed form expressions for $\mathbf{A}_i$ and $\mathbf{E}_k$ are given by

$$\mathbf{A}_i = \sum_{k=0}^{n} K_i(k; n, q) \mathbf{E}_k, \quad \mathbf{E}_k = \frac{1}{q^n} \sum_{i=0}^{n} K_k(i; n, q)\, \mathbf{A}_i$$

Outline of the proof of Lemma 1

Lemma

Let $\mathrm{G} = \overline{\mathcal{G}_{q,n,d}}$. If $q \neq 2$ and $1 < d < n$, or if $q = 2$, $1 < d < n$ and d is even, then G is not edge-transitive.

Outline of the proof.

Let $\mathrm{G} = \overline{\mathcal{G}_{\mathrm{q,n,d}}}$. Then

$$\mathbf{A}(\mathrm{G}) = \sum_{i=d}^{n} \mathbf{A}_i = \sum_{i=d}^{n} \left(\sum_{k=0}^{n} K_i(k; n, q) \mathbf{E}_k \right) = \sum_{k=0}^{n} \left(\underbrace{\sum_{i=d}^{n} K_i(k; n, q)}_{\gamma_k} \right) \mathbf{E}_k$$

Outline of the proof (cont.)

We show that the following polynomial is well defined:

$$p(t) \triangleq \prod_{m \neq 1} \frac{t - \gamma_m}{\gamma_1 - \gamma_m},$$

i.e. that γ_1 is a simple eigenvalue. And that $p(\mathbf{A}(\mathrm{G})) = \mathbf{E}_1$.
Let $\mathbf{v} = \mathbf{1}_n$ and $\mathbf{u} = (0, \mathbf{1}_{n-1})$. Suppose to the contrary that $\exists \varphi \in \mathrm{Aut}(\mathrm{G})$ s.t. $\varphi(\{\mathbf{v}, \mathbf{0}\}) = \{\mathbf{u}, \mathbf{0}\}$. Let $\mathbf{P}_\varphi$ be the corresponding permutation matrix, so $\mathbf{P}_\varphi \mathbf{A}(\mathrm{G}) \mathbf{P}_\varphi^T = \mathbf{A}(\mathrm{G})$. Then,

$$\mathbf{E}_1 = p(\mathbf{A}(\mathrm{G})) = \mathbf{P}_\varphi p(\mathbf{A}(\mathrm{G})) \mathbf{P}_\varphi^T = \mathbf{P}_\varphi \mathbf{E}_1 \mathbf{P}_\varphi^T.$$

$$\Rightarrow (\mathbf{E}_1)_{\mathbf{0},\mathbf{u}} = (\mathbf{E}_1)_{\mathbf{0},\mathbf{v}}$$

Contradicting the closed form expression for $\mathbf{E}_1$. $\square$