Information-Theoretic Bounds on the Parity-Check Density of LDPC Codes: Old and New Results

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Parts of this work are based on joints works with

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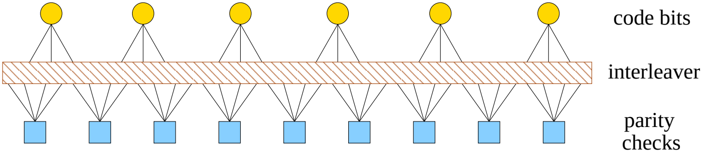
Low-Density Parity-Check Codes



- Low-density parity-check (LDPC) codes are well-known capacity-approaching linear codes which are characterized by sparse parity-check matrices.
- Sparse parity-check matrices
  $\Rightarrow$ Low-complexity encoding and iterative message-passing decoding algorithms.
- For LDPC codes, the sub-optimal iterative decoding algorithm is very efficient, achieving rates close to the Shannon capacity limit with feasible complexity.

Fundamental Questions Regarding LDPC Codes

Question

How good can LDPC codes be, even under ML Decoding ?

Answer to this question ⇒

- Quantitative measure of the inherent loss of sub-optimal and practical iterative message-passing decoding algorithms.

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How sparse can parity-check matrices of binary linear codes be, as a function of their gap (in rate) to capacity?

Answer to this question⇒

- Lower bounds on the decoding complexity per iteration.
- Quantitative measure to the statement that bipartite graphs representing good codes have cycles (even under ML decoding).

Parity-Check Density

Definition

- Let $\mathcal{C}$ be a binary linear code of rate R and block length n, which is represented by a parity-check matrix H.
- The *density* of H, call it $\Delta = \Delta(H)$, is defined as the normalized number of ones in H *per information bit*.
  $\Rightarrow$ The total number of ones in H is therefore equal to $nR\Delta$.

Example

For the (3,6) regular LDPC ensemble, there are $3n$ ones in the parity-check matrix, the design rate is $R = \frac{1}{2}$, so the parity-check density for a code in this ensemble is $\Delta = 6$.

Theorem (Sason and Urbanke, IEEE Trans. on IT, July '03)

Let $\{\mathcal{C}_m\}$ *be a sequence of binary linear block codes achieving a fraction* $1-\varepsilon$ *of the capacity of a memoryless binary-input output-symmetric (MBIOS) channel with vanishing* bit error probability. *Then for every sequence of codes and any representation of the codes by full-rank parity-check matrices*

$$\liminf_{m\to\infty} \Delta_m > \frac{K_1 + K_2 \ln \frac{1}{\varepsilon}}{1-\varepsilon} \,,$$

where

$$K_1 = \frac{(1-C)\cdot \ln\left(\frac{1}{2\ln 2}\cdot\frac{1-C}{C}\right)}{2C\cdot\ln\left(\frac{1}{1-2w}\right)} \qquad K_2 = \frac{1-C}{2C\cdot\ln\left(\frac{1}{1-2w}\right)}.$$

Here, C is the capacity, and $w = \frac{1}{2}\int_{-\infty}^{\infty} \min\left(p(y|0), p(y|1)\right) dy$.

Theorem (Cont.)

For the Binary Erasure Channel (BEC), these coefficients can be improved to

$$\begin{aligned} K_1 &= \frac{p \cdot \ln\left(\frac{p}{1-p}\right)}{(1-p)\cdot \ln\left(\frac{1}{1-p}\right)}, \\ K_2 &= \frac{p}{(1-p)\cdot \ln\left(\frac{1}{1-p}\right)} \end{aligned}$$

where p designates the probability of erasure in the BEC.
This improvement at least doubles the previous lower bound for the BEC.

Some More Questions (Cont.)

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Is the logarithmic behavior of the information-theoretic lower bound in the Theorem true or just an artifact of the bounding technique ?

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If it is indeed the true behavior for an MBIOS channel, then are the coefficients K_1 *and* K_2 *of the bound in Theorem 1 tight in general ?*

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Question

Is the new bound on the parity-check density tight for the BEC ?

Gallager's Ensembles of Regular LDPC Codes

Theorem (Sason and Urbanke, IEEE Trans. on IT, July '03)

For every memoryless binary-input output-symmetric channel, there exists a sequence of Gallager's ensembles of regular LDPC codes which achieves under ML decoding a fraction $1 - \varepsilon$ *of the channel capacity with vanishing block error probability, and*

$$\lim_{n\to\infty} \Delta_n \leq \frac{K_3 + K_4 \ln \frac{1}{\varepsilon}}{1-\varepsilon} ,$$

where K_3 *and* K_4 *are some constants which only depend on the channel.*

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$\Rightarrow$ The general logarithmic behavior of the information-theoretic lower bound on the parity-check density is the correct one, answering Question no. 3 in the affirmative.

Shokrollahi’s Right-Regular LDPC Ensembles

$$\begin{aligned}\lambda_{\alpha,N}(x) &= \frac{\alpha \sum_{k=1}^{N-1} \binom{\alpha}{k} (-1)^{k+1} x^k}{\alpha - N \binom{\alpha}{N} (-1)^{N+1}} \\ \rho_\alpha(x) &= x^{\frac{1}{\alpha}}\,, \quad 0 < \alpha < 1\,.\end{aligned}$$

Right-Regular LDPC Ensembles (Cont.)

Theorem (Sason and Urbanke, IEEE Trans. on IT, July 2003)

For suitable parameters of α *and* N *and under* iterative message-passing decoding, *this sequence achieves asymptotically at least a fraction* $1-\varepsilon$ *of the channel capacity with vanishing bit error probability. The asymptotic density of its parity-check matrices satisfies*

$$\lim_{n\to\infty} \Delta_n \leq \frac{K_1 + K_2 \ln \frac{1}{\varepsilon} + g(\varepsilon, p)}{1-\varepsilon} \, ,$$

where K_1, K_2 *are the coefficients in the lower bound of Theorem 1, and in the limit where the gap to capacity goes to zero*

$$\lim_{\varepsilon\to 0^+} g(\varepsilon, p) \leq 0.5407 \qquad \forall\, 0 < p < 1.$$

Right-Regular LDPC Ensembles (Cont.)

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$$\lim_{\varepsilon\to 0^+} g(\varepsilon, p) \leq 0.5407 \qquad \forall\, 0 < p < 1.$$

$\Rightarrow$ Answering Question no. 5 in the affirmative: The improved bound for the BEC (see Theorem 1) is tight as the gap to capacity vanishes !

Interim Conclusions

- Theorems 1 & 3 $\Rightarrow$ For any iterative decoder based on the standard Tanner graph there is a tradeoff between performance and complexity. This tradeoff cannot be surpassed !

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Question

Can better tradeoffs be achieved by allowing more complicated graphical models (e.g., graphs which involve state nodes, in addition to variable nodes and parity-check nodes used for representing codes by bipartite graphs) ?

Interim Conclusions

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Can better tradeoffs be achieved by allowing more complicated graphical models (e.g., graphs which involve state nodes, in addition to variable nodes and parity-check nodes used for representing codes by bipartite graphs) ?

- Pfister et al. showed that, fortunately, a better tradeoff can be achieved by introducing state nodes in the graph (e.g., punctured bits). For the BEC, ensembles of irregular repeat-accumulate (IRA) codes and variants were constructed so that they achieve capacity with bounded complexity per information bit.

Motivation

- Previous work is based on two-level quantization of the LLR.
  $\Rightarrow$ replaces MBIOS channel with a physically-degraded BSC.
- Bounding technique depends on binary output, by considering the syndrome of the received sequence.

- Is it possible to generalize the results for a finer quantization which gives a more accurate representation of the MBIOS channel ?
- Is it possible to derive a bound directly for the original (or an equivalent) channel ?

A recent work replies both questions in the affirmative, based on a recent joint work with Mr. Gil Wiechman.

"Un-Quantized" Upper Bound on Achievable Rates

Let $\{\mathcal{C}_m\}$ be a sequence of binary linear block codes, and assume

- Communication over an MBIOS channel with capacity $C \ \frac{\text{bits}}{\text{ch. use}}$.
- The block length tends to infinity as $m \rightarrow \infty$

Theorem

A necessary condition for this sequence to achieve vanishing bit error probability as $m \rightarrow \infty$ is that the asymptotic rate R satisfies

$$R \leq 1 - \frac{1 - C}{1 - \frac{1}{2\ln(2)} \sum_{p=1}^{\infty} \frac{\Gamma(g_p)}{p(2p-1)}}.$$

This theorem is valid under ML decoding, and hence, under any sub-optimal decoding algorithm.

"Un-Quantized" Lower Bound on Parity-Check Density

Let $\{\mathcal{C}_m\}$ be a sequence of binary linear block codes, and assume

- Communication over an MBIOS channel with capacity $C\ \frac{\text{bits}}{\text{ch. use}}$.
- The sequence $\{\mathcal{C}_m\}$ achieves a fraction $1-\varepsilon$ of the channel capacity with vanishing bit error probability.

"Un-Quantized" Lower Bound on Parity-Check Density

Theorem

The asymptotic density of their parity-check matrices satisfies

$$\liminf_{m\to\infty} \Delta_m \geq \frac{K_1 + K_2 \ln \frac{1}{\varepsilon}}{1-\varepsilon}$$

where

$$K_1 = K_2 \; \ln\left(\frac{\xi\,(1-C)}{C}\right) \; , \quad K_2 = \frac{1-C}{C}\;\frac{1}{\ln\left(\frac{1}{g_1}\right)} \; .$$

g_1 is introduced above, and

$$\xi \triangleq \begin{cases} 1 & \text{for a BEC} \\ \frac{1}{2\ln(2)} & \text{otherwise} \end{cases} .$$

Numerical Results: Thresholds

Comparison of the bounds for rate-1/2 irregular LDPC-ensembles

- AWGN Channel.
- Average right degree increases with ensemble number.
- Shannon capacity limit for $R = \frac{1}{2}$ is 0.187 dB
- Provides bounds on inherent loss due to message-passing iterative decoding.

Ensemble Number	Lower Bounds				DE Threshold
	2-Level	4-Level	8-Level	Un-Quantized	
1	0.269 dB	0.370 dB	0.404 dB	0.417 dB	0.809 dB
2	0.201 dB	0.226 dB	0.236 dB	0.239 dB	0.335 dB
3	0.198 dB	0.221 dB	0.229 dB	0.232 dB	0.310 dB
4	0.194 dB	0.208 dB	0.214 dB	0.216 dB	0.274 dB

Numerical Results: Thresholds (Cont.)

Question

Does it make sense that the bounds on the thresholds become more pessimistic as the number of quantization levels is increased (as we can see from the table in the previous slide) ?

Answer

Note that the "quantized bounds" rely on quantized values of the LLR as side information used for the derivation of these bounds, but the channel itself is not quantized. Hence, by increasing the number of quantization levels used for these bounds, the corresponding values of the $\frac{E_b}{N_0}$ *thresholds under ML decoding get farther from the channel capacity.*

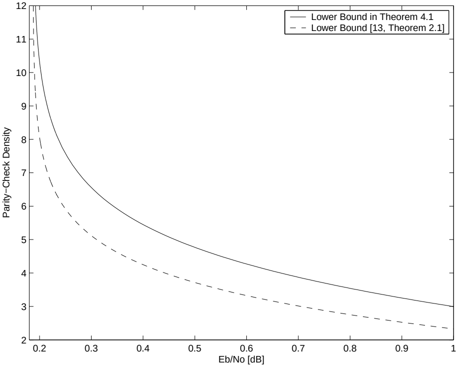
Numerical Results: Parity-Check Density

Setup

- AWGN Channel
- Rate = $\frac{1}{2}$

Observations

- Difference between bounds increases as ε decreases.
- As $\frac{E_\text{b}}{N_\text{o}} \rightarrow 0.187$ dB ($C = \frac{1}{2}$) the bounds tend to infinity.
- The lower bounds and their difference behave like ln $\left(\frac{1}{\varepsilon}\right)$.



Numerical Results

- Original ensemble design rate 1/2.
- Transmission over binary input AWGN channel.
- Puncturing patterns optimized for iterative decoding.
- Provides bound on inherent loss due to iterative decoding.

Design rate	Capacity limit	Lower bound (ML decoding)	Iterative (IT) Decoding	Fractional gap to cap. (ML vs. IT)
0.500	0.187 dB	0.270 dB	0.393 dB	$\geq$ 40.3%
0.592	0.635 dB	0.716 dB	0.857 dB	$\geq$ 36.4%
0.671	1.083 dB	1.171 dB	1.330 dB	$\geq$ 35.6%
0.774	1.814 dB	1.927 dB	2.115 dB	$\geq$ 37.2%
0.838	2.409 dB	2.547 dB	2.781 dB	$\geq$ 37.1%
0.912	3.399 dB	3.607 dB	3.992 dB	$\geq$ 35.1%

Journal Papers Related to this Talk

- I. Sason and R. Urbanke, “Parity-check density versus performance of binary linear block codes over memoryless symmetric channels,” *IEEE Trans. on Information Theory*, vol. 49, no. 7, pp. 1611-1635, July 2003.
- G. Wiechman and I. Sason, “Parity-Check Density versus Performance of binary linear block codes over memoryless symmetric channels: New bounds and applications,” *IEEE Trans. on Information Theory*, vol. 53, no. 2, pp. 550–579, February 2007.
- I. Sason and G. Wiechman, “On achievable rates and complexity of LDPC codes over parallel channels: Bounds and applications,” *IEEE Trans. on Information Theory*, vol. 53, no. 2, pp. 580–598, February 2007.