On the Concentration of the Crest Factor for OFDM Signals

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Orthogonal Frequency Division Multiplexing (OFDM)

- The OFDM modulation converts a high-rate data stream into a number of low-rate steams that are transmitted over parallel narrow-band channels.
- One of the problems of OFDM is that the peak amplitude of the signal can be significantly higher than the average amplitude.

  $\Rightarrow$ Sensitivity to non-linear devices in the communication path (e.g., digital-to-analog converters, mixers and high-power amplifiers).

  $\Rightarrow$ An increase in the symbol error rate and also a reduction in the power efficiency as compared to single-carrier systems.

OFDM (Cont.)

- Given an n-length codeword $\{X_i\}_{i=0}^{n-1}$, a single OFDM baseband symbol is described by

$$s(t; X_0, \ldots, X_{n-1}) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} X_i \exp\left(\frac{j\, 2\pi i t}{T}\right), \quad 0 \leq t \leq T.$$

- Assume that $X_0, \ldots, X_{n-1}$ are i.i.d. complex RVs with $|X_i| = 1$. Since the sub-carriers are orthonormal over $[0, T]$, then a.s. the power of the signal s over this interval is 1.
- The CF of the signal s, composed of n sub-carriers, is defined as

$$\mathrm{CF}_n(s) \triangleq \max_{0 \leq t \leq T} |s(t)|.$$

- The CF scales with high probability like $\sqrt{\log(n)}$ for large n.

Concentration of Measures

- Concentration of measures is a central issue in probability theory, and it is strongly related to information theory and coding.
- In this talk, we consider two of the main approaches for proving concentration inequalities, and apply them to prove concentration for the crest factor of OFDM signals.
  1. The 1st approach is based on martingales (the Azuma-Hoeffding inequality and some refinements).
  2. The 2nd approach is based on Talagrand's inequalities.

A Previously Reported Result

A concentration inequality for the CF of OFDM signals was derived (Litsyn and Wunder, IEEE Trans. on IT, 2006). It states that for every $c \geq 2.5$

$$\mathbb{P}\left(\left|\mathsf{CF}_n(s) - \sqrt{\log(n)}\right| < \frac{c \log\log(n)}{\sqrt{\log(n)}}\right) = 1 - O\left(\frac{1}{\big(\log(n)\big)^4}\right).$$

Concentration via the Martingale Approach

Theorem - **[Azuma-Hoeffding inequality]**

Let $X_0, \ldots, X_n$ be a martingale. If the sequence of differences are bounded, i.e.,

$$|X_i - X_{i-1}| \leq d_i \quad \forall\, i = 1, 2, \ldots, n \quad \text{a.s.}$$

then

$$\mathbb{P}(|X_n - X_0| \geq r) \leq 2 \exp\left(-\frac{r^2}{2\sum_{i=1}^{n} d_i^2}\right), \quad \forall\, r > 0.$$

But, the Azuma-Hoeffding inequality is not tight !.

For example, if $r > \sum_{i=1}^{n} d_i \quad \Rightarrow \quad \mathbb{P}(|X_n - X_0| \geq r) = 0.$

Theorem (Th. 2 – Refined Version I of Azuma-Hoeffding Inequality)

Let $\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}$ *be a discrete-parameter real-valued martingale. Assume that, for some constants* $d, \sigma > 0$*, the following two requirements hold a.s.*

$$|X_k - X_{k-1}| \leq d,$$

$$\mathit{Var}(X_k|\mathcal{F}_{k-1}) = \mathbb{E}\big[(X_k - X_{k-1})^2 \,|\, \mathcal{F}_{k-1}\big] \leq \sigma^2$$

for every $k \in \{1, \ldots, n\}$*. Then, for every* $\alpha \geq 0$*,*

$$\mathbb{P}(|X_n - X_0| \geq \alpha n) \leq 2 \exp\left(-n\, D\left(\frac{\delta+\gamma}{1+\gamma} \Big|\Big| \frac{\gamma}{1+\gamma}\right)\right)$$

where $\gamma \triangleq \frac{\sigma^2}{d^2}$*,* $\delta \triangleq \frac{\alpha}{d}$*, and* $D(p||q) \triangleq p \ln\Big(\frac{p}{q}\Big) + (1-p) \ln\Big(\frac{1-p}{1-q}\Big)$*.*

Corollary

Under the conditions of Theorem 2, for every $\alpha \geq 0$,

$$\mathbb{P}(|X_n - X_0| \geq \alpha n) \leq 2 \exp\left(-n f(\delta)\right)$$

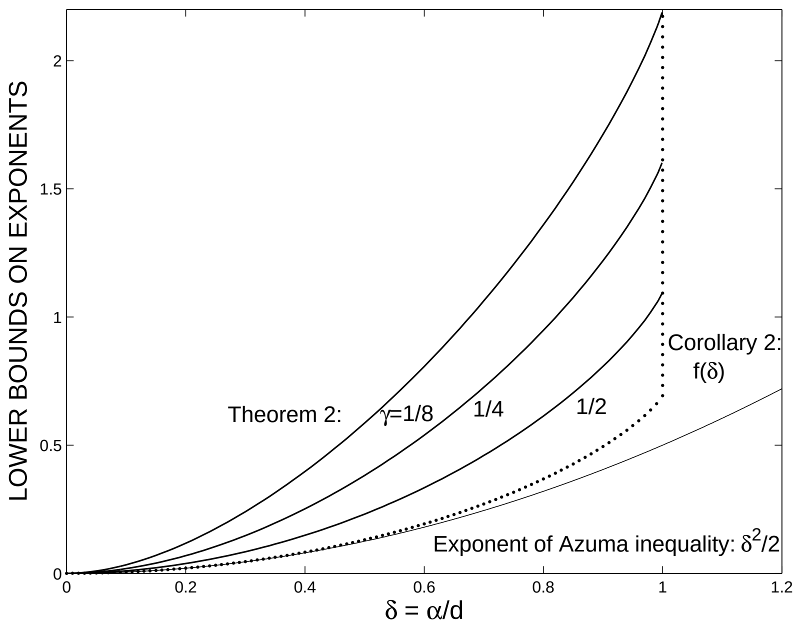
where

$$f(\delta) = \begin{cases} \ln(2)\Big[1 - h_2\left(\frac{1-\delta}{2}\right)\Big], & 0 \leq \delta \leq 1 \\ +\infty, & \delta > 1 \end{cases}$$

and $h_2(x) \triangleq -x \log_2(x) - (1-x) \log_2(1-x)$ for $0 \leq x \leq 1$.

Proof

Set $\gamma = 1$ in Theorem 2.


LOWER BOUNDS ON EXPONENTS
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
δ = α/d
Theorem 2:
γ=1/8
1/4
1/2
Corollary 2:
f(δ)
Exponent of Azuma inequality: δ²/2

Proposition

Let $\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}$ *be a discrete-parameter real-valued martingale. Then, for every* $\alpha \geq 0$*,*

$$\mathbb{P}(|X_n - X_0| \geq \alpha\sqrt{n}) \leq 2\exp\Big(-\frac{\delta^2}{2\gamma}\Big)\Big(1 + O\big(n^{-\frac{1}{2}}\big)\Big).$$

Proof

This follows from Theorem 2 by replacing δ with $\delta' \triangleq \frac{\delta}{\sqrt{n}}$.

Theorem - [McDiarmid's Inequality]

- Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of independent random variables with X_k taking values in a set A_k for each k.
- Suppose that a real-valued function f, defined on $\prod_k A_k$, satisfies

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq c_k$$

  whenever the vectors $\mathbf{x}$ and $\mathbf{x}'$ differ only in the k-th coordinate.
- Let $\mu \triangleq \mathbb{E}[f(X)]$ be the expected value of $f(X)$.

Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|f(X) - \mu| \geq \alpha) \leq 2 \exp\left(-\frac{2\alpha^2}{\sum_k c_k^2}\right).$$

Proving Concentration of the CF for OFDM Signals

Assume that $\{X_j\}_{j=0}^{n-1}$ are independent complex-valued random variables with magnitude 1, attaining the M points of an M-ary PSK constellation with equal probability.

Proving Concentration via the Azuma-Hoeffding Inequality

- Let us define

$$Y_i = \mathbb{E}[\,\mathsf{CF}_n(s)\,|\,X_0, \ldots, X_{i-1}], \quad i = 0, \ldots, n.$$

- $\{Y_i, \mathcal{F}_i\}_{i=0}^n$ is a martingale where $\mathcal{F}_i$ is the σ-algebra generated by $(X_0, \ldots, X_{i-1})$.
- This martingale has bounded jumps: $|Y_i - Y_{i-1}| \leq \frac{2}{\sqrt{n}}$ (revealing the i-th coordinate X_i affects the CF by at most $\frac{2}{\sqrt{n}}$).
- It follows from the Azuma-Hoeffding inequality that, for every $\alpha > 0$,

$$\mathbb{P}(|\mathsf{CF}_n(s) - \mathbb{E}[\mathsf{CF}_n(s)]| \geq \alpha) \leq 2 \exp\left(-\frac{\alpha^2}{8}\right)$$

which demonstrates concentration around the expected value.

Proving Concentration via Martingales (cont.)

- The refined version of the Azuma-Hoeffding inequality improves the exponent by a factor of 2, due to the additional information on the conditional variance.

$$\mathbb{P}(|\mathsf{CF}_n(s) - \mathbb{E}[\mathsf{CF}_n(s)]| \geq \alpha) \leq 2\exp\left(-\frac{\alpha^2}{4}\left(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\right)\right).$$

- McDiarmid's inequality implies that, for every $\alpha \geq 0$,

$$\mathbb{P}(|\mathsf{CF}_n(s) - \mathbb{E}[\mathsf{CF}_n(s)]| \geq \alpha) \leq 2\exp\Big(-\frac{\alpha^2}{2}\Big)$$

  $\Rightarrow$ The exponent improves by a factor of 4 as compared to the Azuma-Hoeffding inequality.

- The same kind of result can be applied to QAM-modulated OFDM signals, since the independent RVs $\{X_j\}$ are bounded.

Proving Concentration via Talagrand's Inequality

- Talagrand's inequality is an approach used for establishing concentration results on product spaces, and this technique was introduced in Talagrand's landmark paper from 1995.
- Talagrand's inequality implies that, for every $\alpha \geq 0$,

$$\mathbb{P}(|\mathsf{CF}_n(s) - m_n| \geq \alpha) \leq 4\exp\left(-\frac{\alpha^2}{16}\right), \quad \forall\, \alpha > 0$$

  where m_n is the median of the crest factor for OFDM signals that are composed of n sub-carriers.
- This inequality demonstrates the concentration of this measure around its median.

Establishing Concentration via Talagrand's Inequality (Cont.)

Corollary

The median and expected value of the crest factor differ by at most a constant, independently of the number of sub-carriers n.

Summary and Conclusions

- Four concentration inequalities for the crest-factor (CF) of OFDM signals under the assumption that the symbols are independent.
  1. The first two concentration inequalities rely on the Azuma-Hoeffding inequality and a refined version of it.
  2. The last two concentration inequalities are based on Talagrand's and McDiarmid's inequalities.
- Although these concentration results when applied to this specific application are weaker than some existing results from the literature, they establish concentration in a rather simple way and provide some insight to the problem.
- For independent symbols, McDiarmid's inequality improves the exponent of the Azuma-Hoeffding inequality by a factor of 4.

Summary and Conclusions (Cont.)

- Talagrand's approach implies that the median and expected value of the CF differ by at most a constant, independently of the number of sub-carriers.
- This talk is aimed to stimulate the use of some refined versions of concentration inequalities, based on the martingale approach and Talagrand's approach, in information-theoretic aspects.

Backup Slides: Martingales and Talagrand's Inequality

Definition - **[Martingale]**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ be a sequence of sub σ-algebras of $\mathcal{F}$. A sequence $X_0, X_1, \ldots$ of RVs is a martingale if

1. $X_i : \Omega \to \mathbb{R}$ is $\mathcal{F}_i$-measurable, i.e.,
$$\{\omega \in \Omega : X_i(\omega) \leq t\} \in \mathcal{F}_i \quad \forall i \in \{0, 1, \ldots\}, t \in \mathbb{R}.$$
2. $\mathbb{E}[|X_i|] < \infty$.
3. $X_i = \mathbb{E}[X_{i+1} | \mathcal{F}_i]$ almost surely.

Example - Random walk

$X_n = \sum_{i=0}^{n} U_i$ where $\mathbb{P}(U_i = +1) = \mathbb{P}(U_i = -1) = \frac{1}{2}$ are i.i.d. RVs.

Martingales – Remarks

Remark 1

Given a RV $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration of sub σ-algebras $\{\mathcal{F}_i\}$, let

$$X_i = \mathbb{E}[X|\mathcal{F}_i] \quad i = 0, 1, \ldots .$$

Then, the sequence $X_0, X_1, \ldots$ forms a martingale.

Remark 2

Choose $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_n = \mathcal{F}$, then it gives a martingale with

$$X_0 = \mathbb{E}[X|\mathcal{F}_0] = \mathbb{E}[X] \quad (\mathcal{F}_0 \text{ doesn't provide information about } X).$$
$$X_n = \mathbb{E}[X|\mathcal{F}_n] = X \text{ a.s.} \quad (\mathcal{F}_n \text{ provides full information about } X).$$

Talagrand's Inequality

- Talagrand's inequality is an approach used for establishing concentration results on product spaces, and this technique was introduced in Talagrand's landmark paper from 1995.
- We provide in the following two definitions that will be required for the introduction of a special form of Talagrand's inequalities.

Talagrand's Inequality (Cont.)

- Let $\mathbf{x}, \mathbf{y}$ be two n-length vectors. The Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ is the number of coordinates where $\mathbf{x}$ and $\mathbf{y}$ disagree, i.e.,

$$d_{\mathsf{H}}(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i=1}^{n} I_{\{x_i \neq y_i\}}$$

where I stands for the indicator function.

Generalization and normalization of the previous distance metric:

- Let $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ (i.e., a is a non-negative vector) satisfy $||a||_2 = 1$. Then, define

$$d_a(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i=1}^{n} a_i I_{\{x_i \neq y_i\}}.$$

Hence, $d_{\mathsf{H}}(\mathbf{x}, \mathbf{y}) = \sqrt{n}\, d_a(\mathbf{x}, \mathbf{y})$ for $a = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$.

Special Form of Talagrand's Inequality (Cont.)

Let the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ be vector of independent random variables with X_k taking values in a set A_k, and let $A \triangleq \prod_{k=1}^{n} A_k$. Let $f : A \to \mathbb{R}$ satisfy the condition that, for every $\mathbf{x} \in A$, there exists a non-negative, normalized n-length vector $a = a(x)$ such that

$$f(\mathbf{x}) \leq f(\mathbf{y}) + \sigma d_a(\mathbf{x}, \mathbf{y}), \quad \forall\, \mathbf{y} \in A$$

for some fixed value $\sigma > 0$. Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|f(X) - m| \geq \alpha) \leq 4 \exp\left(-\frac{\alpha^2}{4\sigma^2}\right)$$

where m is the median of $f(X)$
(i.e., $\mathbb{P}(f(X) \leq m) \geq \frac{1}{2}$ and $\mathbb{P}(f(X) \geq m) \geq \frac{1}{2}$).

Establishing Concentration via Talagrand's Inequality

- Let us assume that $X_0, Y_0, \ldots, X_{n-1}, Y_{n-1}$ are i.i.d. bounded complex RVs, and also for simplicity $|X_i| = |Y_i| = 1$.
- In order to apply Talagrand's inequality to prove concentration, note that

$$
\begin{aligned}
& \max_{0 \leq t \leq T} \bigl| s(t; X_0, \ldots, X_{n-1}) \bigr| - \max_{0 \leq t \leq T} \bigl| s(t; Y_0, \ldots, Y_{n-1}) \bigr| \\
& \leq \max_{0 \leq t \leq T} \bigl| s(t; X_0, \ldots, X_{n-1}) - s(t; Y_0, \ldots, Y_{n-1}) \bigr| \\
& \leq \frac{1}{\sqrt{n}} \left| \sum_{i=0}^{n-1} (X_i - Y_i) \exp\Bigl(\frac{j\, 2\pi i t}{T}\Bigr) \right| \\
& \leq \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |X_i - Y_i| \\
& \leq \frac{2}{\sqrt{n}} \sum_{i=0}^{n-1} I_{\{x_i \neq y_i\}}
\end{aligned}
$$

Establishing Concentration via Talagrand's Inequality (Cont.)

- Talagrand's inequality implies that, for every $\alpha \geq 0$,

$$\mathbb{P}(|\mathsf{CF}_n(s) - m_n| \geq \alpha) \leq 4 \exp\left(-\frac{\alpha^2}{16}\right), \quad \forall\, \alpha > 0$$

  where m_n is the median of the crest factor for OFDM signals that are composed of n sub-carriers.
- This inequality demonstrates the concentration of this measure around its median.

Establishing Concentration via Talagrand's Inequality (Cont.)

Corollary

The median and expected value of the crest factor differ by at most a constant, independently of the number of sub-carriers n.

Proof: From Talagrand's inequality

$$\begin{aligned}
&|\mathbb{E}[\mathsf{CF}_n(s)] - m_n| \\
&\leq \mathbb{E}\,|\mathsf{CF}_n(s) - m_n| \\
&= \int_0^\infty \mathbb{P}(|\mathsf{CF}_n(s) - m_n| \geq \alpha)\, d\alpha \leq 8\sqrt{\pi}.
\end{aligned}$$

where the equality holds since for a non-negative random variable Z

$$\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z \geq t)\, dt.$$