Achieving Marton's Region for Broadcast Channels Using Polar Codes

Marco Mondelli

Joint work with S. Hamed Hassani, Igal Sason, Rüdiger Urbanke

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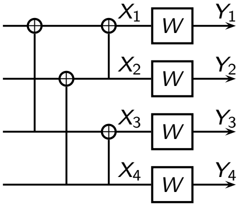
International Symposium on Information Theory, 2014

Polar Coding

Capacity-achieving for binary-input memoryless symmetric channels with complexity $\Theta(n \log n)$ and error probability $\sim 2^{-\sqrt{n}}$.



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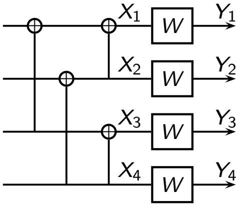

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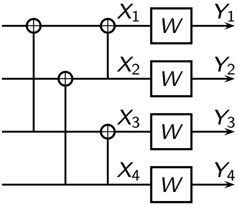

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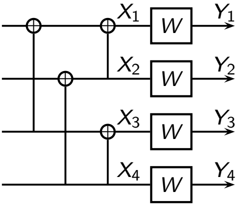

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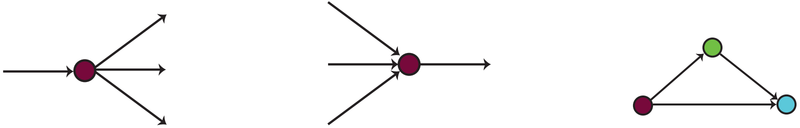
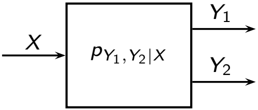
- Lossless and lossy source coding.
- Multi-user channels: **broadcast**, multiple access, relay, interference, wiretap...



Broadcast Channel: Superposition Coding



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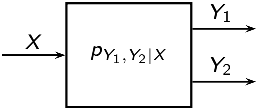

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$$\begin{cases} R_1 < I(X; Y_1 \mid V) \\ R_2 < I(V; Y_2) \\ R_1 + R_2 < I(X; Y_1) \end{cases}$$

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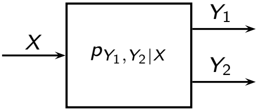

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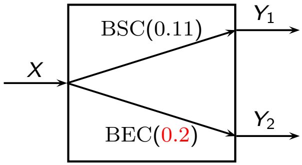
Polar region by Goela et al.*

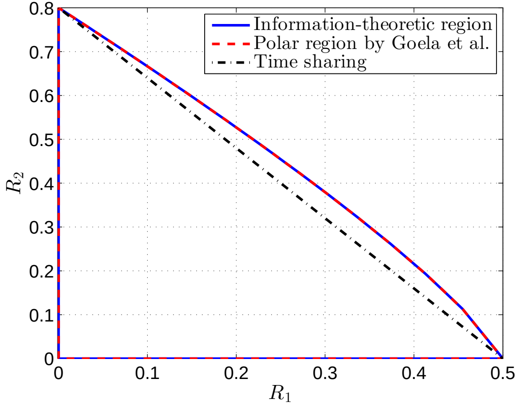
$$\begin{cases} R_1 < I(X; Y_1 \mid V) \\ R_2 < I(V; Y_2) \end{cases}$$

- $p_{Y_1|V} \succ p_{Y_2|V}$

*N. Goela, E. Abbe, M. Gastpar, "Polar Codes for Broadcast Channels", 2013, http://arxiv.org/pdf/1301.6150.pdf.

Broadcast Channel: Comparison of Superposition Regions





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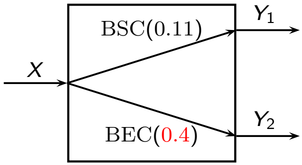

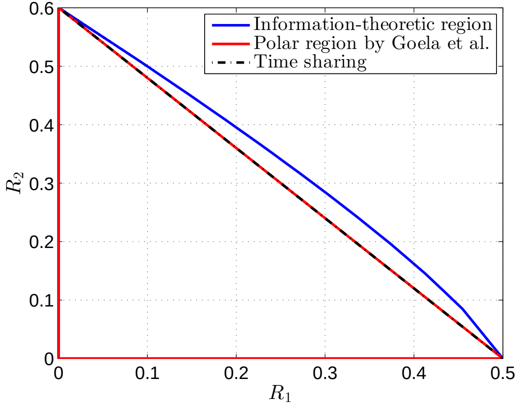

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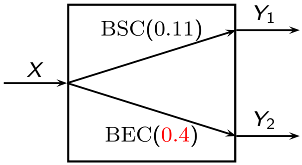

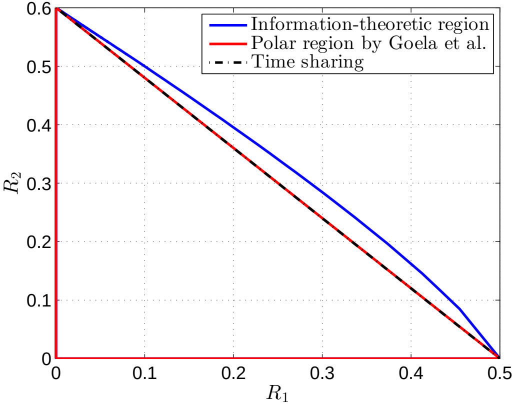

Focus of this talk

Achieve with polar codes the information-theoretic region.

Broadcast Channel: Polar Codes for Marton's Region

Main result

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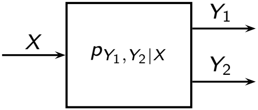

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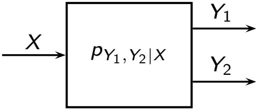

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- Common information at rate R_0
- Best known achievable region
- Combination of superposition and binning

Primitives: Lossless Compression

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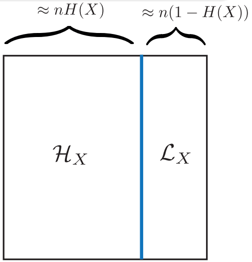

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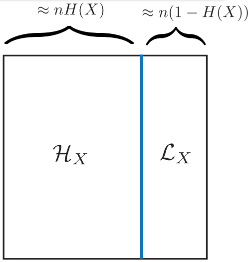

Send bits in $\mathcal{L}_X^c$.

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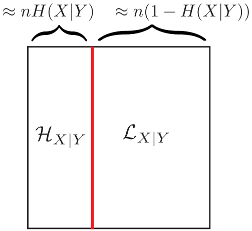

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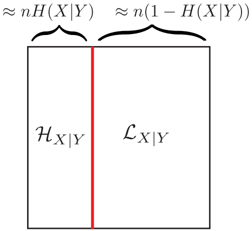

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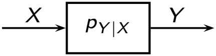

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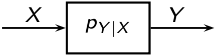

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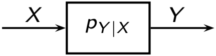

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Information bits in $\mathcal{I} = \mathcal{H}_X \cap \mathcal{L}_{X|Y}$:

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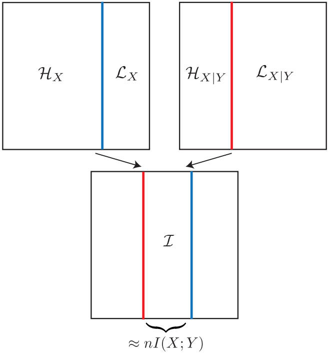

Polar Codes for Superposition Region

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$$\begin{cases} R_1 = I(X; Y_1) - I(V; Y_2) \\ R_2 = I(V; Y_2) \end{cases}$$

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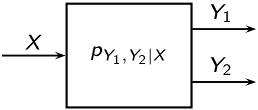

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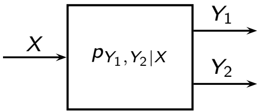

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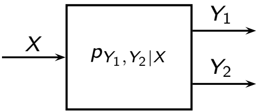

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Basics of superposition coding

- V contains message of user 2 and is decoded by both users.
- X contains message of user 1 and, given V, is decoded by user 1.

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- V = side information on X.
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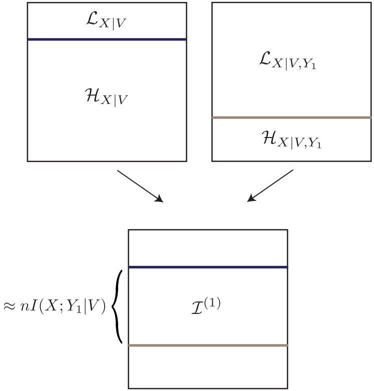

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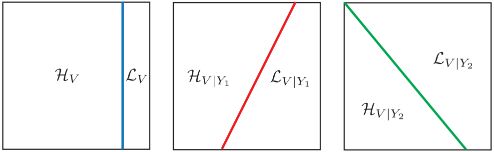

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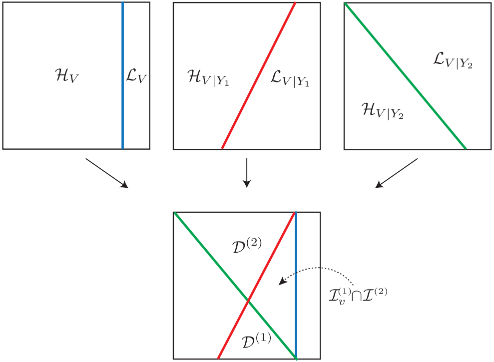
☺ $\mathcal{I}_v^{(1)} \cap \mathcal{I}^{(2)}$: both users decode.

☹ $\mathcal{D}^{(2)} = \mathcal{I}^{(2)} \setminus \mathcal{I}_v^{(1)}$: only user 2 decodes ($p_{Y_1|V} \succ p_{Y_2|V} \Rightarrow \mathcal{D}^{(2)} = \emptyset$).

!! $\mathcal{D}^{(1)} = \mathcal{I}_v^{(1)} \setminus \mathcal{I}^{(2)}$: only user 1 decodes.



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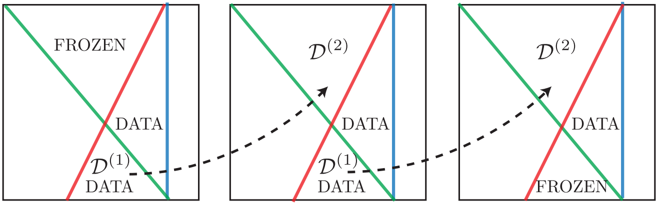

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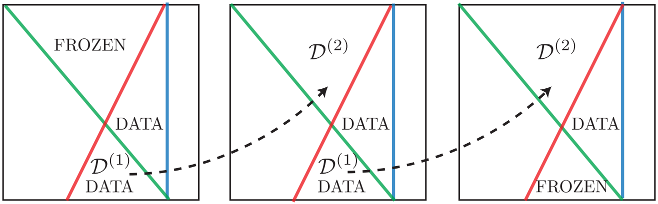

- User 1 decodes "forward" and user 2 decodes "backwards".
- Rate loss $\sim 1/k$.

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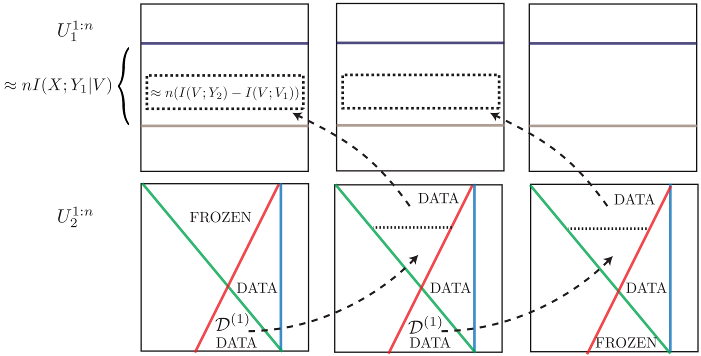

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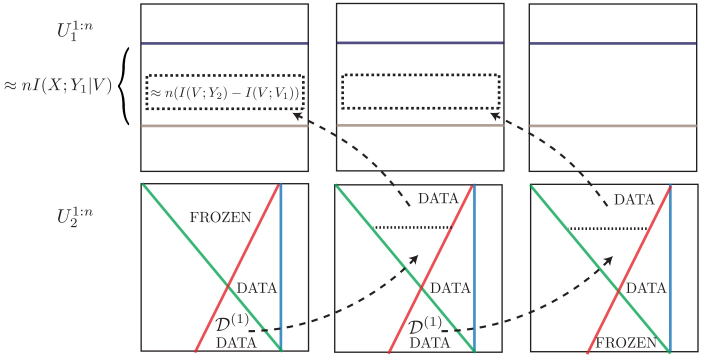

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- Chaining constructions useful in various multi-user scenarios.

M. Mondelli, S. H. Hassani, I. Sason, and R. Urbanke, "Achieving Marton's Region for Broadcast Channels Using Polar Codes", 2014, http://arxiv.org/pdf/1401.6060.pdf.