On Rényi Entropy Power Inequalities

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2016 IEEE International Symposium on Information Theory
Barcelona, Spain

July 10–15, 2016.

The Entropy Power

Let X be a d-dimensional random vector (r.v.) with differential entropy $h(X)$. The entropy power of X is

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The Entropy Power Inequality (EPI)

Let $\{X_k\}_{k=1}^n$ be independent r.v.'s. Then,

$$N\left(\sum_{k=1}^{n} X_k\right) \geq \sum_{k=1}^{n} N(X_k)$$

and equality holds if and only if $\{X_k\}_{k=1}^n$ are Gaussians with proportional covariances.

Applications of the EPI

Converse theorems for...

- The capacity region of the Gaussian broadcast channel - Bergmans, 1974
- The rate-equivocation region of the Gaussian wire-tap channel - Leung-Yan-Cheong & Hellman, 1978
- Multi-terminal rate-distortion theory (the quadratic Gaussian CEO problem) - Oohama, 1998
- The capacity region of the Gaussian broadcast MIMO channel - Weingarten, Steinberg & Shamai, 2006

Rényi's Entropy

- Let X be a d-dimensional r.v. with density f_X.
- Let $\alpha \in (0,1) \cup (1,\infty)$.

The (differential) Rényi entropy of X is

$$h_\alpha(X) = \frac{1}{1-\alpha} \log\left(\int_{\mathbb{R}^d} f_X^\alpha(x)\,\mathrm{d}x \right).$$

Using the L_α norm, $h_\alpha(X) = \frac{\alpha}{1-\alpha} \log \|f_X\|_\alpha$.

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- $h_0(X) = \log \lambda\big(\mathrm{supp}(f_X)\big)$.
- $h_1(X) = h(X) = -\int_{\mathbb{R}^d} f_X(x) \log f_X(x)\, \mathrm{d}x$.
- $h_\infty(X) = -\log\big(\operatorname{ess\,sup}(f_X)\big)$.

where λ is the Lebesgue measure in $\mathbb{R}^d$.

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- Using the L_α norm, for $\alpha \in (0,1) \cup (1, \infty)$,

$$N_\alpha(X) = \left(\|f_X\|_\alpha\right)^{-\frac{\alpha'}{2d}},$$

  where $\alpha' = \frac{\alpha}{\alpha-1}$.
- Homogeneity of order 2: $N_\alpha(\lambda X) = \lambda^2 N_\alpha(X)$, $\forall \lambda \in \mathbb{R}$.
- Monotonically decreasing: $\beta \leq \alpha \implies N_\beta(X) \geq N_\alpha(X)$.

A Rényi EPI (R-EPI)?

1. Let $\{X_k\}_{k=1}^n$ be d–dimensional independent r.v.'s with densities.
2. Let $\alpha \in [0, \infty]$, $n \in \mathbb{N}$.

Is there a positive constant $c_\alpha^{(n,d)}$ such that

$$N_\alpha \left(\sum_{k=1}^n X_k \right) \geq c_\alpha^{(n,d)} \sum_{k=1}^n N_\alpha(X_k)?$$

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Upper and Lower Bounds on $c_\alpha^{(n,d)}$

- For independent Gaussian random vectors with proportional covariances, $N_\alpha\left(\sum_{k=1}^n X_k\right) = \sum_{k=1}^n N_\alpha(X_k)$.
  $\Rightarrow c_\alpha^{(n,d)} \leq 1$
- Trivial Bound - $c_\alpha^{(n,d)} \geq 0$

Related Work

1. Bercher and Vignat, 2002 - For every $\alpha \in [0, \infty]$,
$$N_\alpha \left(\sum_{k=1}^{n} X_k\right) \geq \max_{1 \leq k \leq n} N_\alpha(X_k).$$
2. Wang and Madiman, IEEE Trans. on Info. Theory, 2014 - Conjectures on the optimal R-EPI.
3. Bobkov and Chistyakov, IEEE Trans., 2015 - For every $\alpha > 1$,
$$c_\alpha = \frac{1}{e}\alpha^{\frac{1}{\alpha-1}} \quad \text{(independently of } d \text{ and } n\text{)}.$$
4. Madiman, Melbourne and Xu - Recent survey paper at http://arxiv.org/abs/1604.04225 on EPI, R-EPI and the relation to convex geometry.

Sharpened Young's Inequality

Let $p, q, r \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ be non-negative functions. Then

$$\|f * g\|_r \leq \left(\frac{A_p A_q}{A_r} \right)^{\frac{d}{2}} \|f\|_p \, \|g\|_q,$$

where $A_t = t^{\frac{1}{t}} \, t'^{-\frac{1}{t'}}$ and $t' = \frac{t}{t-1}$.

- Equality holds if and only if f and g are Gaussians.
- Reversed for $p, q, r \in (0, 1)$.

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Monotonicity of Rényi Entropy Power in its Order

Let $\alpha, \beta \in (0,1) \cup (1, \infty)$ such that $\beta \leq \alpha$ and let $f \in L^\alpha \cap L^1$ be a non-negative function such that $\|f\|_1 = 1$. Then,

$$\|f\|_\beta^{\beta'} \leq \|f\|_\alpha^{\alpha'}.$$

Combining the Two Inequalities

Proposition 1: Let $\mathcal{P}^n = \{\underline{t} \in \mathbb{R}^n \colon t_k \geq 0, \, \sum_{k=1}^n t_k = 1\}$ be the probability simplex and let $\alpha > 1$. If $\sum_{k=1}^n N_\alpha(X_k) = 1$, then

$$\log N_\alpha \left(\sum_{k=1}^n X_k \right) \geq f_0(\underline{t}), \; \forall \, \underline{t} \in \mathcal{P}^n,$$

where

- $f_0(\underline{t}) = \frac{\log \alpha}{\alpha - 1} - D(\underline{t} \| \underline{N}_\alpha) + \alpha' \sum_{k=1}^n \left(1 - \frac{t_k}{\alpha'}\right) \log \left(1 - \frac{t_k}{\alpha'}\right).$
- $\underline{N}_\alpha = (N_\alpha(X_1), \ldots, N_\alpha(X_n)).$
- $D(\underline{t} \| \underline{N}_\alpha) = \sum_{k=1}^n t_k \log \left(\frac{t_k}{N_\alpha(X_k)} \right).$

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- $D(\underline{t} \| \underline{N}_\alpha) = \sum_{k=1}^n t_k \log \left(\frac{t_k}{N_\alpha(X_k)} \right).$

$\Rightarrow$ The bound can be tightened by maximizing $f_0(\underline{t})$.

Theorem 1: A New R-EPI

Let

- $\{X_k\}_{k=1}^n$ be d–dimensional independent r.v's with densities.
- $\alpha > 1$, $\alpha' = \frac{\alpha}{\alpha-1}$.
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$$N_\alpha\left(\sum_{k=1}^n X_k\right) \geq c_\alpha^{(n)} \sum_{k=1}^n N_\alpha(X_k),$$

with

$$c_\alpha^{(n)} = \alpha^{\frac{1}{\alpha-1}} \left(1 - \frac{1}{n\alpha'}\right)^{n\alpha'-1}.$$

Theorem 1: A New R-EPI (Cont.)

$$N_\alpha\Big(\sum_{k=1}^{n} X_k\Big) \geq c_\alpha^{(n)} \sum_{k=1}^{n} N_\alpha(X_k), \quad c_\alpha^{(n)} = \alpha^{\frac{1}{\alpha-1}} \left(1 - \frac{1}{n\alpha'}\right)^{n\alpha'-1}.$$

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3. If $\alpha \downarrow 1$, it coincides with the EPI.
4. $n = 2$ and $\alpha \to \infty \;\Rightarrow\; c_\alpha^{(n)}$ tends to $\frac{1}{2}$ which is optimal (Rogozin 1988); achieved when X_1 and X_2 are uniformly distributed in the cube $[0,1]^d$.

Theorem 1: A New R-EPI (Cont.)

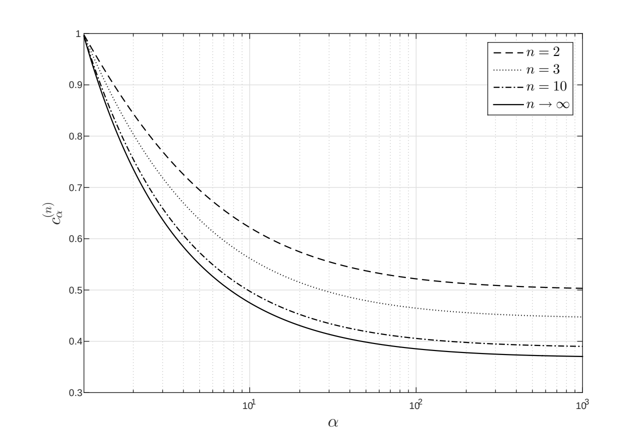

Figure: A plot of $c_\alpha^{(n)}$ as a function of α, for $n = 2, 3, 10$ and $n \to \infty$.

Motivation

- Bercher and Vignat -
  $N_\alpha(X_1 + X_2) \geq \max(N_\alpha(X_1), N_\alpha(X_2)),\ \alpha \in [0, \infty]$

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- $\frac{1}{2}\left(N_\infty(X_1) + N_\infty(X_2)\right) \leq \max(N_\infty(X_1), N_\infty(X_2))$

$\Rightarrow$ Bercher and Vignat's bound is tighter for $n = 2$ and large enough α.

The Optimization Problem

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$$\begin{array}{ll} \text{maximize} & f_0(t_1, t_2, \ldots, t_{n-1}, t_n) \\ \text{subject to} & t_k \geq 0, \quad k \in \{1, \ldots, n\}, \\ & \sum_{k=1}^n t_k = 1 \end{array}$$

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$$\begin{array}{ll} \text{maximize} & f_0\left(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k\right) \\ \text{subject to} & t_k \geq 0, \quad k \in \{1, \ldots, n-1\}, \\ & \sum_{k=1}^{n-1} t_k \leq 1 \end{array}$$

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- This problem can be shown to be convex by a non trivial use of the next result from matrix theory (Bunch *et al.* 1978).

Rank–One Modification Theorem (Bunch *et al.* 1978)

Let

- $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix with the eigenvalues $d_1 \leq d_2 \leq \ldots \leq d_n$.
- $z \in \mathbb{R}^n$ such that $\|z\|_2 = 1$, $\rho \in \mathbb{R}$ and $C = D + \rho z z^T$.
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Then,

1. $\lambda_i = d_i + \rho \mu_i$, where $\sum_{i=1}^n \mu_i = 1$ and $\mu_i \geq 0$ for all $i \in \{1, \ldots, n\}$.
2. If $\rho > 0$, then $d_1 \leq \lambda_1 \leq d_2 \leq \lambda_2 \leq \ldots \leq d_n \leq \lambda_n$.
   If $\rho < 0$, then $\lambda_1 \leq d_1 \leq \lambda_2 \leq d_2 \leq \ldots \leq \lambda_n \leq d_n$.
3. If $d_j \neq d_i$ and $z_i, \rho \neq 0$, then the inequalities are strict, and for every $i \in \{1, \ldots, n\}$, λ_i is a zero of $W(x) = 1 + \rho \sum_{j=1}^n \frac{z_i^2}{d_j - x}$.

Applying The Rank–One Modification Theorem

1. The Rank–One Modification Theorem is used to prove that the Hessian matrix of $f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k)$ is negative semi-definite.

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   is convex.
3. The solution can be found by solving the KKT conditions.

Theorem 2

- Let $X_1, \ldots, X_n$ be d–dimensional independent r.v's with densities.
- Assume, w.l.g, that $N_\alpha(X_k) \leq N_\alpha(X_n), \quad k \in \{1, \ldots, n-1\}$.
- Let $c_k = \frac{N_\alpha(X_k)}{N_\alpha(X_n)}, \qquad k \in \{1, \ldots, n-1\}$.
- let $t_n \in [0,1]$ be the unique solution of $t_n + \sum_{k=1}^{n-1} \psi_k(t_n) = 1$ with $\psi_k(x) = \frac{\alpha' - \sqrt{\alpha'^2 - 4c_k\, x(\alpha' - x)}}{2}, \quad x \in [0,1]$.
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$$N_\alpha\left(\sum_{k=1}^{n} X_k\right) \geq e^{f_0(t_1, \ldots, t_n)} \sum_{k=1}^{n} N_\alpha(X_k),$$

with f_0 defined in Proposition 1.

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- Improves the R-EPI in Theorem 1 unless $N_\alpha(X_k)$ is independent of k; in the latter case, the two R-EPIs coincide.

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$$N_\alpha\left(\sum_{k=1}^{n} X_k\right) \geq e^{f_0(t_1,\ldots,t_n)} \sum_{k=1}^{n} N_\alpha(X_k).$$

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- It coincides with the EPI as $\alpha \downarrow 1$.
- For $n = 2$ it leads to a closed-form bound in the next Corollary.

Further Tightening The Bound for $n = 2$

Corollary 1: Let

- X_1 and X_2 be d–dimensional independent r.v's with densities.
- $\alpha > 1$, $\alpha' = \frac{\alpha}{\alpha-1}$.
- $\beta_\alpha = \frac{\min\{N_\alpha(X_1),N_\alpha(X_2)\}}{\max\{N_\alpha(X_1),N_\alpha(X_2)\}}$, $\quad d(x\|y) = x\log\left(\frac{x}{y}\right) + (1-x)\log\left(\frac{1-x}{1-y}\right)$.
- $t_\alpha = \begin{cases} \frac{\alpha'(\beta_\alpha+1)-2\beta_\alpha-\sqrt{(\alpha'(\beta_\alpha+1))^2-8\alpha'\beta_\alpha+4\beta_\alpha}}{2(1-\beta_\alpha)} & \text{if } \beta_\alpha < 1 \\ \frac{1}{2} & \text{if } \beta_\alpha = 1 \end{cases}$

The following R-EPI holds:

$$N_\alpha(X_1 + X_2) \geq c_\alpha \left(N_\alpha(X_1) + N_\alpha(X_2)\right),$$

with

$$c_\alpha = \alpha^{\frac{1}{\alpha-1}} \exp\left\{-d\left(t_\alpha \,\middle\|\, \frac{\beta_\alpha}{\beta_\alpha+1}\right)\right\} \left(1 - \frac{t_\alpha}{\alpha'}\right)^{\alpha'-t_\alpha} \left(1 - \frac{1-t_\alpha}{\alpha'}\right)^{\alpha'-1+t_\alpha}.$$

Corollary 1 (Cont.)

$$N_\alpha(X_1 + X_2) \geq c_\alpha \left(N_\alpha(X_1) + N_\alpha(X_2)\right).$$

All the properties of our tightest bound in Theorem 2 hold:

- Improves the bound in Theorem 1 for $n = 2$. Both bounds coincide if and only if $N_\alpha(X_1) = N_\alpha(X_2)$.

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In addition,

- $\alpha \to \infty \;\Rightarrow$ the bound is tight and achieved by two independent d-dimensional random vectors uniformly distributed in the cubes $[0, \sqrt{N_1}]^d$ and $[0, \sqrt{N_2}]^d$.

Corollary 1 (Cont.)

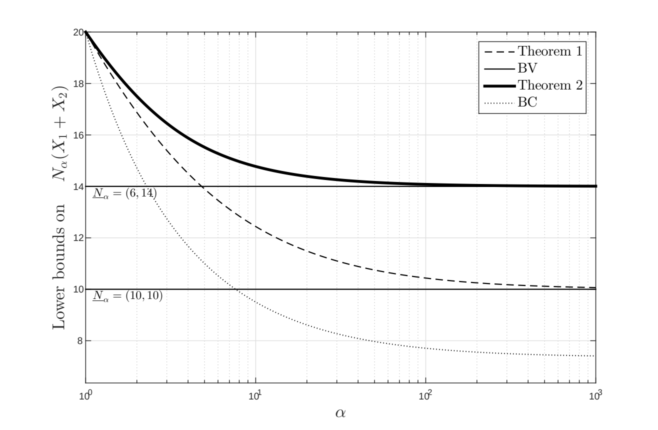

Figure: A comparison of the R-EPIs, for $n = 2$.

Summary

Three improved R-EPIs for independent random vectors in $\mathbb{R}^d$:

$$N_\alpha \left(\sum_{k=1}^n X_k\right) \geq c_{\alpha,d}^{(n)} \sum_{k=1}^n N_\alpha(X_k), \quad \alpha \in (1, \infty).$$

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   - $c_\alpha^{(n)} = \alpha^{\frac{1}{\alpha-1}} \left(1 - \frac{1}{n\alpha'}\right)^{n\alpha'-1}$ with $\alpha' = \frac{\alpha}{\alpha-1}$.
   - Improves the R-EPI by Bobkov & Chistyakov ($c_\alpha = \frac{1}{e}\alpha^{\frac{1}{\alpha-1}}$ for $\alpha > 1$).
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3. Corollary 1 ($n = 2$):
   - Closed form bound;
   - Best known R-EPI for $N_\alpha(X_1 + X_2)$ with $\alpha > 1$.

Summary (Cont.)

Theorem 1 is obtained by tightening the recent R-EPI by Bobkov and Chistyakov with the same analytical tools:

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Theorem 2, providing a further improvement of the R-EPI, also relies on the following analytical tools:

- Solution of the Karush-Kuhn-Tucker (KKT) equations of the related optimization problem;
- Strong Lagrange duality in convex optimization where convexity is asserted by invoking a theorem in matrix theory regarding the rank-one modification of a real-valued symmetric matrix.

Applications

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Full Paper Version

E. Ram and I. Sason, "On Rényi Entropy Power Inequalities," submitted to the *IEEE Trans. on Information Theory* in January 2016, and revised in June 2016. http://arxiv.org/abs/1601.06555.

A Discussion for $\alpha \in (0,1)$

- $\alpha \in (0,1) \implies \alpha' < 0 \quad (\alpha' = \frac{\alpha}{\alpha-1})$.
- **Reverse Sharpened Young's Inequality**: for $0 < \alpha, \nu_1, \nu_2 < 1$ such that $\frac{1}{\alpha'} = \frac{1}{\nu_1'} + \frac{1}{\nu_2'}$ and $f, g \geq 0$,

$$\|f * g\|_\alpha \geq \left(\frac{A_{\nu_1} A_{\nu_2}}{A_\alpha}\right)^{\frac{d}{2}} \|f\|_{\nu_1} \|g\|_{\nu_2}.$$

- $\alpha \in (0,1) \implies 0 < \alpha < \nu_1, \nu_2 \implies N_\alpha(X_k) \geq N_{\nu_k}(X_k)$.

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Question

In view of these reversed inequalities, can we derive a reversed R-EPI for $\alpha \in (0,1)$?

Discussion for $\alpha \in (0,1)$ (Cont.)

- Unfortunately, our bounding technique is not extendable for $\alpha \in (0,1)$. Since $\alpha' < 0$ then $-\frac{2\alpha'}{d} > 0$, and with $A = \frac{1}{A_\alpha} \prod_{k=1}^{n} A_{\nu_k}$:

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N_\alpha\left(\sum_{k=1}^{n} X_k\right) &= \left(\|f_{X_1} * \ldots * f_{X_n}\|_\alpha\right)^{-\frac{2\alpha'}{d}} \\
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  - The reverse Young's inequality still leads to **a lower bound**.
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  - The reverse Young's inequality still leads to **a lower bound**.
  - 2nd reverse inequality $\Longrightarrow$ **an upper bound to a lower bound :(**.
- The Bercher & Vignat (BV) bound still holds for $\alpha \in (0,1)$:

$$N_\alpha\left(\sum_{k=1}^{n} X_k\right) \geq \max_{1 \leq k \leq n} N_\alpha(X_k).$$

- For independent Gaussian random vectors with proportional covariances, $N_\alpha\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} N_\alpha(X_k)$ also for $\alpha \in (0,1)$.
- $\Longrightarrow \frac{1}{n} \leq c_\alpha^{(n,d)} \leq 1$ for $\alpha \in (0,1)$.