On Rényi Entropy Power Inequalities

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***Abstract*—This work introduces Rényi entropy power inequalities (R-EPIs) for sums of independent random vectors, improving recent R-EPIs by Bobkov and Chistyakov. The latter work inspired the derivation of the improved bounds.**

Keywords: Differential Rényi entropy, entropy power inequality, Rényi entropy power.

1. Introduction

One of the well-known inequalities in information theory is Shannon's celebrated entropy power inequality (EPI) [13, Theorem 15]. Let X be a d-dimensional random vector with density, let $h(X)$ be its differential entropy, and let

$$N(X) = \exp\left(\frac{2}{d}\,h(X)\right) \tag{1}$$

designate the entropy power of X. The EPI states that for independent random vectors $\{X_k\}_{k=1}^n$

$$N\left(\sum_{k=1}^n X_k\right) \geq \sum_{k=1}^n N(X_k) \tag{2}$$

where equality in (2) holds if and only if $\{X_k\}_{k=1}^n$ are Gaussians with proportional covariances.

Due to the importance of the EPI, various insightful information-theoretic proofs of this inequality have been obtained (e.g., [2], [7], [14], [15]). More studies on the theme include EPIs for discrete random variables and some analogies (e.g., [17] and references therein), generalized EPIs (e.g., [8], [9], [18]), reversed inequalities [5], and related inequalities to the EPI in terms of rearrangements [16]. We refer the reader to a very recent survey in [10].

The Rényi entropy and divergence evidence a long track record of usefulness in information theory. For completeness, we introduce the differential Rényi entropy and Rényi entropy power which are used throughout this work.

Definition 1 (Differential Rényi entropy)**.** Let X be a d-dimensional random vector with density designated by f_X. The differential Rényi entropy of X of order $\alpha \in (0,1) \cup (1,\infty)$ is given by

$$\begin{aligned} h_\alpha(X) &= \frac{1}{1-\alpha}\,\log\left(\int_{\mathbb{R}^d} f_X^\alpha(x)\,\mathrm{d}x\right) \\ &= \frac{\alpha}{1-\alpha}\,\log\|f_X\|_\alpha. \end{aligned} \tag{3}$$

The differential Rényi entropies of orders $\alpha = 0, 1, \infty$ are defined as the continuous extension of $h_\alpha(X)$ for $\alpha \in (0,1) \cup (1,\infty)$, which yields

$$h_0(X) = \log \lambda\big(\mathrm{supp}(f_X)\big), \tag{4}$$

$$h_1(X) = h(X) = -\int_{\mathbb{R}^d} f_X(x)\,\log f_X(x)\,\mathrm{d}x, \tag{5}$$

$$h_\infty(X) = -\log\big(\operatorname{ess\,sup}(f_X)\big), \tag{6}$$

where λ is the Lebesgue measure in $\mathbb{R}^d$.

Definition 2 (Rényi entropy power)**.** For a d-dimensional random vector X with density, the Rényi entropy power of order $\alpha \in [0,\infty]$ is given by

$$N_\alpha(X) = \exp\left(\frac{2}{d}\,h_\alpha(X)\right). \tag{7}$$

Since the differential Rényi entropy, $h_\alpha(X)$, generalizes the differential Shannon entropy, $h(X)$, the question of generalizing the EPI for Rényi entropy powers (call it, R-EPI) has emerged.

Question 1. For independent random vectors $\{X_k\}$ with densities, $\alpha \in [0,\infty]$ and $n \in \mathbb{N}$, does an R-EPI of the form

$$N_\alpha\left(\sum_{k=1}^n X_k\right) \geq c_\alpha^{(n,d)} \sum_{k=1}^n N_\alpha(X_k) \tag{8}$$

hold for some positive constant $c_\alpha^{(n,d)}$?

Question 1 has been recently addressed in [4], showing that (8) holds with

$$c_\alpha = \tfrac{1}{e}\,\alpha^{\frac{1}{\alpha-1}}, \quad \forall\,\alpha > 1 \tag{9}$$

independently of the values of n and d. It is the purpose of this paper to derive some improved R-EPIs. A study of Question 1 for $\alpha \in (0,1)$ is currently an open problem (see [4, p. 709]), and this paper introduces new R-EPIs for $\alpha > 1$ (the case of $\alpha = 1$ refers to the EPI (2)). A conjecture with respect to tight R-EPIs has been stated in [16, Conjecure 4.4].

In view of the close relation between the differential Rényi entropy and the L_α norm (see (3)), inequalities from functional analysis play a key role in proving the EPI and the R-EPI in [4]. One of these inequalities is a sharpened Young's inequality which has been used by Dembo *et al.* [7] in proving the EPI, and also by Bobkov and Chistyakov in a derivation of an R-EPI [4].

This work introduces some R-EPIs which improve recent results in [4]. The reader is referred to the full paper version in [11] for proofs, and further discussions on the results.

2. A New Rényi EPI

In the following, a new R-EPI is introduced. This inequality, which is expressed in closed-form, is tighter than the R-EPI in [4, Theorem I.1].

Proposition 1. Let $\{X_k\}_{k=1}^n$ be a sequence of independent random vectors with densities defined on $\mathbb{R}^d$, and let $n \in \mathbb{N}$, $\alpha > 1$, $\alpha' = \frac{\alpha}{\alpha-1}$ and $S_n = \sum_{k=1}^n X_k$. Let $\mathcal{P}^n = \{\underline{t} \in \mathbb{R}^n \colon t_k \geq 0, \sum_{k=1}^n t_k = 1\}$ be the probability simplex. Then,

$$\log N_\alpha(S_n) \geq f_0(\underline{t}), \ \forall \underline{t} \in \mathcal{P}^n \tag{10}$$

where

$$\begin{aligned} f_0(\underline{t}) &= \frac{\log \alpha}{\alpha - 1} - D(\underline{t} \| \underline{N}_\alpha) \\ &\quad + \alpha' \sum_{k=1}^n \left(1 - \frac{t_k}{\alpha'}\right) \log \left(1 - \frac{t_k}{\alpha'}\right) \end{aligned} \tag{11}$$

and

$$\underline{N}_\alpha = (N_\alpha(X_1), \ldots, N_\alpha(X_n)) \tag{12}$$

$$D(\underline{t} \| \underline{N}_\alpha) = \sum_{k=1}^n t_k \log \left(\frac{t_k}{N_\alpha(X_k)} \right). \tag{13}$$

Proof: See [11, Section III]. ∎

In view of Proposition 1, one can tighten the R-EPI in [4, Theorem I.1] by maximizing the right side of (10). This leads to the optimization problem

$$\begin{aligned} &\text{maximize} && f_0(\underline{t}) \\ &\text{subject to} && t_k \geq 0, \quad k \in \{1, \ldots, n\}, \\ &&& \textstyle\sum_{k=1}^n t_k = 1 \end{aligned} \tag{14}$$

which then yields the following R-EPI (see [11]):

Theorem 1. Let $\{X_k\}_{k=1}^n$ be independent random vectors with densities defined on $\mathbb{R}^d$, let $n \in \mathbb{N}$, $\alpha > 1$, $\alpha' = \frac{\alpha}{\alpha-1}$ and $S_n = \sum_{k=1}^n X_k$. Then, the following R-EPI holds:

$$N_\alpha(S_n) \geq c_\alpha^{(n)} \sum_{k=1}^n N_\alpha(X_k) \tag{15}$$

with

$$c_\alpha^{(n)} = \alpha^{\frac{1}{\alpha-1}} \left(1 - \frac{1}{n\alpha'}\right)^{n\alpha' - 1}. \tag{16}$$

Furthermore, the R-EPI in (15) has the following properties:

1) Eq. (15) improves the R-EPI in [4, Theorem I.1] for every $\alpha > 1$ and $n \in \mathbb{N}$.
2) For all $\alpha > 1$, it asymptotically coincides with the R-EPI in [4, Theorem I.1] as $n \to \infty$.
3) In the other limiting case where $\alpha \downarrow 1$, it coincides with the EPI (similarly to [4]).
4) If $n = 2$ and $\alpha \to \infty$, then $c_\alpha^{(n)}$ in (16) tends to $\frac{1}{2}$ which is optimal; this constant is achieved when X_1 and X_2 are independent random vectors which are uniformly distributed in the cube $[0,1]^d$.

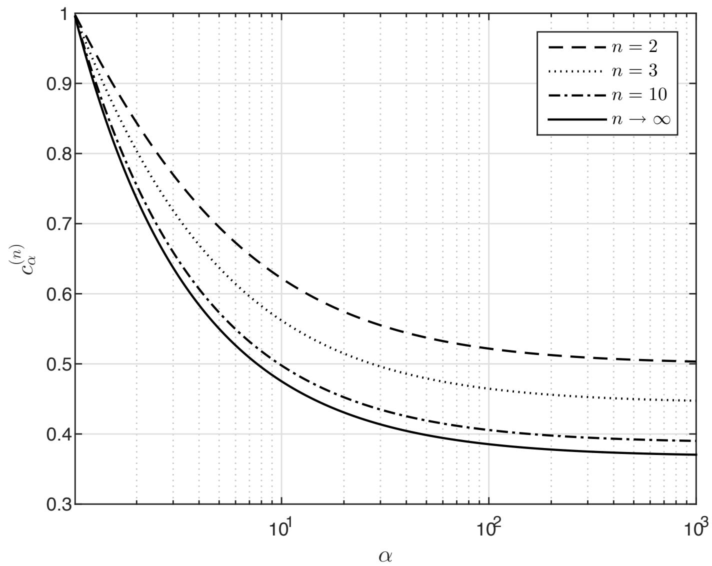

Fig. 1. A plot of $c_\alpha^{(n)}$ in (16), as a function of α, for $n = 2, 3, 10$ and $n \to \infty$.

Proof: See [11, Section III]. The proof relies on Proposition 1 and a sub-optimal choice in the optimization problem (14). ∎

Figure 1 plots $c_\alpha^{(n)}$ as a function of α, for some values of n, verifying numerically Items 1)–4) in Theorem 1. In [4, Theorem I.1], $c_\alpha^{(n)}$ is independent of n; it is given by c_α in (8) which is equal to the limit of $c_\alpha^{(n)}$ in (16) as $n \to \infty$ (the solid curve in Figure 1).

Remark 1. Let X be a d-dimensional random vector with density f_X, and define

$$M(X) := \operatorname{ess\,sup}(f_X). \tag{17}$$

From (6), (7) and (17), it follows that

$$\begin{aligned} N_\infty(X) &:= \lim_{\alpha \to \infty} N_\alpha(X) && (18) \\ &= M^{-\frac{2}{d}}(X). && (19) \end{aligned}$$

Note that from (16), since $\alpha' \to 1$ as $\alpha \to \infty$,

$$\lim_{\alpha \to \infty} c_\alpha^{(n)} = \left(1 - \frac{1}{n}\right)^{n-1}. \tag{20}$$

By assembling (15), (19) and (20), it follows that if $X_1, \ldots, X_n$ are independent d-dimensional random vectors with densities then

$$M^{-\frac{2}{d}}(S_n) \geq \left(1 - \frac{1}{n}\right)^{n-1} \sum_{k=1}^n M^{-\frac{2}{d}}(X_k). \tag{21}$$

Moreover, if $d = 1$, (21) can be strengthened to (see [3, p. 105] and [12])

$$\frac{1}{M^2(S_n)} \geq \tfrac{1}{2} \sum_{k=1}^n \frac{1}{M^2(X_k)}. \tag{22}$$

3. A Further tightening of the R-EPI

A. A Tightened R-EPI for $n = 2$

This sub-section forms a preparatory stage towards a tightening in [11] of the R-EPI in Theorem 1 for a general $n \geq 2$ independent random vectors with densities. We consider in the following the case where $n = 2$, leading to a closed form bound in this special case. To this end, the constant $c_\alpha^{(2)}$ in (16) is replaced by a constant which depends not only on $\alpha > 1$, but also on the ratio $\frac{N_\alpha(X_1)}{N_\alpha(X_2)}$.

Define the binary relative entropy function as the continuous extension to $[0,1]^2$ of

$$d(x\|y) = x \log\left(\frac{x}{y}\right) + (1-x)\log\left(\frac{1-x}{1-y}\right). \tag{23}$$

Proposition 2. Let X_1 and X_2 be independent random vectors with densities defined on $\mathbb{R}^d$, and let $N_\alpha(X_1)$, $N_\alpha(X_2)$ be their Rényi entropy powers of order $\alpha > 1$. Let

$$\alpha' = \frac{\alpha}{\alpha - 1}, \tag{24}$$

$$\beta_\alpha = \frac{\min\{N_\alpha(X_1), N_\alpha(X_2)\}}{\max\{N_\alpha(X_1), N_\alpha(X_2)\}}, \tag{25}$$

$$t_\alpha = \begin{cases} \frac{\alpha'(\beta_\alpha+1) - \sqrt{(\alpha'(\beta_\alpha+1))^2 - 8\alpha'\beta_\alpha + 4\beta_\alpha} - 2}{2(\beta_\alpha - 1)} & \text{if } \beta_\alpha < 1 \\ \frac{1}{2} & \text{if } \beta_\alpha = 1. \end{cases} \tag{26}$$

Then, the following R-EPI holds:

$$N_\alpha(X_1 + X_2) \geq c_\alpha \left(N_\alpha(X_1) + N_\alpha(X_2)\right) \tag{27}$$

with

$$\begin{aligned} c_\alpha = {} & \alpha^{\frac{1}{\alpha-1}} \exp\left(-d\Big(t_\alpha \,\|\, \frac{\beta_\alpha}{\beta_\alpha + 1}\Big)\right) \\ & \left(1 - \frac{t_\alpha}{\alpha'}\right)^{\alpha' - t_\alpha} \left(1 - \frac{1 - t_\alpha}{\alpha'}\right)^{\alpha' - 1 + t_\alpha}. \end{aligned} \tag{28}$$

The R-EPI in (27) satisfies the following properties:

1) Eq. (27) improves the bound in (15) for $n = 2$, and both bounds coincide if and only if $N_\alpha(X_1) = N_\alpha(X_2)$.
2) Eq. (27) improves the bound [1]
$$N_\alpha(X_1 + X_2) \geq \max\{N_\alpha(X_1), N_\alpha(X_2)\} \tag{29}$$
and the bounds in (27) and (29) asymptotically coincide as $\alpha \to \infty$.
3) In the limit where $\alpha \to \infty$, the bound is tight and it is achieved by letting X_1 and X_2 be independent d-dimensional random vectors which are uniformly distributed in the cubes $[0, \sqrt{N_1}]^d$ and $[0, \sqrt{N_2}]^d$, respectively; in this case, $N_\infty(X_k) = N_k$ for $k = 1, 2$.
4) In the limit where $\alpha \downarrow 1$, it coincides with the EPI which is tight when X_1 and X_2 are independent Gaussian random vectors.

Proof: See [11, Section IV.A]. ■

Figure 2 plots the four bounds presented so far for $n = 2$: the abbreviations 'BC' and 'BV' stand, respectively, for the bounds in [4] (see (9)) and [1] (see (29)), and this figure also refers to the improvement of the BC bound in Theorem 1, and the tightest bound among the four in Proposition 2. Note that the bound in Proposition 2 asymptotically coincides with the BV bound as $\alpha \to \infty$, and with the EPI as $\alpha \downarrow 1$. The bounds are exemplified in Figure 2 for a symmetric case where $(N_\alpha(X_1), N_\alpha(X_2)) = (10, 10)$, and an asymmetric case where $(N_\alpha(X_1), N_\alpha(X_2)) = (6, 14)$; note that, in both cases, $N_\alpha(X_1) + N_\alpha(X_2) = 20$.

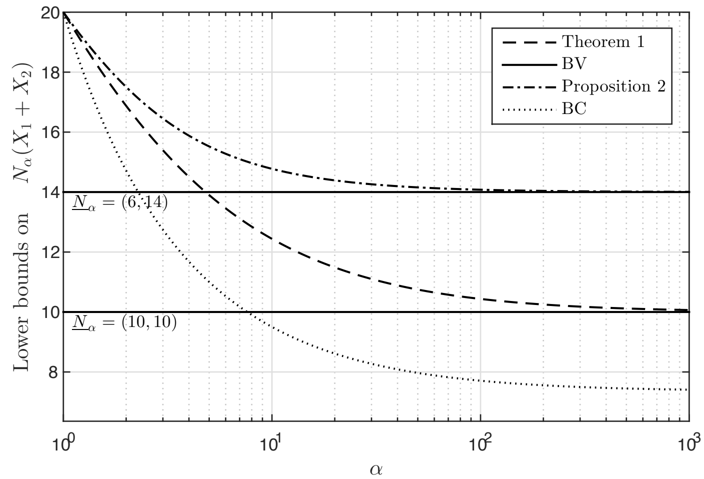

Fig. 2. A comparison of the R-EPIs, for $n = 2$, according to the BC and BV bounds in [4] and [1], respectively, Theorem 1, and the tightest bound for $n = 2$ in Proposition 2. The bounds are shown in a symmetric case where $(N_\alpha(X_1), N_\alpha(X_2)) = (10, 10)$ and an asymmetric case where $(N_\alpha(X_1), N_\alpha(X_2)) = (6, 14)$.

B. A Generalization of the Tightened R-EPI for $n \geq 2$

This sub-section introduces (without proof) a non-trivial generalization of the R-EPI in Proposition 2 for $n \geq 2$ independent random vectors with probability densities.

The analysis which leads to the generalized R-EPI in [11, Section IV.B] relies on a sharpened Young's inequality, Hölder's inequality, convex optimization (Lagrange duality and KKT conditions), and Fact 1 below which provides some useful and interesting properties from matrix theory [6].

Fact 1. Let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix with the eigenvalues $d_1 \leq d_2 \leq \ldots \leq d_n$. Let $z \in \mathbb{R}^n$ such that $\|z\|_2 = 1$ and let $\rho \in \mathbb{R}$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the rank-one modification of D which is given by $C = D + \rho z z^T$. Then,

1) $\lambda_i = d_i + \rho\mu_i$, where $\sum_{i=1}^n \mu_i = 1$ and $\mu_i \geq 0$ for all $i \in \{1, \ldots, n\}$.
2) If $\rho > 0$, then the following interlacing property holds:
$$d_1 \leq \lambda_1 \leq d_2 \leq \lambda_2 \leq \ldots \leq d_n \leq \lambda_n \tag{30}$$
and, if $\rho < 0$, then
$$\lambda_1 \leq d_1 \leq \lambda_2 \leq d_2 \leq \ldots \leq \lambda_n \leq d_n. \tag{31}$$

3) If all eigenvalues of D are different, all the entries of z are non-zero, and $\rho \neq 0$, then inequalities (30) and (31) are strict, and for $i \in \{1, \ldots, n\}$, the eigenvalue λ_i is a zero of

$$W(x) = 1 + \rho \sum_{j=1}^{n} \frac{z_i^2}{d_j - x}. \tag{32}$$

The generalization of Proposition 2 for $n \geq 2$ is introduced in the following:

Theorem 2. Let $X_1, \ldots, X_n$ be independent random vectors with densities defined on $\mathbb{R}^d$, let $N_\alpha(X_1), \ldots, N_\alpha(X_n)$ be their Rényi entropy powers of order $\alpha > 1$, and let $\alpha' = \frac{\alpha}{\alpha-1}$. Let the indices of $X_1, \ldots, X_n$ be ordered such that

$$N_\alpha(X_k) \leq N_\alpha(X_n), \quad k \in \{1, \ldots, n-1\}. \tag{33}$$

Let

$$c_k = \frac{N_\alpha(X_k)}{N_\alpha(X_n)}, \qquad k \in \{1, \ldots, n-1\} \tag{34}$$

and let $t_n \in [0,1]$ be the unique solution of the equation

$$t_n + \sum_{k=1}^{n-1} \psi_k(t_n) = 1 \tag{35}$$

with

$$\psi_k(x) = \frac{\alpha' - \sqrt{\alpha'^2 - 4 c_k \, x(\alpha' - x)}}{2}, \quad x \in [0,1]. \tag{36}$$

Define

$$t_k = \psi_k(t_n), \quad k \in \{1, \ldots, n-1\}. \tag{37}$$

Then, the following R-EPI holds:

$$N_\alpha\left(\sum_{k=1}^{n} X_k\right) \geq e^{f_0(t_1, \ldots, t_n)} \sum_{k=1}^{n} N_\alpha(X_k) \tag{38}$$

with f_0 given in (11). Furthermore, the R-EPI in (38) satisfies the following properties:

1) It improves the R-EPI in Theorem 1 unless $N_\alpha(X_k)$ is independent of k; in the latter case, the two R-EPIs coincide.
2) It improves the BV bound in [1] which states that

$$N_\alpha\left(\sum_{k=1}^{n} X_k\right) \geq \max_{1 \leq k \leq n} N_\alpha(X_k) \tag{39}$$

and the bounds in (38) and (39) asymptotically coincide as $\alpha \to \infty$ if and only if

$$\sum_{k=1}^{n-1} N_\infty(X_k) \leq N_\infty(X_n) \tag{40}$$

where $N_\infty(X)$ is defined in (19).
3) For $n = 2$, it coincides with the closed-form expression of the R-EPI in Proposition 2.
4) It coincides with the EPI and the two R-EPIs in [4, Theorem I.1] and Theorem 1 as $\alpha \downarrow 1$.

Theorem 2 provides the tightest version of R-EPIs in our work, and it forms a major part of [11]. The lower bound in Theorem 2 is not given in closed form (in contrast to its specialization in Proposition 2 for two independent random vectors); however, an efficient algorithm for the calculation of the generalized R-REPI in Theorem 2 is provided in [11, Section IV.B], and it is exemplified numerically in Section 4.

4. Example: The Rényi Entropy Difference Between Data and its Filtering

Let $\{\underline{X}(n)\}$ be i.i.d. d-dimensional random vectors (note that the entries of a vector $\underline{X}(n)$ need not be independent), with arbitrary densities on $\mathbb{R}^d$. Let

$$\underline{Y}(n) = \sum_{k=0}^{L-1} \mathbf{H}_k \, \underline{X}(n-k) \tag{41}$$

be the filtered data at the output of a finite impulse response (FIR) filter where $\mathbf{H}_0, \ldots, \mathbf{H}_{L-1}$ are fixed non-singular $d \times d$ matrices.

In the following, the tightness of R-EPIs is exemplified by obtaining universal lower bounds on the difference $h_\alpha\big(\underline{Y}(n)\big) - h_\alpha\big(\underline{X}(n)\big)$, being also compared with the actual value of this difference when the i.i.d. inputs are d-dimensional Gaussian random vectors with i.i.d. entries.

For $k \in \{0, \ldots, L-1\}$ and every n, we have

$$h_\alpha\big(\mathbf{H}_k \, \underline{X}(n-k)\big) = h_\alpha\big(\underline{X}(n)\big) + \log\big|\det(\mathbf{H}_k)\big| \tag{42}$$

and

$$N_\alpha\big(\mathbf{H}_k \, \underline{X}(n-k)\big) = \big|\det(\mathbf{H}_k)\big|^{\frac{2}{d}} \, N_\alpha\big(\underline{X}(n)\big). \tag{43}$$

In view of (42), (43), and the R-EPI of Theorem 2, it follows that for every n, $\alpha > 1$

$$\begin{aligned} & h_\alpha\big(\underline{Y}(n)\big) - h_\alpha\big(\underline{X}(n)\big) \\ & \geq \frac{d}{2}\left(\frac{\log \alpha}{\alpha - 1} + \sum_{k=0}^{L-1} g(t_k)\right) + \sum_{k=0}^{L-1} t_k \, \log\big|\det(\mathbf{H}_k)\big| \end{aligned} \tag{44}$$

where the function g is given by

$$g(x) = (\alpha' - x)\log\left(1 - \frac{x}{\alpha'}\right) - x \log x, \quad x \in [0,1] \tag{45}$$

and the sequence $\{t_k\}_{k=0}^{L-1}$ is calculated by (35)–(37).

In view of the analysis in [11], it is easy to verify that the R-EPI in Theorem 1 is equivalent to the following looser bound, which is expressed in closed form:

$$\begin{aligned} & h_\alpha\big(\underline{Y}(n)\big) - h_\alpha\big(\underline{X}(n)\big) \\ & \geq \frac{d}{2} \cdot \log\left(\sum_{k=0}^{L-1} \big|\det(\mathbf{H}_k)\big|^{\frac{2}{d}}\right) \\ & \quad + \frac{d}{2}\left(\frac{\log \alpha}{\alpha - 1} + \left(\frac{L\alpha}{\alpha - 1} - 1\right) \log\left(1 - \frac{\alpha - 1}{L\alpha}\right)\right). \end{aligned} \tag{46}$$

The R-EPI of [4, Theorem I.1] leads to the following loosened bound in comparison to (46):

$$\begin{aligned} & h_\alpha\big(\underline{Y}(n)\big) - h_\alpha\big(\underline{X}(n)\big) \\ & \geq \frac{d}{2}\left[\log\left(\sum_{k=0}^{L-1}\big|\det(\mathbf{H}_k)\big|^{\frac{2}{d}}\right) + \frac{\log\alpha}{\alpha-1} - \log e\right]. \end{aligned} \tag{47}$$

and, finally, the BV bound in [1] leads to the following loosening in comparison to (44):

$$h_\alpha\big(\underline{Y}(n)\big) - h_\alpha\big(\underline{X}(n)\big) \geq \log\left(\max_{0\leq k\leq L-1}\big|\det(\mathbf{H}_k)\big|\right). \tag{48}$$

The differential Rényi entropy of order $\alpha \neq 1$ for a d-dimensional multivariate Gaussian distribution is given by

$$h_\alpha\big(\underline{X}(n)\big) = \frac{d\log\alpha}{2(\alpha-1)} + \tfrac{1}{2}\,\log\Big((2\pi)^d\,\det\big(\mathrm{Cov}(\underline{X}(n))\big)\Big) \tag{49}$$

Hence, for the specific case where the entries of the Gaussian random vector $\underline{X}(n)$ are i.i.d.

$$h_\alpha\big(\underline{Y}(n)\big) - h_\alpha\big(\underline{X}(n)\big) = \tfrac{1}{2}\,\log\left(\det\left(\sum_{k=0}^{L-1}\mathbf{H}_k\,\mathbf{H}_k^T\right)\right). \tag{50}$$

Example 1. Let

$$Y(n) = 2X(n) - X(n-1) - X(n-2) \tag{51}$$

for every n where $\{X(n)\}$ are i.i.d. random variables, and consider the difference $h_2(Y) - h_2(X)$ in the quadratic differential Rényi entropy. In this example $\alpha = 2$, $d = 1$, $L = 3$, and $H_0 = 2$, $H_1 = -1$, $H_2 = -1$. The lower bounds in (44), (46), (47), (48) are equal to 0.8195, 0.7866, 0.6931 and 0.7425 nats, respectively (recall that the first and second lower bounds correspond to Theorems 2 and 1, respectively, and the third and fourth lower bounds correspond to [4] and [1] respectively). These lower bounds are compared to the achievable value in (50), for an i.i.d. Gaussian input, which is equal to 0.8959 nats.

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