On Universal LDPC Code Ensembles

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Universal LDPC Code Ensembles

Density Evolution enables to design capacity-approaching degree distributions of LDPC code ensembles for a particular channel.

Question

How to design LDPC code ensembles that will **provably** *operate reliably over a range of channels?*

Selected Previous Work on Universal LDPC Codes

- M. Franceschini, G. Ferrari, and R. Raheli, *"Does the performance of LDPC codes depend on the channel?,"* IEEE Trans. on COM., vol. 54, no. 12, pp. 2129-2132, December 2006.
- F. Peng, W. E. Ryan and R. D. Wesel, *"Surrogate-channel design of universal LDPC codes,"* IEEE COM. Letters, vol. 10, no. 6, pp. 480–482, June 2006.
- A. Sanaei, M. Ramezani and M. Ardakani, *"Identical-capacity channel decomposition for design of universal LDPC codes,"* IEEE Trans. on COM., vol. 57, no. 7, pp. 1972–1981, July 2009.
- A. Yedla, H. D. Pfister and K. R. Narayanan, *"Can iterative decoding for erasure decoded sources be universal?,"* Proc. of the 47th Allerton Conference, September 2009.

Universal LDPC Code Ensembles

In this work

- *Communication over various families of memoryless, binary-input, output-symmetric (MBIOS) channels is considered.*
- *Analytic conditions for universality of LDPC code ensembles under Belief Propagation (BP) decoding over these families.*
- *Universal Conditions on the Bhattacharyya parameter (B-parameter) for Good/Bad Communications under BP decoding are derived.*
- *Analytic bounds on thresholds under BP decoding that are simple to calculate and can be tight for some ensembles.*
- *Universality under Maximum Likelihood (ML) decoding.*

Useful Functionals

Let a designate the L-density of an MBIOS channel. The symmetry property for MBIOS channels yields

- *The capacity functional*:

$$C \triangleq C(a) = \int_{-\infty}^{\infty} a(x)\big(1 - \log_2(1 + e^{-x})\big)\, \mathrm{d}x.$$

- *The Bhattacharyya functional*:

$$\mathcal{B} \triangleq \mathcal{B}(a) = \int_{-\infty}^{\infty} a(x) e^{-\frac{x}{2}}\, \mathrm{d}x.$$

- *The error probability functional*:

$$\mathcal{E}(a) = \frac{1}{2} \int_{-\infty}^{\infty} a(x)\, e^{-(|\frac{x}{2}| + \frac{x}{2})}\, \mathrm{d}x.$$

A Condition for Convergence

- Consider an arbitrary MBIOS channel with L-density a_0, and LDPC code ensemble (λ, ρ).
- From density evolution

$$a_l = a_0 \circledast \lambda\Big(\Gamma^{-1}\Big(\rho\big(\Gamma(a_{l-1})\big)\Big)\Big), \quad l = 1, 2, \dots$$

- Let us denote $x_l = \mathcal{B}(a_l)$. It can be shown that:

$$x_l \leq \mathcal{B}(a_0)\, \lambda\big(1 - \rho(1 - x_{l-1})\big), \quad l = 1, 2, \dots$$

- Since $2\mathcal{E}(a) \leq \mathcal{B}(a) \leq 2\sqrt{\mathcal{E}(a)\big(1 - \mathcal{E}(a)\big)}$, the bit error probability vanishes as the number of iterations grow iff $x_l \to 0$.

General Approach for Universal Achievability

- Consider an arbitrary *set* of MBIOS channels and let $\mathcal{A}$ designate the corresponding set of its L-densities.
- We wish to design an LDPC code ensemble with degree distributions (λ, ρ) that asymptotically achieve vanishing bit-error probability over the entire set.
- Let

$$B \triangleq \max_{a \in \mathcal{A}} \mathcal{B}(a).$$

- Consider the recursive equation

$$y_l = B\,\lambda\big(1 - \rho(1 - y_{l-1})\big), \quad l = 1, 2, \ldots$$

  with initial condition $y_0 = B$.
  - This refers to the density evolution of a BEC with erasure probability B.

General Approach for Universal Achievability (cont.)

- By induction, for every $l \geq 0$ and $a \in \mathcal{A}$,

$$0 \leq x_l \leq y_l.$$

- Therefore, selecting a pair of degree distributions (λ, ρ) so that $\lim_{l \to \infty} y_l = 0$, yields $\lim_{l \to \infty} x_l = 0$ for *every* MBIOS channel from the set $\mathcal{A}$.
  $\Rightarrow$ the LDPC code ensemble (λ, ρ) is universal over the set of channels.

Construct a capacity-achieving LDPC code ensemble for a BEC with erasure probability B. It achieves bit error probability $\rightarrow 0$ under BP decoding for all MBIOS channels in the set $\mathcal{A}$.

Theorem (Universality of LDPC Codes under BP Decoding for Equi-Capacity MBIOS Channels)

- *Let $\mathcal{A}$ be a set of MBIOS channels that exhibit a given capacity C, and let B denote the maximal Bhattacharyya parameter over this set.*
- *Let $\{(n, \lambda, \rho)\}$ form a capacity-achieving sequence of LDPC code ensembles for BEC(B), achieving vanishing bit erasure probability under BP decoding.*
- *Then, this sequence universally achieves vanishing bit error probability under BP decoding for the entire set $\mathcal{A}$, and the design rate of this sequence forms a fraction that is at least $\frac{1-B}{C}$ of the channel capacity.*

Application of the Theorem:

- *The family of equi-capacity MBIOS channels with capacity C:*
  - The BSC exhibits the maximal B-parameter.
  - The asymptotic achievable fraction of channel capacity is:

$$\mu_1(C) \triangleq \frac{R_{\mathsf{d}}}{C} = \frac{1 - \sqrt{4h_2^{-1}(1-C)\left(1 - h_2^{-1}(1-C)\right)}}{C}.$$

- *BEC and BIAWGNC with capacity C:*
  - The BIAWGNC channel exhibits the maximal B-parameter, B (computed numerically).
  - The asymptotic achievable fraction of channel capacity is:

$$\mu_2(C) = \frac{1-B}{C}.$$

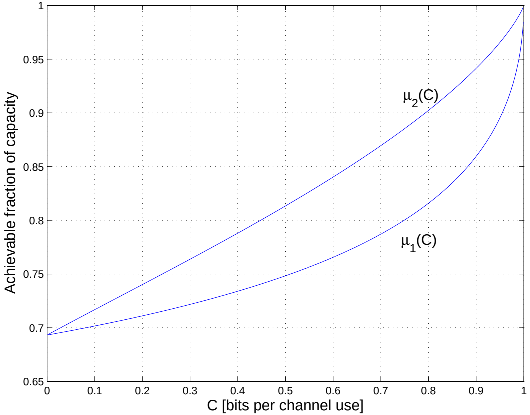

Figure: Universal achievable fraction of capacity under BP decoding for two sets of MBIOS channels which exhibit a given capacity. The values of μ_1 and μ_2 correspond, respectively, to the entire set of equi-capacity MBIOS channels, and the subset of a BEC and BIAWGNC with capacity C bits per channel use.

Theorem (Universal Conditions on the Bhattacharyya parameter for Good/ Bad Communications under BP Decoding)

Let $\{(n, \lambda, \rho)\}$ be a sequence of LDPC code ensembles whose block lengths tend to infinity. The following universal properties hold under BP decoding:

- *This sequence achieves vanishing bit error probability under BP decoding for every MBIOS channel whose Bhattacharyya parameter (B-parameter) is less than*

$$B_0(\lambda, \rho) \triangleq \inf_{x \in (0,1]} \frac{x}{\lambda\big(1-\rho(1-x)\big)}.$$

- *For a right-regular sequence, it does not achieve reliable communications over any MBIOS channel whose B-parameter is greater than*

$$B_1(\lambda, \rho) \triangleq \inf_{x \in (0,1]} \frac{x}{\lambda\Big(\sqrt{1-\rho(1-x^2)}\Big)}.$$

Corollary

The left-to-right message error probability stays bounded away from zero under BP decoding for every MBIOS channel whose B-parameter is greater than

$$B_2(\lambda, \rho) \triangleq \min \left\{ B_1(\lambda, \rho), \frac{1}{\lambda'(0)\rho'(1)}, \sqrt{1 - R_d^2} \right\}$$

where R_d is the design rate of the ensemble.

Note: If $B > B_1(\lambda, \rho)$ then we can specify an explicit positive lower bound on the left-to-right message error probability under BP decoding.

Example

$\lambda(x) = \sum_i \lambda_i x^{i-1}$	$\rho(x) = \sum_i \rho_i x^{i-1}$	B_0	B_2	B_1
$\lambda_3 = 1$	$\rho_6 = 1$	0.4294	0.6553	0.6553
$\lambda_2 = 0.4127,\ \lambda_3 = 0.1762,\ \lambda_4 = 0.1177$ $\lambda_7 = 0.1202,\ \lambda_8 = 0.1731$	$\rho_6 = 1$	**0.4816**	**0.4846**	0.7066
$\lambda_4 = 1$	$\rho_8 = 1$	0.3834	0.6192	0.6192
$\lambda_2 = 0.2879,\ \lambda_3 = 0.1222,\ \lambda_4 = 0.0905$ $\lambda_6 = 0.1174,\ \lambda_7 = 0.0300,\ \lambda_{12} = 0.0807$ $\lambda_{13} = 0.0831,\ \lambda_{32} = 0.0050,\ \lambda_{33} = 0.1831$	$\rho_8 = 1$	**0.4962**	**0.4962**	0.7146
$\lambda_5 = 1$	$\rho_{10} = 1$	0.3416	0.5884	0.5884
$\lambda_2 = 0.2226,\ \lambda_3 = 0.1013,\ \lambda_4 = 0.0504,\ \lambda_5 = 0.0646$ $\lambda_6 = 0.0445,\ \lambda_{10} = 0.1219,\ \lambda_{11} = 0.0117$ $\lambda_{24} = 0.0903,\ \lambda_{25} = 0.0678,\ \lambda_{100} = 0.2248$	$\rho_{10} = 1$	**0.4988**	**0.4992**	0.7123

Table: Numerical results for some right-regular LDPC ensembles with design rate $1/2$. BP Decoding converges if $B < B_0$. If $B > B_2$ then the left-to-right message error probability stays bounded away from 0 under BP decoding.

Universality under ML decoding

- The results for universality under BP decoding automatically extend to the ML decoding case.
- However, under ML decoding stronger results are possible:
  - Capacity can be approached *arbitrarily closely* for the entire set of equi-capacity channels.
  - ML decoding can achieve vanishing *block* error probability.

Universality under ML decoding (Cont.)

Theorem

Under ML decoding, Gallager's regular LDPC code ensembles can be made universal for the set $\mathcal{A}$ of MBIOS channels that exhibit a given capacity C. More explicitly,

- *For any $\varepsilon > 0$, there exists a sequence of these code ensembles whose design rate forms at least a fraction $1 - \varepsilon$ of the channel capacity with vanishing block error probability for the entire set $\mathcal{A}$.*
- *The right degree of this sequence scales like $\log \frac{1}{\varepsilon}$.*

Proof Outline

- Upper bounds on the ML decoding error probability, which depend on the weight distribution of the code (ensemble), are applied to expurgated ensembles of Gallager's LDPC codes.[1]
- Tight bounds on the average weight distribution of these ensembles serve to determine the ensemble parameters that will achieve a desired fraction of the channel capacity of an arbitrary MBIOS channel.[2]
- The analysis solely depends on the channel capacity, making the result universal for the entire set of equi-capacity channels.

[1] G. Miller and D. Burshtein, *"Bounds on the maximum-likelihood decoding error probability of LDPC codes,"* IEEE Trans. on I-T, vol. 47, no. 7, Nov. 2001.

[2] I. Sason and R. Urbanke, *"Parity-check density versus performance of binary linear block codes over memoryless symmetric channels,"* IEEE Trans. on I-T, vol. 49, no. 7, July 2003.

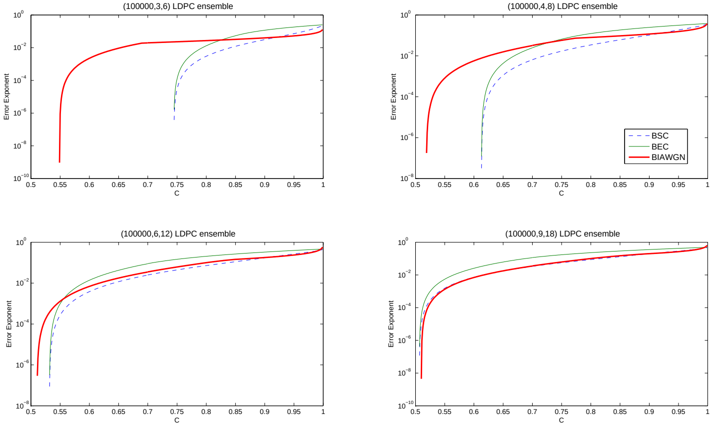

Figure: Lower bounds on the error exponent for expurgated Gallager's LDPC code ensembles on various MBIOS channels. The results were computed for block length $n = 100000$, and for codes with design rate $1/2$ and increasing variable and check node degrees.

Full Paper Version

I. Sason and B. Shuval, “On Universal LDPC Code Ensembles,” *submitted to the IEEE Trans. on Information Theory*, April 2010.