ON UNIVERSAL LDPC CODE ENSEMBLES

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***Abstract*—A universal design of low-density parity-check (LDPC) code ensembles which enables to operate reliably over various channels is of great theoretical and practical interest. This paper considers the universality of LDPC code ensembles over multitude memoryless binary-input output-symmetric (MBIOS) channels, addressing their universality under belief-propagation (BP) decoding. Based on the density evolution approach, analytical results related to the universality of LDPC code ensembles under BP decoding are derived; these results are expressed in closed form, and are easy to calculate. The full paper version related to this work [11] provides further results, full proofs, additional discussions on the theorems, and it also considers the universality issue under ML decoding.**

***Index Terms*—Belief propagation (BP), Bhattacharyya parameter (B-parameter), density evolution (DE), low-density parity-check (LDPC) codes, memoryless binary-input output-symmetric (MBIOS) channels.**

I. INTRODUCTION

Low-density parity-check (LDPC) codes are linear block codes that can be represented by a sparse parity-check matrix. This sparse structure enables to decode these codes by sub-optimal iterative decoding algorithms. These low-complexity algorithms are remarkable in that they achieve rates close to capacity for properly designed code ensembles.

The density evolution approach serves as a main tool for the asymptotic analysis of the performance of LDPC code ensembles under iterative message-passing decoding [7]. Using this approach, it is possible to numerically optimize LDPC codes for specific MBIOS channels. The goal is to find degree distributions that asymptotically ensure convergence to error-free communications for a given channel model, and that are optimal in the sense of either achieving maximal rate for specific channel parameters, or exhibiting maximum threshold for a specific chosen rate or other constraints on the degree distributions.

Density evolution is a powerful tool for numerical optimization of degree distributions (see [7]), but it does not lend itself, in general, for *analytical* design of degree distributions. An exception to this is the case of the binary erasure channel (BEC), where density evolution is greatly simplified to a single-dimensional equation. Based on this, several explicit expressions of capacity-approaching sequences for BEC have been derived (see, e.g. [5]). To this date, no explicit expressions for capacity-approaching code ensembles under iterative decoding for other MBIOS channel models have been found.

It is of great interest, both practically and theoretically, to design a code that will operate reliably over a range of channels. Such robust codes are termed *universal*. This subject has been addressed in several recent studies (see, e.g., [2], [3], [6], [9]). Further universal properties of capacity-approaching LDPC code ensembles have been studies in [10].

In this paper, we consider the universality of LDPC code ensembles under belief propagation (sum-product) decoding over MBIOS channels. We use density evolution to derive some conditions for universality of LDPC codes over various MBIOS channels.

This paper is structured as follows: Section II presents briefly some preliminaries and notation, and section III introduces some of the main results of this work, provides an outline of the proofs, and shows some numerical results. The interested reader is referred to the full paper version [11].

II. PRELIMINARIES

This section follows the notation in [8, Chapter 4], and introduces briefly some preliminaries on memoryless binary-input output-symmetric (MBIOS) channels.

Consider an MBIOS channel whose input and output are designated by X and Y, respectively, and let $p_{Y|X}(\cdot|\cdot)$ be its transition probability. The associated log-likelihood ratio (LLR) $l(y)$ when the channel output is $Y = y$ is given by $l(y) = \ln\left(\frac{p_{Y|X}(y|0)}{p_{Y|X}(y|1)}\right)$. The LLR associated with the random variable Y is defined as $L = l(Y)$. Let a designate the conditional *pdf* of the random variable L given that the channel input is $X = 0$ (to be referred to as the L-density function). This density function satisfies the symmetry property $a(x) = e^x\, a(-x)$ for every $x \in \mathbb{R}$ (see [8, Theorem 4.26]).

This paper relies on the following three functionals (various other functionals are presented in [8, Section 4.1]).

Proposition 1. [Capacity functional] Consider an MBIOS channel whose symmetric L-density function is denoted by a. The capacity of this channel in units of bits per channel use, $C \triangleq C(a)$, is given by

$$C = \int_{-\infty}^{\infty} a(x)\big(1 - \log_2(1 + e^{-x})\big)\, \mathrm{d}x. \quad (1)$$

Proof: See [8, p. 193]. ■

Definition 1. [The Bhattacharyya functional] The Bhattacharyya parameter (B-parameter), $B \triangleq \mathcal{B}(a)$, which is associated with the symmetric L-density function a is given by

$$B = \int_{-\infty}^{\infty} a(x) e^{-\frac{x}{2}} \, \mathrm{d}x. \tag{2}$$

A direct consequence of [1, Proposition 1] states that for every MBIOS channel

$$\log_2\left(\frac{2}{1+B}\right) \leq C \leq \sqrt{1-B^2}. \tag{3}$$

This conforms with the fact that the B-parameter of a perfect MBIOS channel tends to zero. On the other hand, the B-parameter of a very noisy channel tends to 1.

Definition 2. [The error probability functional] The bit error probability that is associated with a symmetric L-density function a is given by

$$\mathcal{E}(a) = \frac{1}{2}\int_{-\infty}^{+\infty} a(x)\, e^{-(|\frac{x}{2}|+\frac{x}{2})} \, \mathrm{d}x.$$

The following inequality ([8, Lemma 4.64]) holds for an arbitrary symmetric L-density a:

$$2\mathcal{E}(a) \leq \mathcal{B}(a) \leq 2\sqrt{\mathcal{E}(a)\big(1-\mathcal{E}(a)\big)}. \tag{4}$$

Note that the upper and lower bounds on the B-parameter, as given in (4), are satisfied with equality for a BSC and BEC, respectively.

III. Universality under Belief Propagation Decoding

A. Universal Achievability Results

In the following, we consider the suitability of LDPC code ensembles to operate reliably over a set of MBIOS channels under BP decoding. We rely here on the density evolution approach, and our goal is to construct LDPC code ensembles which achieve vanishing bit error probability, in the asymptotic case where the block length tends to infinity, uniformly over a set of MBIOS channels.

To this end, let us consider first an arbitrary MBIOS channel, and let a_0 denote the *pdf* of the LLR at the channel output given that the channel input is zero. Let λ and ρ designate the degree distributions of the variable and parity-checks, respectively, from the edge perspective. Based on density evolution, the densities at every iteration of the BP decoder satisfy the recursive equation

$$a_l = a_0 \circledast \lambda\bigg(\Gamma^{-1}\Big(\rho\big(\Gamma(a_{l-1})\big)\Big)\bigg), \quad l = 1, 2, \ldots \tag{5}$$

where the mapping Γ and its inverse Γ^{-1} are introduced in [7], and $\circledast$ denotes the convolution of densities in the L-domain (see [8]). The densities a_l are symmetric functions for every $l \geq 0$, i.e., $a_l(x) = e^x a_l(-x)$ for all $x \in \mathbb{R}$. Let $x_l = \mathcal{B}(a_l)$ for $l \geq 0$ where $\mathcal{B}(a)$ designates the B-parameter that is associated with the L-density a. Based on [4, Theorem 4.2], it follows that

$$x_l \leq \mathcal{B}(a_0)\, \lambda\big(1-\rho(1-x_{l-1})\big), \quad l = 1, 2, \ldots \tag{6}$$

(see also [8, p. 234]). From (4), an LDPC code ensemble obtains asymptotically vanishing bit error probability as the number of iterations grows if and only if $\lim_{l\to\infty} x_l = 0$.

Let us now consider an arbitrary set of MBIOS channels, and let $\mathcal{A}$ designate the corresponding set of its L-densities. Suppose that one wishes to design an LDPC code ensemble with degree distributions (λ, ρ) in order to asymptotically achieve vanishing bit error probability under BP decoding for every channel in this set. Let us designate by B the maximal B-parameter over the MBIOS channels of the considered set, i.e.,

$$B \triangleq \max_{a \in \mathcal{A}} \mathcal{B}(a). \tag{7}$$

Let us consider the recursive equation

$$y_l = B\, \lambda\big(1-\rho(1-y_{l-1})\big), \quad l = 1, 2, \ldots \tag{8}$$

with the initial value $y_0 = B$. This recursive equation refers to the density evolution of a BEC whose erasure probability is equal to B. By comparing (6) and (8), it is straightforward to show (e.g., by induction) that $0 \leq x_l \leq y_l$ for every $l \geq 0$ and $a \in \mathcal{A}$. If the pair of degree distributions (λ, ρ) is selected in a way where $\lim_{l\to\infty} y_l = 0$, then we get that $\lim_{l\to\infty} x_l = 0$ in (6) for *every MBIOS channel* from the set $\mathcal{A}$. Hence, the universality of the LDPC code ensemble whose degree distribution is (λ, ρ) follows with respect to the considered set of channels.

Since explicit constructions of capacity-achieving sequences of LDPC code ensembles are well-known for the BEC (see, e.g., [5] and references therein), one can rely on (8) to construct a sequence of LDPC code ensembles which achieves vanishing bit error probability, under BP decoding, for all the MBIOS channels of the considered set. By this approach, the asymptotic design rate of this capacity-achieving sequence of LDPC code ensembles is equal to

$$R_{\mathrm{d}} = 1 - B \tag{9}$$

where B is given in (7). We study the following particular cases of this approach.

1) Universal LDPC Code Ensembles for Equi-Capacity MBIOS Channels: Among all MBIOS channels which exhibit a given capacity C, the B-parameter that is associated with the L-densities of this set of channels attains its maximal and minimal values for the BSC and BEC, respectively (this follows readily from (4)). By referring to the set of all equi-capacity MBIOS channels, one therefore gets from (7) that $B = \sqrt{4h_2^{-1}(1-C)\left(1-h_2^{-1}(1-C)\right)}$, where h_2^{-1} designates the inverse of the binary entropy function on base 2. From (9), the asymptotic design rate of the corresponding sequence of LDPC code ensembles is equal to $R_{\mathrm{d}} = 1-B$. The achievable fraction of the channel capacity by this approach is therefore equal to

$$\mu_1(C) \triangleq \frac{R_{\mathrm{d}}}{C} = \frac{1-\sqrt{4h_2^{-1}(1-C)\left(1-h_2^{-1}(1-C)\right)}}{C}. \tag{10}$$

We note that the function μ_1 is monotonic increasing over the interval $(0, 1]$, and

$$\lim_{C\to 0} \mu_1(C) = \ln 2 \approx 69.3\%, \qquad \lim_{C\to 1} \mu_1(C) = 1.$$

This analysis implies that at least 69.3% of any MBIOS channel can be achieved by designing a capacity-achieving sequence of LDPC code ensembles for a BEC; the erasure probability of this BEC is set to be equal to the B-parameter of a BSC whose capacity matches our channel. Moreover, as the value of the capacity is increased, a larger fraction of the channel capacity is achievable uniformly for the entire considered set of equi-capacity MBIOS channels (see Fig. 1).

This presents an analytical approach for the design of universal LDPC code ensembles for equi-capacity MBIOS channels where a provable (non-vanishing) fraction of capacity is universally achieved, and the value of this fraction gets larger as the value of capacity is increased. We note however that numerical optimization via density evolution enables to design universal LDPC code ensembles in [9] achieving a significantly larger fraction of the channel capacity, though the considered approach here is purely analytical, and it is not subject to numerical optimizations.

2) Universal LDPC Code Ensembles for BEC and BI-AWGNC: We consider here an achievability result where one wishes to design an LDPC code ensemble which achieves asymptotically vanishing bit error probability under BP decoding for both the BEC and the binary-input AWGN channel (BIAWGNC) with the same capacity. Since among all equi-capacity MBIOS channels, the BEC possesses the minimal B-parameter (this follows from (4)), then the parameter B in (7) corresponds to the B-parameter of the BIAWGNC. The conversion from the channel capacity to the B-parameter for this channel is done numerically based on (1) and (2) applied for the AWGN channel. From (9), the asymptotic achievable fraction of the capacity is equal to

$$\mu_2(C) = \frac{1-B}{C}. \tag{11}$$

Since the universality in this example applies to a subset of the equi-capacity MBIOS channels, the inequality $\mu_2(C) \geq \mu_1(C)$ is expected to hold for $0 \leq C \leq 1$ (as is exemplified in Fig. 1).

The following theorem summarizes the discussion above; to this end, we denote by $\text{BEC}(\varepsilon)$ the binary erasure channel whose erasure probability is ε:

Theorem 1. [Universality of LDPC Codes under BP Decoding for Equi-Capacity MBIOS Channels] Consider a set $\mathcal{A}$ of MBIOS channels that exhibit a given capacity C, and let B denote the maximal Bhattacharyya parameter over this set (see (7)). Let $\{(n, \lambda, \rho)\}$ form a capacity-achieving sequence of LDPC code ensembles for $\text{BEC}(B)$, achieving vanishing bit erasure probability under BP decoding. Then, this sequence universally achieves vanishing bit error probability under BP decoding for the entire set $\mathcal{A}$, and the design rate of this sequence forms a fraction that is at least $\frac{1-B}{C}$ of the channel capacity. As a consequence, the following results hold:

- For the entire set of equi-capacity MBIOS channels, the universal achievable design rate forms at least a fraction $\mu_1(C)$ of capacity (see (10)). Moreover, μ_1 forms a monotonic increasing function of the capacity C (see Fig. 1), getting the extreme values $\ln 2 \approx 69.3\%$ and 100% at the endpoints where $C \to 0$ or $C \to 1$, respectively.
- For some sub-classes of equi-capacity MBIOS channels, the results for the universal achievable design rate significantly improve (see, e.g., (11) and μ_2 in Fig. 1).

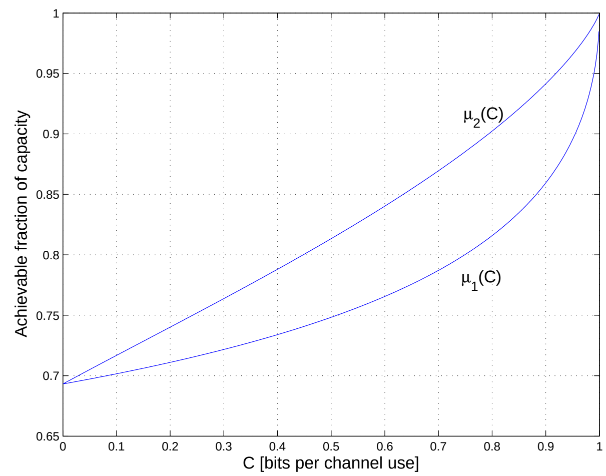

Fig. 1: Universal achievable fraction of capacity under BP decoding for two sets of MBIOS channels which exhibit a given capacity (see Theorem 1). The values of μ_1 in (10) and μ_2 in (11) correspond, respectively, to the entire set of equi-capacity MBIOS channels, and the subset of a BEC and BIAWGNC with capacity C bits per channel use.

Fig. 1 compares the achievable fractions of capacity, μ_1 and μ_2, as a function of the channel capacity.

B. A Necessary Condition for Universality of LDPC Code Ensembles under BP Decoding

We consider in the following *right-regular* LDPC code ensembles where d_c designates the degree of the parity-check nodes (i.e., $\rho(x) = x^{d_c - 1}$).

Theorem 2. [A Necessary Condition for Universality of LDPC Code Ensembles under BP Decoding] Let $\{(n, \lambda, \rho)\}$ be a right-regular sequence of LDPC code ensembles, universally achieving vanishing bit error probability under BP decoding for a set of MBIOS channels $\mathcal{A}$. Then, the following condition holds

$$B\lambda\left(\sqrt{1-\rho(1-x^2)}\right) < x, \quad \forall\, x \in (0, B] \tag{12}$$

where B designates the maximal B-parameter over the set $\mathcal{A}$ (see (7)).

Proof: Due to space limitations, we only outline the proof here. The proof relies on applying an extension of [8, Problem 4.62] to the density evolution equation (5) to obtain the inequality:

$$x_l \triangleq \mathcal{B}(a_l) \geq \mathcal{B}(a_0)\,\lambda\left(\sqrt{1-\rho\left(1-x_{l-1}^2\right)}\right). \tag{13}$$

Thus, if the sequence $\{x_l\}$, whose initial value is $x_0 = \mathcal{B}(a_0)$, tends asymptotically to zero, then the sequence

$$z_l = \mathcal{B}(a_0)\,\lambda\left(\sqrt{1-\rho(1-z_{l-1}^2)}\right), \quad l = 1, 2, \ldots \tag{14}$$

with the initial value $z_0 = \mathcal{B}(a_0)$, should also tend to zero. Moreover, (4) implies that the convergence of the sequence $\{x_l\}$ forms a necessary and sufficient condition for achieving vanishing bit error probability as we let the number of iterations grow.

The proof then follows by considering a sequence of right-regular LDPC code ensembles which universally achieves vanishing bit error probability under BP decoding over a set $\mathcal{A}$ of MBIOS channels, where the maximal B-parameter over the entire set is B. ■

C. Universal Conditions for Reliable Communications under Belief Propagation Decoding

We introduce in this sub-section the following theorem and exemplify its use:

Theorem 3. [Universal Conditions on the Bhattacharyya parameter for Good/ Bad Communications under BP Decoding] Let $\{(n, \lambda, \rho)\}$ be a sequence of LDPC code ensembles whose block lengths tend to infinity. The following universal properties hold under BP decoding:

- This sequence achieves vanishing bit error probability under BP decoding for *every* MBIOS channel whose Bhattacharyya parameter (B-parameter) is less than
$$B_0(\lambda, \rho) \triangleq \inf_{x\in(0,1]} \frac{x}{\lambda\big(1-\rho(1-x)\big)}. \tag{15}$$
- For a right-regular sequence, it does not achieve reliable communications over *any* MBIOS channel whose B-parameter is greater than
$$B_1(\lambda, \rho) \triangleq \inf_{x\in(0,1]} \frac{x}{\lambda\big(\sqrt{1-\rho(1-x^2)}\big)}. \tag{16}$$
For every MBIOS channel whose B-parameter B satisfies $B > B_1(\lambda, \rho)$, BP decoding is not reliable in the sense that the left-to-right message error probability (i.e., the average probability of error for a message emanating from a variable node to a parity-check node) is greater than the positive value
$$\left(\frac{1}{2}\max\left\{x\in(0,1]:\ \frac{x}{\lambda\big(\sqrt{1-\rho(1-x^2)}\big)} \le B\right\}\right)^2$$
irrespective of the number of iterations performed by the BP decoder.

Proof: Due to space limitations, we only outline the proof here. The first part of the theorem follows from the analysis in section III-A. The results of that section imply that if an LDPC code ensemble asymptotically achieves vanishing bit error probability under BP decoding for a BEC whose channel erasure probability is B, then the ensemble converges for any other MBIOS channel with B-parameter less than B. Considering an arbitrary LDPC code ensemble (λ, ρ), its threshold under BP decoding for a BEC is $B_0(\lambda, \rho)$ (this follows from [8, Theorem 3.59]). The threshold determines a necessary and sufficient condition for the asymptotic convergence of the LDPC code ensemble under BP decoding over BEC channels: the ensemble will converge if and only if the erasure probability is less than $B_0(\lambda, \rho)$.

The second part of the theorem refers to a sequence of right-regular LDPC code ensembles and relies on the necessary condition of Theorem 2; if the necessary condition is violated then reliable communication is not possible. ■

Corollary 1. For every MBIOS channel with B-parameter $B > B_1(\lambda, \rho)$, let $x(B)$ be defined as

$$x(B) \triangleq \max\left\{x\in(0,1]:\ \frac{x}{\lambda\big(\sqrt{1-\rho(1-x^2)}\big)} \le B\right\}. \tag{17}$$

Then the (average) left-to-right message error probability is bounded away from zero by the universal bound

$$\eta \triangleq \lim_{B\to B_1(\lambda,\rho)^+} \left(\frac{x(B)}{2}\right)^2 \tag{18}$$

irrespective of the number of iterations of the BP decoder.

Proof: By definition, x in (17) is an increasing function of B, and therefore we take the limit $B \to B_1(\lambda, \rho)$, where the limit is from the right side, in order to obtain a lower bound on the left-to-right message error probability for the case where $B > B_1(\lambda, \rho)$. ■

Corollary 2. Let $\{(n, \lambda, \rho)\}$ be a sequence of right-regular LDPC code ensembles whose block lengths tend to infinity. Then, the left-to-right message error probability stays bounded away from zero under BP decoding for every MBIOS channel whose B-parameter is greater than

$$B_2(\lambda, \rho) \triangleq \min\left\{B_1(\lambda, \rho), \frac{1}{\lambda'(0)\rho'(1)}, \sqrt{1-R_{\mathrm{d}}^2}\right\} \tag{19}$$

where B_1 is introduced in (16), and

$$R_{\mathrm{d}} \triangleq 1 - \frac{\int_0^1 \rho(x)\,\mathrm{d}x}{\int_0^1 \lambda(x)\,\mathrm{d}x}$$

designates the design rate.

Proof: (outline) The proof follows by combining the second part of Theorem 3, the stability condition (see [8, Theorem 4.125], and the right-hand side of (3). ■

Remark 1. Note that if the B-parameter is above $B_2(\lambda, \rho)$ (see (19)), then Corollary 2 does not specify an explicit positive lower bound on the left-to-right message error probability under BP decoding. However, if $B > B_1(\lambda, \rho)$ (where it readily follows from (19) that $B_1(\lambda, \rho) \ge B_2(\lambda, \rho)$), then the second part of Theorem 3 determines an explicit positive lower bound on the left-to-right message error probability that is valid universally for all MBIOS channels. As shown in Example 1, the value of this lower bound η (see (18)) is typically large, irrespective of the number of iterations of the BP decoder, and this lower bound holds for all MBIOS channels whose B-parameter is above $B_1(\lambda, \rho)$.

$\lambda(x)=\sum_i \lambda_i x^{i-1}$	$\rho(x)=\sum_i \rho_i x^{i-1}$	B_0	B_2	B_1	η
$\lambda_3=1$	$\rho_6=1$	0.4294	0.6553	0.6553	$6.50\cdot 10^{-2}$
$\lambda_2=0.4127,\ \lambda_3=0.1762,\ \lambda_4=0.1177$ $\lambda_7=0.1202,\ \lambda_8=0.1731$	$\rho_6=1$	**0.4816**	**0.4846**	0.7066	$8.45\cdot 10^{-2}$
$\lambda_4=1$	$\rho_8=1$	0.3834	0.6192	0.6192	$6.58\cdot 10^{-2}$
$\lambda_2=0.2879,\ \lambda_3=0.1222,\ \lambda_4=0.0905$ $\lambda_6=0.1174,\ \lambda_7=0.0300,\ \lambda_{12}=0.0807$ $\lambda_{13}=0.0831,\ \lambda_{32}=0.0050,\ \lambda_{33}=0.1831$	$\rho_8=1$	**0.4962**	**0.4962**	0.7146	$1.02\cdot 10^{-1}$
$\lambda_5=1$	$\rho_{10}=1$	0.3416	0.5884	0.5884	$6.18\cdot 10^{-2}$
$\lambda_2=0.2226,\ \lambda_3=0.1013,\ \lambda_4=0.0504,\ \lambda_5=0.0646$ $\lambda_6=0.0445,\ \lambda_{10}=0.1219,\ \lambda_{11}=0.0117$ $\lambda_{24}=0.0903,\ \lambda_{25}=0.0678,\ \lambda_{100}=0.2248$	$\rho_{10}=1$	**0.4988**	**0.4992**	0.7123	$1.08\cdot 10^{-1}$

TABLE I: Numerical results based on Theorem 3 and its corollaries for some right-regular LDPC ensembles whose design rate is one-half. BP Decoding is guaranteed to converge if the MBIOS channel's B-parameter, B, is less than B_0 in (15). If $B > B_2$ (see (16) and (19)) then the left-to-right message error probability stays bounded away from zero under BP decoding. If, in addition, $B > B_1$, then η in (18) provides a lower bound for the average left-to-right message error probability.

Example 1. Table I shows the numerical values of B_0, B_1, B_2, and η from Theorem 3 and its corollaries for some LDPC code ensembles with design rate one-half. We consider both regular and numerically optimized right-regular LDPC code ensembles. The irregular ensembles are numerically optimized for the BEC under BP decoding based on the approach described in [8, Section 3.18] for a design rate of one-half and for a maximal degree of the variable nodes of one hundred ($d_{\text{v}}^{\max}=100$).

For the regular LDPC code ensembles, our results are not tight compared to [12], where the authors used an information-combining approach to obtain bounds on the threshold of the channel. For example, for the $(3,6)$ regular ensemble over the BIAWGN channel, the bounds developed in this work imply the following bounds on the threshold: $0.7222 < \sigma < 1.0214$, whereas the bounds of [12] are tighter: $0.7707 < \sigma < 0.9553$.

For the numerically optimized codes it is evident from Table I that $B_0 \approx B_2$ (the respective values are bolded in this table). Hence, for these optimized LDPC code ensembles, the first part of Theorem 3 states that these LDPC code ensembles are reliable under BP decoding, in the sense of achieving vanishing bit error probability, for every MBIOS channel whose B-parameter is below B_0; on the other hand, Corollary 2 implies that these code ensembles are not reliable under BP decoding for every MBIOS channel whose B-parameter is slightly above B_0 or greater than this value. This is a universal result that applies to all MBIOS channels, and it separates them into two sets of good or bad channels for which the reliability of these code ensembles under BP decoding solely depends on the B-parameter of the communication channel *without any relevance to its channel model* (as long as it is MBIOS, and it exhibits a given B-parameter).

In the irregular case, the results of this work are tighter than those of [12]. For example, for the irregular ensemble with right-degree 6 over the BIAWGN channel, the bounds developed in this work imply that the threshold lies between $0.8272 < \sigma < 0.8308$, whereas [12] bounds the threshold more loosely: $0.5182 < \sigma < 1.1209$.

Acknowledgment

A discussion with H. Pfister is acknowledged. This work was supported by the Israel Science Foundation (ISF 1070/07).

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