Linear Programming Bounds on the Degree Distributions of LDPC Code Ensembles

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Degree Distributions of LDPC Code Ensembles

Question

How do the degree distributions of LDPC code ensembles behave as a function of their achievable gap (in rate) to capacity ?

Degree Distributions of LDPC Code Ensembles

In this work

- *Communication over a memoryless binary-input output-symmetric (MBIOS) channel is assumed.*
- *Linear programming (LP) upper bounds on the degree distributions of LDPC code ensembles are expressed in terms of their gap (in rate) to capacity and the bit/ block error probability.*
- *Analytical solutions of these bounds are obtained via Lagrange duality, and these bounds are easy to calculate.*
- *These information-theoretic bounds give an indication on the behavior of degree distributions of capacity-approaching LDPC code ensembles.*

Degree Distributions of LDPC Code Ensembles

Consider the case where transmission takes place over a memoryless, binary-input output-symmetric (MBIOS) channel.

- From the symmetry property

$$a(l) = e^{l} a(-l), \quad \forall\, l \in \mathbb{R}$$

  where a is the pdf of the log-likelihood ratio (LLR) at the channel output given that the channel input is zero.

- Consider LDPC code ensembles whose design rate forms a fraction $1-\varepsilon$ of the channel capacity with a target bit error probability P_{b}.

Question

What can be said about the degree distributions of the LDPC code ensembles in this setting ?

An Outline of the Derivation of the LP Bounds

- A lower bound on the conditional entropy for binary linear block codes transmitted over MBIOS channels:

$$\frac{H(\mathbf{X}|\mathbf{Y})}{n} \geq R - C + \frac{1-R}{2\ln 2} \sum_{p=1}^{\infty} \frac{\Gamma(g_p)}{p(2p-1)}$$

where

$$g_p \triangleq \int_0^{\infty} a(l)(1+e^{-l}) \tanh^{2p}\left(\frac{l}{2}\right) dl, \quad p \in \mathbb{N}.$$

and $\Gamma(x) \triangleq \sum_k \Gamma_k x^k$ forms the degree distribution of the parity-check nodes, from the node perspective, of an arbitrary representation of the code by a **full-rank** parity-check matrix (Wiechman & Sason, IEEE Trans. on IT, Feb. 2007).

Question

But what if the parity-check matrix has some linearly dependent parity-check equations ? (e.g., parity-check matrices of some LDPC code ensembles).

Theorem (Sason, '09)

For (regular and irregular) LDPC code ensembles of binary LDPC codes, the above lower bound on the conditional entropy stays valid for every code from the ensemble when

- *The rate R of the code is replaced by the design rate of the ensemble.*
- *The sequence $\{\Gamma_k\}$ denotes the degree distribution of the parity-check nodes of the ensemble (where the representation of a code by a parity-check matrix, with the given degree distribution, possibly includes some linearly dependent rows).*

Some additional things needed for the derivation of the LP bounds in this work:

- Fano inequality.
- The inequality
$$g_p \geq (g_1)^p$$
holds for every $p \in \mathbb{N}$, with equality for the BSC.
- The derivation of the LP bounds finally relies on the equality
$$\frac{1}{2\ln 2}\sum_{k=1}^{\infty}\frac{u^k}{k(2k-1)} = 1 - h_2\left(\frac{1-\sqrt{u}}{2}\right), \; \forall\, u \in [0,1].$$
where h_2 designates the binary entropy function on base 2.

LP1 Bound for the Degree Distribution of the Parity-Check Nodes for LDPC Code Ensembles

$$\text{maximize} \quad \sum_{i=1}^{k} \rho_i, \quad k = 1, 2, \ldots$$

subject to

$$\begin{cases} \displaystyle\sum_{i=1}^{\infty} \left\{ \left[1 - h_2\left(\frac{1 - g_1^{\frac{i}{2}}}{2} \right) \right] \frac{\rho_i}{i} \right\} \leq \frac{\varepsilon C + h_2(P_{\mathrm{b}})}{1 - (1-\varepsilon) C} \sum_{i=1}^{\infty} \frac{\rho_i}{i} \\ \displaystyle\sum_{i=1}^{\infty} \rho_i = 1 \\ \rho_i \geq 0, \quad i = 1, 2, \ldots \end{cases}$$

where the optimization variables are $\{\rho_i\}_{i \geq 1}$. The quantity g_1 above depends on the channel statistics only.

Closed-Form Solution of the LP1 Bound

- Since strong duality holds for a feasible LP problem, then the LP1 problem can be solved via Lagrange duality.
- The dual problem gets the form

$$\boxed{\begin{array}{l} \text{minimize } \lambda_2 \\ \text{subject to} \\ \begin{cases} -1+\lambda_1 d_i+\lambda_2-\theta_i=0, \quad i=1,2,\ldots,k \\ \lambda_1 d_i+\lambda_2-\theta_i=0, \quad i=k+1,k+2,\ldots \\ \lambda_1,\lambda_2 \geq 0 \\ \theta_i \geq 0, \quad i=1,2,\ldots \end{cases} \end{array}}$$

where $d_i \triangleq \frac{1}{i}\left[1-h_2\left(\frac{1-g_1^{\frac{i}{2}}}{2}\right)-\frac{\varepsilon C+h_2(P_b)}{1-(1-\varepsilon)C}\right]$ for $i \geq 1$.

Closed-Form Solution of the LP1 Bound (Cont.)

- The sequence $\{d_i\}$ is non-negative if and only if $i \leq k_0$ where

$$k_0 \triangleq \alpha \ln \left(\frac{1}{1 - 2h_2^{-1}\left(\frac{1-C-h_2(P_\mathrm{b})}{1-(1-\varepsilon)C}\right)} \right), \quad \alpha \triangleq \frac{2}{\ln\left(\frac{1}{g_1}\right)}$$

- For $k \leq k_0$, the sequence $\{d_i\}_{i=1}^k$ is non-negative and monotonic decreasing: $d_1 > d_2 > \ldots, > d_k > 0, \quad \forall k \leq k_0$.
- $d_i < 0$ for $i > k_0$, and $\lim_{i\to\infty} d_i = 0$. Let $d^* \triangleq \min_{i\geq 1} d_i$ where the minimum of the sequence $\{d_i\}$ is attained for some index $i > k_0$, and $d^* < 0$.
- The optimal value of the dual LP is equal to $-\frac{d^*}{d_k - d^*}$ (which is indeed bounded between 0 and 1) for $k \leq k_0$, and 1 for $k > k_0$.

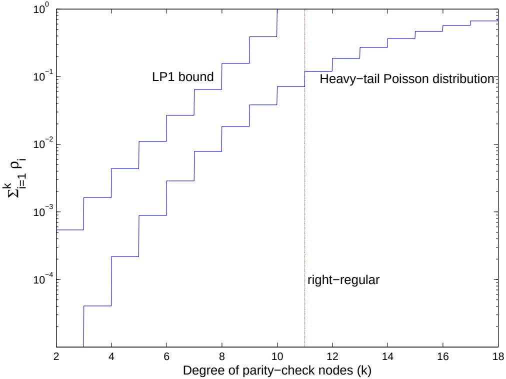

Figure: The LP1 bound for a BEC is compared to the heavy-tail Poisson distribution, and the right-regular LDPC ensemble. This figure refers to the fraction of edges which are attached to parity-check nodes of degree $\leq k$ for an integer $k \geq 2$. We refer to a BEC of capacity $\frac{1}{2}$ bit per channel use, and consider the case where 99.9% of capacity is achieved under iterative message-passing decoding with vanishing bit erasure probability.

LP2 Bound for the Degree Distribution: Universal for All Equi-Capacity MBIOS Channels

Consider the set of all equi-capacity MBIOS channels whose capacity is equal to C. We wish to get here a universal bound on the parity-check node degree which applies to all MBIOS channels from this set.

- Replace the parameter g_1 with the channel capacity C (where, in general, $g_1 \geq C$ with equality for the BEC).
- $\Rightarrow$ Same Solution as that of the LP1 bound, except of the replacement of g_1 with C.

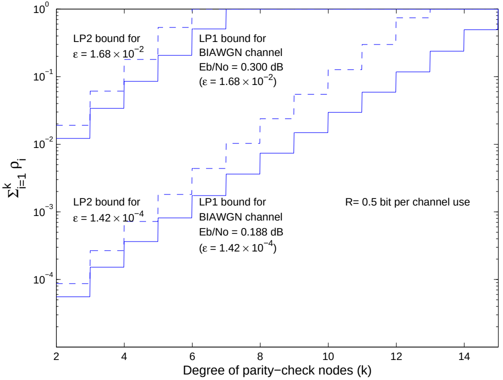

Figure: LP1 versus LP2 bounds for LDPC code ensembles whose design rate is $R = \frac{1}{2}$. The stair functions show upper bounds on the fraction of edges connected to parity-check nodes of degree at most k ($k \geq 1$). All bounds refer to a target bit error probability of $P_{\mathrm{b}} = 10^{-10}$. The universal bound corresponds to all the MBIOS channels whose capacity is fixed and is equal to the capacity of the considered BIAWGN channel.

Closed-Form Solution of the LP Upper Bound on the Degree Distribution of Variable Nodes

$$\sum_{i=1}^{k} \lambda_i \leq \min \left\{ 1, \frac{k \ln\left(\frac{1}{g_1}\right)}{2(1-C)\left(1+\frac{\varepsilon C}{1-C}\right) \ln\left(\frac{1}{1-2h_2^{-1}\left(\frac{1-C-h_2(P_b)}{1-(1-\varepsilon)C}\right)}\right)} \right\}$$

and for the universal bound for equi-capacity MBIOS channels, the parameter g_1 is replaced by the capacity (C).

Example (the Degree Distribution of the Variable Nodes)

- Consider LDPC code ensembles whose design rate is $\frac{1}{2}$ bits per channel use.
- Transmission over a Binary-input AWGN channel.
- A target bit error probability of $P_b = 10^{-10}$ at $\frac{E_b}{N_0} = 0.188$ dB.

From our analytical LP bound, we get

$$
\begin{aligned}
&\lambda_2 \leq 0.2683 \\
&\lambda_2 + \lambda_3 \leq 0.4025 \\
&\lambda_2 + \lambda_3 + \lambda_4 \leq 0.5367 \\
&\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 0.6709 \\
&\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \leq 0.8051 \\
&\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 \leq 0.9392 \\
&\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 \leq 1.0000.
\end{aligned}
$$

Asymptotic Behavior of the Degree Distributions

Corollary

If the asymptotic bit error/ erasure probability vanishes, then the following properties hold for an arbitrary finite degree i

$$L_i = O(1)\,, \qquad R_i = O(\varepsilon)\,,$$
$$\lambda_i = O\left(\frac{1}{\ln \frac{1}{\varepsilon}}\right), \quad \rho_i = O\left(\frac{\varepsilon}{\ln \frac{1}{\varepsilon}}\right).$$

where $\{L_i\}$ *and* $\{R_i\}$ *are the degree distributions of the variable and parity-check nodes, respectively, and* $\{\lambda_i\}$ *and* $\{\rho_i\}$ *are the corresponding degree distributions from the edge perspective.*

- These bounds hold under ML decoding (or any other algorithm).
- The upper bounds on the left degree distribution look at first glance looser than those for the right degree distribution (due to the additional factor ε in the latter case).
- However, it is not an artifact of the bounding technique, as it indeed reflects reality, e.g.:
  - For various capacity-achieving degree distributions on the BEC with iterative message-passing decoding, the fraction of degree-2 variable nodes tends to $\frac{1}{2}$.
  - The upper bound on the fraction of edges connected to degree-2 variable nodes (λ_2) is shown in this work to be obtained for the right-regular LDPC code ensemble of Shokrollahi which achieves capacity on the BEC under iterative decoding.

Full Paper Version

I. Sason, “On Universal Properties of Capacity-Approaching LDPC Code Ensembles,” *IEEE Trans. on Information Theory*, vol. 55, no. 7, pp. 2956–2990, July 2009.