Lower Bounds on the Graphical Complexity of Finite-Length LDPC Codes

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2009 IEEE International Symposium on Information Theory
Seoul, Korea

June 28– July 3, 2009.

Graphical Complexity versus Performance

Question

Consider the representation of a finite-length binary linear block code by an arbitrary bipartite graph. How simple can such a graphical representation be as a function of the channel model, target block error probability, and code rate ?

Answer $\Rightarrow$

- An information-theoretic measure of the inherent graphical complexity of finite-length LDPC codes as a function of their gap to capacity.
- Provides a measure of the sub-optimality of explicit constructions of LDPC codes by comparing to some lower bounds.

Tools Used for This Work

- Sphere-packing bounds and some recent improvements for finite-length codes $\Rightarrow$ A lower bound on the required block length given the channel model, target block error probability and code rate.
- An improved (information-theoretic) lower bound on the average degree of the variable nodes $\Rightarrow$ A lower bound on the number of edges normalized per code bit.

The graphical complexity is defined as the number of edges in the bipartite graph $\Rightarrow$ The graphical complexity is lower bounded by the product of the two lower bounds above.

Tools Used for This Work (Cont.)

- A lower bound on the graphical complexity relies on two previous information-theoretic results:
  - A new lower bound on the average left/ right degrees in the bipartite graph as a function of the channel, target block/ bit error probability, and the achievable gap to capacity (Sason, IEEE Trans. on IT, July 2009).
  - Sphere-packing lower bounds: We rely here on the classical 1959 sphere-packing bound of Shannon, and the recently introduced ISP bound (Wiechman & Sason, IEEE Trans. on IT, May 2008).
- The lower bound on the graphical complexity is compared here to various capacity-approaching LDPC code ensembles under ML and message-passing iterative decoding.

Sphere-Packing Bounds

- Lower bounds on the decoding error probability of optimal block codes, given in terms of
  1. block length
  2. rate
  3. communication channel
- Based on geometrical properties of the decoding regions.
- Decay to zero exponentially with the block length.

The 1967 sphere-packing (SP67) bound (Shannon et al.)

- Applies to codes transmitted over DMCs.
- Valid under optimal ML decoding or even under list decoding.
- Error exponent is exact between the critical rate and channel capacity.

Theorem (The 1967 Sphere-Packing Bound)

- Let $\mathcal{C}$ be a block code consisting of M codewords each of length N.
- Assume communication over a DMC, and let $P(j|k)$ designate the transition probabilities where $k \in \{1, \ldots, K\}$ and $j \in \{1, \ldots, J\}$ are the channel input and output alphabets, respectively.
- Assume a list decoder where the size of the list is limited to L.
- Define

$$R \triangleq \frac{\ln\left(\frac{M}{L}\right)}{N} \quad - \quad \text{code rate in nats per channel use}$$

$$P_{\min} \quad - \quad \text{smallest non-zero transition probability of the DMC.}$$

- Then, the *average decoding error probability* is lower bounded by

$$P_e(N, M, L) \geq \exp\left\{-N\left[E_{\text{sp}}\left(R - O_1\left(\frac{\ln N}{N}\right)\right) + O_2\left(\frac{1}{\sqrt{N}}\right)\right]\right\}$$

Sphere-Packing Bounds (Cont.)

- The original focus in the derivation of the SP67 bound was on asymptotic analysis.
- The aim was to make the derivation as simple as possible, as long as there is no loss in the asymptotic behavior.
- Problem: The SP67 bound is in general very loose for codes of short to moderate block lengths.
- Goal: Improve the tightness of the sphere-packing bound for finite-length codes.

This direction was studied by Valembios and Fossorier (IEEE Trans. on Information Theory, December 2004), followed by Wiechman and Sason (IEEE Trans. on Information Theory, May 2008).

Theorem (ISP Bound, IEEE Trans. on IT, May 2008)

- Let $\mathcal{C}$ be an arbitrary block code consisting of M codewords, each of length N.
- Assume communication over a symmetric memoryless channel specified by the transition probabilities (or densities) $P(j|k)$.
- Assume a list decoder where the size of the list is limited to L.
- Then, the *average decoding error probability* is lower bounded by

$$P_{\mathrm{e}}(N,M,L) \geq \exp\Big\{-NE_{\mathrm{ISP}}(R,N)\Big\}$$

where

$$E_{ISP}(R,N) \triangleq \inf_{x>\frac{\sqrt{2}}{2}} \left\{E_0(\rho_x) - \rho_x \left(R - O_1\left(\frac{1}{N}, x\right)\right) + O_2\left(\frac{1}{\sqrt{N}}, x, \rho_x\right)\right\}$$

On the average degree of the parity-check nodes

- In this work, we introduce an information-theoretic lower bound which is related to the average right degree of binary linear block codes which are represented by an arbitrary bipartite graph.
- This forms an improvement to some bounds previously reported by Sason and Urbanke (July 2003) and Wiechman and Sason (Fab. 2007).
- The proof of the new bound refers however to
  - Finite-length code (as opposed to the asymptotic case of infinite block length in the bounds above).
  - A rigorous adaptation of the theorem to LDPC code ensembles.
- An improvement of the new bound over previously reported bounds will be exemplified.

Theorem (On the average degree of the parity-check nodes, IEEE Trans. on IT, July 2009)

- Let $\mathcal{C}$ be a binary linear block code of block length n whose transmission takes place over an MBIOS channel.
- Let $\mathcal{G}$ be a bipartite graph which corresponds to a full-rank parity-check matrix of $\mathcal{C}$.
- Let C designate the capacity of the channel, in bits per channel use, and a be the L-density function of this channel (L is the LLR at the channel output given that the binary input is zero).
- Assume that the code rate is (at least) a fraction $1 - \varepsilon$ of the channel capacity (where $0 < \varepsilon < 1$), and the code achieves a block error probability P_{B} or a bit error probability P_{b} under some decoding algorithm.

Theorem (Cont.)

Then, the average right degree of the bipartite graph (i.e., the average degree of the parity-check nodes in $\mathcal{G}$) satisfies

$$a_R \geq \frac{2\ln\left(\frac{1}{1-2h_2^{-1}\left(\frac{1-C-\delta}{1-(1-\varepsilon)C}\right)}\right)}{\ln\left(\frac{1}{g_1}\right)} \tag{1}$$

where $g_1 \triangleq \mathbb{E}[\tanh^2(L/2)]$, and

$$\delta \triangleq \begin{cases} P_B + \frac{h_2(P_B)}{n} & \text{for a block error probability } P_B \\ h_2(P_b) & \text{for a bit error probability } P_b \end{cases} . \tag{2}$$

Theorem (Cont.)

Furthermore, among all the MBIOS channels which exhibit a given capacity C and for which a target block error probability (P_{B}) or a bit error probability (P_{b}) is obtained under some decoding algorithm, a universal lower bound on a_{R} holds by replacing g_1 on the RHS of (1) with C.

Theorem (Cont.)

For LDPC code ensembles, where the parity-check matrices are not necessarily full-rank (i.e., there is some linear dependence between the parity-check equations), the theorem still holds for **for every code from the ensemble** when the rate of a code is replaced by the design rate of the ensemble.

A proof of this theorem (IEEE Trans. on Information Theory, July 2009) follows from

- A lower bound on the conditional entropy for binary linear block codes transmitted over MBIOS channels:

$$\frac{H(\mathbf{X}|\mathbf{Y})}{n} \geq R - C + \frac{1-R}{2\ln 2} \sum_{p=1}^{\infty} \frac{\Gamma(g_p)}{p(2p-1)}$$

  where $g_p \triangleq \mathbb{E}[\tanh^{2p}(L/2)]$ for $p \in \mathbb{N}$, and $\Gamma(x) \triangleq \sum_k \Gamma_k x^k$ forms the degree distribution of the parity-check nodes for an arbitrary representation of the code by a **full-rank** parity-check matrix (Wiechman & Sason, IEEE Trans. on IT, Feb. 2007).
- This inequality was extended for the case where the parity-check matrix is not necessarily full-rank.
- The bound on the average right degree was then tightened as compared to its previous form in the IEEE Trans. on IT, Feb. 2007.

Comparison of the new lower bound on the average right degree with previously reported bounds

- Communications takes place over a binary-input AWGN channel (BIAWGNC).
- The LDPC code ensembles in each sequence are specified by the following pairs of degree distributions, followed by their design rates and thresholds under belief-propagation (BP) decoding:

Ensemble 1:

$$\lambda(x) = x, \quad \rho(x) = x^{19}, \quad R_{\text{d}} = 0.9000.$$

$$\sigma_{\text{BP}} = 0.4156590 \Rightarrow C = 0.9685 \text{bits per channel use.}$$

Ensemble 2:

$$\lambda(x) = 0.4012x + 0.5981x^2 + 0.0007x^{29}, \quad \rho(x) = x^{24}$$

$$R_{\text{d}} = 0.9000, \quad \sigma_{\text{BP}} = 0.4741840 \Rightarrow C = 0.9323 \text{bits per channel}$$

- The corresponding gap (in rate) to capacity $\varepsilon = 1 - \frac{R_\mathrm{d}}{C}$ is equal to $\varepsilon = 7.1 \cdot 10^{-2}$ and $3.5 \cdot 10^{-2}$ for Ensembles 1 and 2, respectively.
- For the first ensemble which is a (2,20) regular LDPC code ensemble, the new lower bound on the average right degree is equal to 9.949 whereas the previously reported lower bound (IEEE Trans. on IT, Feb. 2007) is equal to 2.392.
- For the second ensemble whose fixed right degree is equal to 25, the new lower bound on the average right degree is 16.269 whereas the previous lower bound is 14.788.

This shows that the improvement obtained in this theorem is of practical use for a calculation of the lower bound on the graphical complexity.

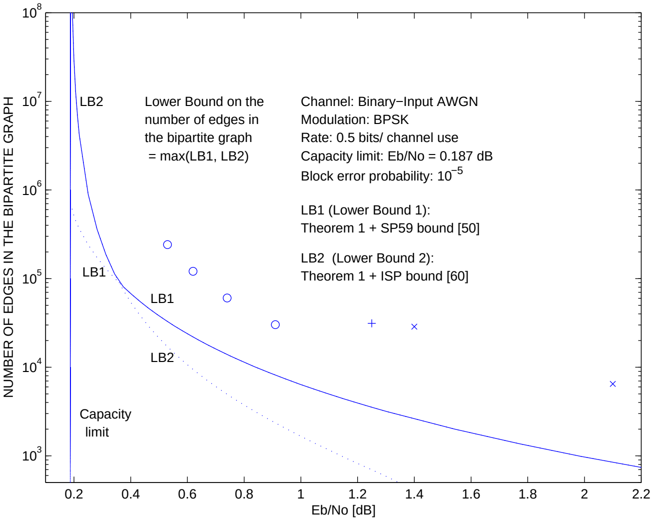

Figure: A comparison between the graphical complexity of various efficient LDPC code ensembles and an information-theoretic lower bound.

Full Paper Version

I. Sason, "On Universal Properties of Capacity-Approaching LDPC Code Ensembles," *IEEE Trans. on Information Theory*, vol. 55, no. 7, pp. 2956–2990, July 2009.