Schrijver's ϑ-function need not upper bound the Shannon capacity of a graph

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Strong Product of Graphs

Let G and H be two graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with

- vertex set: $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$,
- two distinct vertices (g, h) and (g', h') in $\mathsf{G} \boxtimes \mathsf{H}$ are adjacent if one of the following three conditions holds:
  1. $g = g'$ and $\{h, h'\} \in \mathsf{E}(\mathsf{H})$,
  2. $\{g, g'\} \in \mathsf{E}(\mathsf{G})$ and $h = h'$,
  3. $\{g, g'\} \in \mathsf{E}(\mathsf{G})$ and $\{h, h'\} \in \mathsf{E}(\mathsf{H})$.

Strong products are commutative and associative.

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Strong Powers of Graphs

Let

$$\mathsf{G}^{\boxtimes k} \triangleq \underbrace{\mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}}_{\mathsf{G} \text{ appears } k \text{ times}}, \quad k \in \mathbb{N} \tag{1.1}$$

denote the k-fold strong power of a graph G.

Shannon Capacity of a Graph (Cont.)

- The Shannon capacity of a graph G is given by

$$\Theta(\mathsf{G}) = \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}. \tag{2.1}$$

where $\alpha(\cdot)$ denotes the independence number of the graph.

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The last equality holds by Fekete's Lemma: the sequence $\{\alpha(\mathsf{G}^{\boxtimes k})\}_{k=1}^{\infty}$ is super-multiplicative, i.e.,

$$\alpha(\mathsf{G}^{\boxtimes (k_1+k_2)}) \geq \alpha(\mathsf{G}^{\boxtimes k_1})\, \alpha(\mathsf{G}^{\boxtimes k_2}). \tag{2.2}$$

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Alas, the Shannon capacity can be rarely computed exactly !

Lovász ϑ-function $\vartheta(\mathsf{G})$

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\vartheta(\mathsf{G})$:

$$\begin{array}{l} \text{maximize} \;\; \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \;\; \mathrm{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i,j \in [n]. \end{cases} \end{array}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Lovász Bound on the Shannon Capacity of Graphs (1979)

Theorem 2.1

For every finite, simple and undirected graph G,

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.3}$$

Schrijver's ϑ-function $\vartheta'(\mathsf{G})$

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Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta'(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Theorem 2.2

For every graph G,

$$\alpha(\mathsf{G}) \leq \vartheta'(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.4}$$

Question

Can the upper bound on the Shannon capacity,

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G})$$

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Our work resolves this query regarding the variant of the ϑ-function by Schrijver (1978). The answer is negative.

Example 2.3 (I.S., '25)

Let G be the Gilbert graph on 32 vertices, where

$$\mathsf{V}(\mathsf{G}) = \{0,1\}^5, \qquad \mathsf{E}(\mathsf{G}) = \Big\{\underline{u}, \underline{v} \in \{0,1\}^5 : \; 1 \leq d_{\mathrm{H}}(\underline{u}, \underline{v}) \leq 2\Big\},$$

so, every two vertices are adjacent if and only if the Hamming distance of their corresponding 5-tuples binary vectors is either 1 or 2.

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- G is 15-regular, vertex-transitive, edge-transitive, distance-regular.
- The complement $\overline{\mathsf{G}}$ is 16-regular, vertex-transitive, but not edge-transitive nor distance-regular.
- $\alpha(\mathsf{G}) = 4.$ An example of such a maximal independent set of G:

$$\Big\{(1,0,0,1,0), \, (0,1,1,1,0), \, (0,0,0,0,1), \, (1,1,1,0,1)\Big\}.$$

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- Solving the SDP problem for $\vartheta'(\mathsf{G})$ gives
$$\vartheta'(\mathsf{G}) = 4 = \alpha(\mathsf{G}).$$

Example 2.3 (cont. - I.S., '25)

- G is 15-regular and edge-transitive on 32 vertices, with $\lambda_{\min}(\mathsf{G}) = -3$, so

$$\vartheta(\mathsf{G}) = -\frac{n\lambda_{\min}(\mathsf{G})}{d(\mathsf{G}) - \lambda_{\min}(\mathsf{G})} = \tfrac{32 \cdot 3}{15+3} = 5\tfrac{1}{3}.$$

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- Hence, for this graph,

$$4 = \alpha(\mathsf{G}) = \vartheta'(\mathsf{G}) < \vartheta(\mathsf{G}) = 5\tfrac{1}{3},$$

so $\vartheta'(\mathsf{G})$ coincides with the independence number of G, and it is strictly smaller than $\vartheta(\mathsf{G})$.

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so $\vartheta'(\mathsf{G})$ coincides with the independence number of G, and it is strictly smaller than $\vartheta(\mathsf{G})$.
- It can be verified that
$$\alpha(\mathsf{G} \boxtimes \mathsf{G}) = 20,$$
and the strong product graph $\mathsf{G} \boxtimes \mathsf{G}$ has $368,640$ such maximal independent sets of size 20.

Example 2.3 (cont. - I.S., '25)

- An example of a maximal independent set (of size 20) for $G \boxtimes G$:

$$\begin{aligned}\big\{((1,1,0,0,0),(1,1,1,1,1)), &\quad ((1,0,1,0,0),(1,1,0,0,0)),\\ ((0,1,1,0,0),(0,0,1,1,0)), &\quad ((1,1,1,0,0),(0,0,0,0,1)),\\ ((1,0,0,1,0),(0,0,1,0,1)), &\quad ((0,1,0,1,0),(1,0,0,0,0)),\\ ((1,1,0,1,0),(0,1,0,1,0)), &\quad ((0,0,1,1,0),(0,1,0,1,1)),\\ ((1,0,1,1,0),(1,0,1,1,0)), &\quad ((0,1,1,1,0),(1,1,1,0,1)),\\ ((1,0,0,0,1),(0,0,0,1,0)), &\quad ((0,1,0,0,1),(0,1,0,0,1)),\\ ((1,1,0,0,1),(1,0,1,0,0)), &\quad ((0,0,1,0,1),(1,0,1,0,1)),\\ ((1,0,1,0,1),(0,1,1,1,1)), &\quad ((0,1,1,0,1),(1,1,0,1,0)),\\ ((0,0,0,1,1),(1,1,1,1,0)), &\quad ((1,0,0,1,1),(1,1,0,0,1)),\\ ((0,1,0,1,1),(0,0,1,1,1)), &\quad ((0,0,1,1,1),(0,0,0,0,0))\big\}.\end{aligned}$$

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- An example of a maximal independent set (of size 20) for $\mathsf{G} \boxtimes \mathsf{G}$:

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\end{aligned}
$$

- Consequently, we get

$$\Theta(\mathsf{G}) \geq \sqrt{\alpha(\mathsf{G} \boxtimes \mathsf{G})} = \sqrt{20} > 4 = \vartheta'(\mathsf{G}).$$

Summary and Takeaways

- Shannon capacity $\Theta(\mathsf{G})$ is notoriously difficult to compute.
- Lovász's $\vartheta(\mathsf{G})$ provides a polynomial-time computable upper bound:

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}).$$

- Schrijver's variant $\vartheta'(\mathsf{G})$ gives a polynomial-time upper bound on the independence number of a graph, but does *not* upper bound $\Theta(\mathsf{G})$.
- This resolves a 1978 query.
- Concrete example: Gilbert graph on 32 vertices, where $\Theta(\mathsf{G}) > \vartheta'(\mathsf{G})$.

Journal Paper

I. S., "An example showing that Schrijver's ϑ'-function need not upper bound the Shannon capacity of a graph," AIMS Mathematics, vol. 10, no. 7, pp. 15294–15301, May 2025.
https://www.aimspress.com/article/doi/10.3934/math.2025685.