On the Corner Points of the Capacity Region of a Two-User Gaussian Interference Channel

Igal Sason

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel

2014 International Symposium on Information Theory
Honolulu, Hawaii, USA

July 4, 2014.

Two-User Gaussian Interference Channel (GIC)

- Model in *standard form*:

$$Y_1 = X_1 + \sqrt{a_{12}}\, X_2 + Z_1$$
$$Y_2 = \sqrt{a_{21}}\, X_1 + X_2 + Z_2$$

- The cross-link gains a_{12} and a_{21} are fixed in time.
- Input and output signals are real-valued.
- No cooperation between transmitters.
- Power constraints: $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_{1,i}^2] \leq P_1$ and $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_{2,i}^2] \leq P_2$.
- $Z_1, Z_2 \sim \mathcal{N}(0,1)$ i.i.d. and independent of the inputs X_1 and X_2.
- No cooperation between receivers.
- Full synchronization $\Rightarrow$ capacity region is convex.

Two-User GIC (Cont.)

- This channel has been studied during the last 4 decades.
- Full characterization of the capacity region is still unknown, except for
  - **Very strong interference** (Carleial, 1975):
  $$a_{12} \geq 1 + P_1, \quad a_{21} \geq 1 + P_2.$$
  The capacity region is the same as if there is no interference:
  $$\left\{(R_1, R_2) : 0 \leq R_1 \leq \frac{1}{2} \log(1 + P_1),\; 0 \leq R_2 \leq \frac{1}{2} \log(1 + P_2)\right\}.$$
  - **Strong interference** (Sato (1981), Han & Kobayashi (1981)):
  $$1 \leq a_{12} \leq 1 + P_1, \quad 1 \leq a_{21} \leq 1 + P_2.$$
  The capacity region is the intersection of the capacity regions of the two MACs $(X_1, X_2) \rightarrow Y_1$ and $(X_1, X_2) \rightarrow Y_2$.

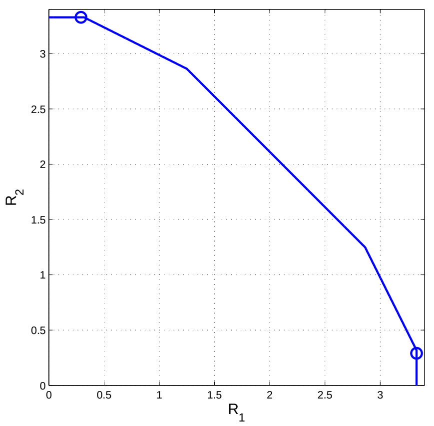
Two-User GIC (Cont.)

Capacity region is unknown for a two-user GIC with either

1. **Weak interference** $(0 < a_{12}, a_{21} < 1)$
2. **Mixed interference** ($a_{12} \geq 1$ and $a_{21} \leq 1$ or vice versa).
3. **One-sided interference** (a.k.a. Gaussian Z-interference) with weak interference (i.e., $a_{12} = 0$ and $a_{21} = a \in (0, 1)$ or vice versa).

Corner Points of the Capacity Region

For a 2-user GIC, the rate pairs where one user sends its data at the single-user capacity, and the other at the largest rate for which reliable communication is possible are called corner points.



Conjecture (Originated by Costa, 1985)

For a two-user GIC with positive cross-link gains, let

$$C_1 \triangleq \frac{1}{2}\log(1+P_1), \quad C_2 \triangleq \frac{1}{2}\log(1+P_2)$$

be the capacities of the single-user AWGN channels, and

$$R_1^* \triangleq \frac{1}{2}\log\left(1+\frac{a_{21}P_1}{1+P_2}\right), \quad R_2^* \triangleq \frac{1}{2}\log\left(1+\frac{a_{12}P_2}{1+P_1}\right).$$

Then, the following is conjectured to hold for reliable communication:

1. If $R_2 \geq C_2 - \varepsilon$, then $R_1 \leq R_1^* + \delta_1(\varepsilon)$ where $\lim_{\varepsilon \to 0} \delta_1(\varepsilon) = 0$.
2. If $R_1 \geq C_1 - \varepsilon$, then $R_2 \leq R_2^* + \delta_2(\varepsilon)$ where $\lim_{\varepsilon \to 0} \delta_2(\varepsilon) = 0$.

Interpretation of this conjecture for weak GIC

If one user transmits at its maximal possible rate, the other user should decrease its rate such that both decoders can reliably decode its message.

Theorem 1 - Bounds on the Corner Points, Weak Interference

Consider a weak two-user GIC (where $(0 < a_{12}, a_{21} < 1)$), and let C_1, C_2, R_1^* and R_2^* be as defined earlier. If $R_1 \geq C_1 - \varepsilon$ for an arbitrary $\varepsilon > 0$, then reliable communication requires that

$$R_2 \leq \min\left\{R_2^* + \frac{1}{2}\log\left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)}\right) + 2\varepsilon,\right.$$
$$\left.\frac{1}{2}\log\left(1 + \frac{P_2}{1 + P_1}\right) + \left(1 + \frac{1 + P_1}{a_{21}P_2}\right)\varepsilon\right\}.$$

A similar bound holds if $R_2 \geq C_2 - \varepsilon$.

Theorem 1 - Bounds on the Corner Points, Weak Interference

Consider a weak two-user GIC (where $(0 < a_{12}, a_{21} < 1)$), and let C_1, C_2, R_1^* and R_2^* be as defined earlier. If $R_1 \geq C_1 - \varepsilon$ for an arbitrary $\varepsilon > 0$, then reliable communication requires that

$$R_2 \leq \min\left\{R_2^* + \frac{1}{2}\log\left(1 + \frac{P_2}{(1 + a_{21}P_1)(1 + a_{12}P_2)}\right) + 2\varepsilon,\right.$$
$$\left.\frac{1}{2}\log\left(1 + \frac{P_2}{1 + P_1}\right) + \left(1 + \frac{1 + P_1}{a_{21}P_2}\right)\varepsilon\right\}.$$

A similar bound holds if $R_2 \geq C_2 - \varepsilon$.

In the limit where P_1 and P_2 tend to infinity:

1. The conjecture holds, and it gives an asymptotically tight bound.
2. The rate pairs (C_1, R_2^*) and (R_1^*, C_2) form the corner points.

Tools used for proving Theorem 1

Upper bound on the corner points: The proof relies on two outer bounds on the capacity region of a GIC:

- The outer bound of Etkin, Tse and Wang (2008) that is within the capacity region up to a difference of at most 1 bit per channel use in the sum-rate.
- The outer bound of Kramer (2004) that relies on the capacity region of the broadcast Gaussian channel when the transmitters cooperate.

Tools used for proving Theorem 1

Upper bound on the corner points: The proof relies on two outer bounds on the capacity region of a GIC:

- The outer bound of Etkin, Tse and Wang (2008) that is within the capacity region up to a difference of at most 1 bit per channel use in the sum-rate.
- The outer bound of Kramer (2004) that relies on the capacity region of the broadcast Gaussian channel when the transmitters cooperate.

Lower bound: A standard approach for the achievability:

- Using Gaussian codebooks with i.i.d. entries, subject to the power constraints.
- First decoding the interference by treating the transmitted signal of the intended user as an additive Gaussian noise.
- Subtracting the interference from the received signal.
- Finally, decoding the signal of the intended user.

Excess Rate for the Sum-Rate w.r.t. the Corner Points

- The sum-rate of a GIC is attained at one of the corner points of the capacity region for mixed, strong or one-sided interference.
- This is in contrast to a GIC with two-sided weak interference.

Excess Rate for the Sum-Rate w.r.t. the Corner Points

- The sum-rate of a GIC is attained at one of the corner points of the capacity region for mixed, strong or one-sided interference.
- This is in contrast to a GIC with two-sided weak interference.
- $\Rightarrow$ Study the **excess rate for the sum-rate w.r.t. these corner points**, defined to be

$$\Delta \triangleq C_{\text{sum}} - \max\big\{R_1 + R_2 \colon (R_1, R_2) \text{ is a corner point}\big\}.$$

Excess Rate for the Sum-Rate w.r.t. the Corner Points

- The sum-rate of a GIC is attained at one of the corner points of the capacity region for mixed, strong or one-sided interference.
- This is in contrast to a GIC with two-sided weak interference.
- $\Rightarrow$ Study the **excess rate for the sum-rate w.r.t. these corner points**, defined to be

$$\Delta \triangleq C_{\text{sum}} - \max\bigl\{R_1 + R_2 \colon (R_1, R_2) \text{ is a corner point}\bigr\}.$$

- $\Delta = 0$ for a GIC with mixed/ strong/ one-sided interference.
- $\Rightarrow$ We focus on a GIC with two-sided weak interference.

An Analogous Measure to Generalized Degrees of Freedom (GDOF)

- This GDOF refers to the case where the SNR (P) tends to infinity, and $a = P^{\alpha-1}$ for some $\alpha \geq 0 \quad \Rightarrow \quad \frac{\log(\mathsf{INR})}{\log(\mathsf{SNR})} = \alpha$.

An Analogous Measure to Generalized Degrees of Freedom (GDOF)

- This GDOF refers to the case where the SNR (P) tends to infinity, and $a = P^{\alpha-1}$ for some $\alpha \geq 0 \quad \Rightarrow \quad \frac{\log(\text{INR})}{\log(\text{SNR})} = \alpha$.
- **Theorem 2**: For an arbitrary $\alpha \geq 0$, the following limit exists:

$$\delta(\alpha) \triangleq \lim_{P \to \infty} \frac{\Delta(P, P^{\alpha-1})}{\log P}$$

and it admits the following closed-form expression:

$$\delta(\alpha) = \begin{cases} \left|\frac{1}{2} - \alpha\right|, & \text{if } 0 \leq \alpha < \frac{2}{3} \\ \frac{1-\alpha}{2}, & \text{if } \frac{2}{3} \leq \alpha < 1 \\ 0, & \text{if } \alpha \geq 1 \end{cases}.$$

An Analogous Measure (Cont.)

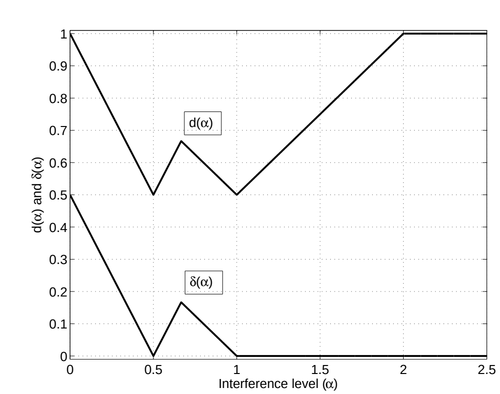


Figure: A comparison of $d(\alpha) \triangleq \lim_{P\to\infty} \frac{C_{\text{sum}}(P, P^{\alpha-1})}{\log P}$ (the symmetric GDOF) and $\delta(\alpha)$. We have $d(\alpha) - \delta(\alpha) = \frac{1}{2}, \quad \forall\, \alpha \in [0, 1]$.

Some Implications

- $\delta(\alpha)$ is non-monotonic over $[0, 1]$, the cross-link gain $a = P^{\alpha-1}$ is increasing in α for $P \geq 1$, and it is a mapping from [0,1] to itself. $\Rightarrow \Delta(P, a)$ is not monotonic decreasing in $a \in [0, 1]$ for large P.

Some Implications

- $\delta(\alpha)$ is non-monotonic over $[0,1]$, the cross-link gain $a = P^{\alpha-1}$ is increasing in α for $P \geq 1$, and it is a mapping from [0,1] to itself. $\Rightarrow \Delta(P,a)$ is not monotonic decreasing in $a \in [0,1]$ for large P.
- This forms a **stronger property** than the non-monotonicity of the sum-rate, for large P.

Some Implications

- $\delta(\alpha)$ is non-monotonic over $[0,1]$, the cross-link gain $a = P^{\alpha-1}$ is increasing in α for $P \geq 1$, and it is a mapping from [0,1] to itself. $\Rightarrow \Delta(P,a)$ is not monotonic decreasing in $a \in [0,1]$ for large P.
- This forms a **stronger property** than the non-monotonicity of the sum-rate, for large P.
- The excess rate $\Delta(P,a)$, for large P, jumps drastically when the cross-link gain just varies slightly from $\frac{1}{\sqrt{P}}$ to $\frac{1}{\sqrt[3]{P}}$; in this case, for $P \gg 1$, the excess rate $\Delta(P,a)$ is increased from ≈ 0 to $\approx \frac{1}{6} \log P$.

New Simple Bounds on Δ for finite SNR and INR

Theorem 3: Consider a two-user symmetric GIC with weak interference in standard form where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0, 1]$. Then,

$$\Delta(P,a) \leq \frac{1}{2}\left[\min\left\{\log(1+P) + \log\left(1+\frac{P}{1+aP}\right),\ 2\log\left(1+aP+\frac{P}{1+aP}\right)\right\} - \log\big(1+(1+a)P\big)\right]$$

and, if $P \geq 2.551$,

$$\Delta(P,a) \geq \frac{1}{2}\left[\min\left\{\log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{1+aP}\right),\ 2\log\left(1+aP+\frac{P}{1+aP}\right)\right\} - \min\left\{\log\big(1+(a+1)P\big) + \log\left(1+\frac{P}{(1+aP)^2}\right), \log(1+2P)\right\}\right] - 1$$

The upper and lower bounds on $\Delta(P, P^{\alpha-1})$ are asymptotically tight, as we let P tend to infinity, in the sense of achieving the asymptotic limit $\delta(\alpha)$ for an arbitrary $\alpha \geq 0$.

Corollary - Asymptotic Bounds on Δ

Consider a two-user symmetric GIC with weak interference in standard form where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0, 1]$. Then,

$$\frac{1}{2} \log\left(1 + \frac{1}{a}\right) - 1 \leq \lim_{P \to \infty} \Delta(P, a) \leq \frac{1}{2} \log\left(\frac{1}{a}\right), \quad \forall\, a \in (0, 1].$$

The base of the log is 2.

Corollary - Asymptotic Bounds on Δ

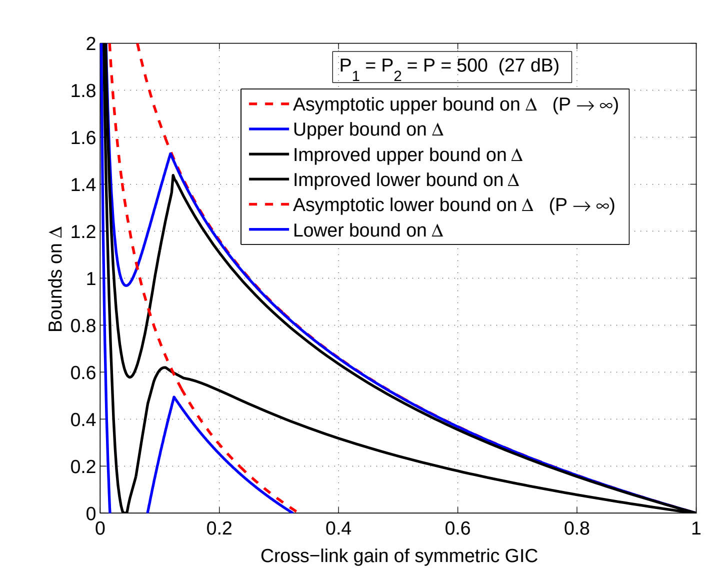
Consider a two-user symmetric GIC with weak interference in standard form where $P_1 = P_2 = P$ and $a_{12} = a_{21} = a \in (0, 1]$. Then,

$$\frac{1}{2} \log\left(1 + \frac{1}{a}\right) - 1 \leq \lim_{P \to \infty} \Delta(P, a) \leq \frac{1}{2} \log\left(\frac{1}{a}\right), \quad \forall\, a \in (0, 1].$$

The base of the log is 2.

- This provides the correct scaling of $\Delta(P, a)$ for large P and fixed cross-link gain a.
- It is consistent with $\delta(1) = 0$.

Two-user symmetric GIC with weak interference



Two-user symmetric GIC with weak interference

Table: Comparison of the asymptotic approximation of the excess rate for the sum-rate w.r.t. the corner points (Δ) with its improved upper bound on Δ.

Power	Value of a achieving minimum of Δ		Normalized Δ by $\log P$		Value of a achieving maximum of Δ for $a \geq \frac{1}{\sqrt{P}}$		Normalized Δ by $\log P$	
	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value	Asymptotic approximation	Exact value
(P)	$(a = \frac{1}{\sqrt{P}})$				$(a = \frac{1}{\sqrt[3]{P}})$			
27 dB	0.045	0.050	0	0.065	0.126	0.140	0.167	0.154
40 dB	0.010	0.011	0	0.046	0.046	0.042	0.167	0.164
60 dB	0.001	0.001	0	0.032	0.010	0.010	0.167	0.166

Summary

- This work derives simple and informative bounds on the corner points of the capacity region of a two-user GIC with weak interference.
- Analysis of the ε-**proximity to these corner points** is provided.
- Upper and lower bounds on the **excess rate for the sum-rate w.r.t. the corner points** of the capacity region are derived (denoted by Δ).
- An asymptotic analysis of this gap is provided, analogously to the study of the GDoF (where the SNR and INR scalings are coupled).
- $\Rightarrow$ **Tight in the whole range of this scaling ($\alpha \geq 0$).**
- Upper and lower bounds on Δ are derived for finite SNR and INR.
- The full paper version is available at http://arxiv.org/abs/1306.4934 (submitted to the IEEE Trans. on IT).
- A work by Bustin et al. studies the corner points via the connection between MMSE and MI (ISIT '14, draft: arxiv.org/abs/1404.6690). It provides another support (endorsement) to Costa's conjecture.