Tight Bounds on the Rényi Entropy via Majorization with Applications to Guessing and Compression

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A Mathematical Theory of Communication

By C. E. SHANNON

(a) $H(p_1, p_2, \cdots, p_n)$ *is a symmetric function of its variables for* $n = 2, 3, \cdots$.
(b) $H(p, 1-p)$ *is a continuous function of* p *for* $0 \leqq p \leqq 1$.
(c) $H(1/2, 1/2) = 1$.
(d) $H[tp_1, (1-t)p_1, p_2, \cdots, p_n] = H(p_1, p_2, \cdots, p_n) + p_1 H(t, 1-t)$
for any distribution $\mathcal{P} = (p_1, p_2, \cdots, p_n)$ *and for* $0 \leqq t \leqq 1$.

1961

ON MEASURES OF ENTROPY AND INFORMATION

ALFRÉD RÉNYI

Lemma (Erdös, 1946; Rényi, 1961 (a simplified proof))

Let $f \colon \mathbb{N} \to \mathbb{R}$, and suppose that

$$f(nm) = f(n) + f(m), \quad \forall\, n, m \in \mathbb{N},$$

$$\lim_{n \to \infty} \bigl(f(n+1) - f(n)\bigr) = 0.$$

Then, there exists a constant c such that

$$f(n) = c \log n, \quad \forall\, n \in \mathbb{N}.$$

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Shannon Entropy

This enabled Rényi to show that the Shannon entropy, which satisfies Postulates a)–d), is of the form:

$$H(p_1, \ldots, p_n) = -\sum_{k=1}^{n} p_k \log p_k.$$

(a) $H(p_1, p_2, \cdots, p_n)$ *is a symmetric function of its variables for* $n = 2, 3, \cdots$.
(b) $H(p, 1 - p)$ *is a continuous function of* p *for* $0 \leqq p \leqq 1$.
(c) $H(1/2, 1/2) = 1$.

(d') If X and Y are independent, then $H(X, Y) = H(X) + H(Y)$

Rényi entropy

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_{x \in \mathcal{A}} P_X^\alpha(x), \quad \alpha \in (0,1) \cup (1,\infty)$$

$$\tfrac{\alpha}{1-\alpha} \log \|P_X\|_\alpha$$

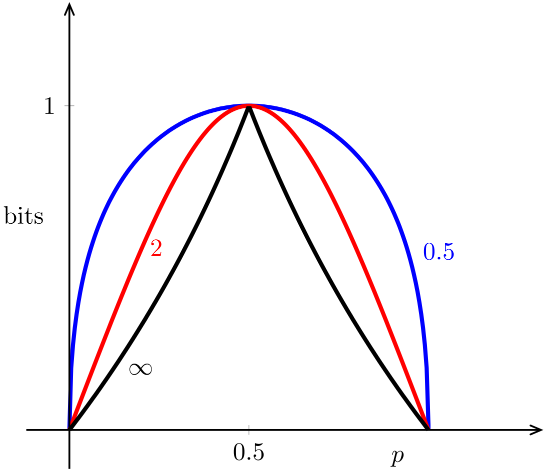
$$H_1(X) = H(X),$$

$$H_\infty(X) = \min_{x \in \mathcal{A}} \log \frac{1}{P_X(x)}$$

$$H_0(X) = \log \left| \{x \in \mathcal{A} \colon P_X(x) > 0\} \right|$$

$$H_2(X) = \log \frac{1}{\sum_{x \in \mathcal{A}} P_X^2(x)}$$

Rényi entropy: binary distribution



Applications of Rényi Entropy

- Random search (Rényi, 1965).
- Statistical physics (Tsallis, 1988).
- Secret-key generation (Renner-Wolf, 2005).
- Data compression (Campbell, 1965).
- Hypothesis testing and coding theorems (Csiszár, 1995).
- Guessing (Arikan, 1996).

Majorization

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Additional Books

- R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, Springer, 1997 (Ch. 2).
- J. M. Steele, *The Cauchy-Schwarz Master Class*, Cambridge University Press, 2004 (Ch. 13).

Setting

Let

- P be a probability mass function (PMF) defined on a finite set $\mathcal{X}$;
- $p_{\max}$ be the maximal mass of P;
- $G_P(k)$ be the sum of the k largest masses of P for $k \in \{1, \ldots, |\mathcal{X}|\}$.

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Definition: Majorization

Let P and Q be two PMFs defined on a finite set $\mathcal{X}$.
It is said that P is majorized by Q, and it is denoted by $P \prec Q$, if

$$G_P(k) \leq G_Q(k), \quad \forall\, k \in \{1, \ldots, |\mathcal{X}|\}.$$

(Recall that in our case $G_P(|\mathcal{X}|) = G_Q(|\mathcal{X}|) = 1$).

Examples

- A unit mass majorizes any other distribution;
- The equiprobable distribution on a finite set is majorized by any other distribution defined on the same set.

The relation $\prec$ is transitive.

Theorem (a Classical Result)

Let P and Q be PMFs defined on a finite set $\mathcal{A}$. Then, $P \prec Q$ if and only if there exists a doubly-stochastic transformation $W_{Y|X} \colon \mathcal{A} \to \mathcal{A}$:

$$\begin{aligned}
&\sum_{x \in \mathcal{A}} W_{Y|X}(y|x) = 1, \quad \forall\, y \in \mathcal{A}, \\
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&W_{Y|X}(\cdot|\cdot) \geq 0,
\end{aligned}$$

such that $Q \to W_{Y|X} \to P$.

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such that $Q \to W_{Y|X} \to P$.

Equivalently, $P \prec Q$ if and only if in their representation as column vectors, there exists a doubly-stochastic matrix $\mathbf{W}$ such that $P = \mathbf{W}Q$.

Let $\mathcal{P}_n$, for an integer $n \geq 2$, be the set of all PMFs defined on $\mathcal{A}_n := \{1, \ldots, n\}$.

Definition: Schur Convexity

A function $f \colon \mathcal{P}_n \to \mathbb{R}$ is said to be Schur-convex if for every $P, Q \in \mathcal{P}_n$ such that $P \prec Q$, we have $f(P) \leq f(Q)$.

Likewise, f is said to be Schur-concave if $-f$ is Schur-convex, i.e., $P, Q \in \mathcal{P}_n$ and $P \prec Q$ imply that $f(P) \geq f(Q)$.

A Schur-convex function is not necessarily convex.

Schur's Criterion

If a real-valued function f is continuously differentiable and symmetric, it is Schur-convex if and only if

$$0 \leq (x_j - x_k) \left(\frac{\partial f(\mathbf{x})}{\partial x_j} - \frac{\partial f(\mathbf{x})}{\partial x_k} \right) \geq 0, \quad \forall \, 1 \leq j < k \leq n.$$

Rényi Entropy

Schur's condition $\Rightarrow$

The Rényi entropy $H_\alpha(\cdot)$, for all $\alpha > 0$ is Schur concave.

Motivation

- Cicalese *et al.* (IEEE T-IT, April '18):

  If X is a RV taking n possible values, and the size of the support of $f(X)$ equals m with $m < n$, how close $H\big(f(X)\big)$ can be to $H(X)$?

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- Their goal: computing

$$\max_f H\big(f(X)\big) = \max_f \Big\{ H\big(f(X)\big) - H\big(f(X)|X\big) \Big\} = \max_f I\big(X; f(X)\big)$$

  with max. over all functions mapping a set of cardinality n to a set of cardinality $m < n$.

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- Useful in the context of data clustering.

Motivation (Cont.)

- Rényi measures are very useful, so how about ...

  Generalizing this question to $H_{\alpha}\big(f(X)\big)$ for any $\alpha > 0$ (not trivial).

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- Rényi measures are very useful, so how about ...

  Generalizing this question to $H_\alpha\big(f(X)\big)$ for any $\alpha > 0$ (not trivial).
- Possible Applications:
  - Guessing (Arikan '96);
  - Lossless compression problems (Campbell '65).

Setting

Let

- $\alpha > 0$;
- $\mathcal{X}$ and $\mathcal{Y}$ be finite sets of cardinalities

$$|\mathcal{X}| = n, \quad |\mathcal{Y}| = m, \quad n > m \geq 2;$$

  without any loss of generality, let

$$\mathcal{X} = \{1, \ldots, n\}, \quad \mathcal{Y} = \{1, \ldots, m\};$$

- $\mathcal{P}_n$ $(n \geq 2)$ be the set of probability mass functions (PMF) on $\mathcal{X}$;
- X be a RV taking values on $\mathcal{X}$ with a PMF $P_X \in \mathcal{P}_n$;
- $\mathcal{F}_{n,m}$ be the set of deterministic functions $f \colon \mathcal{X} \to \mathcal{Y}$;
- $f \in \mathcal{F}_{n,m}$ is not one-to-one since $m < n$.

Bad News

For an arbitrary $\alpha > 0$, the maximization problem

$$\max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big) \qquad (2 \leq m < n)$$

is strongly NP-hard.

- Unless P = NP, there is no poly. time algorithm which, for any $\varepsilon > 0$, computes an admissible deterministic function $f_\varepsilon \in \mathcal{F}_{n,m}$ such that

$$H_\alpha\big(f_\varepsilon(X)\big) \geq (1-\varepsilon) \max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big).$$

Good News

We can efficiently construct (by the use of Huffman algorithm) an admissible function $f^* \in \mathcal{F}_{n,m}$ s.t.

$$H_\alpha\big(f^*(X)\big) \geq \max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big) - v(\alpha), \quad \alpha > 0$$

where

$$v(\alpha) := \begin{cases} \log\left(\dfrac{\alpha-1}{2^\alpha-2}\right) - \dfrac{\alpha}{\alpha-1}\,\log\left(\dfrac{\alpha}{2^\alpha-1}\right), & \alpha \neq 1, \\ \log\left(\dfrac{2}{e\,\ln 2}\right) \approx 0.08607\,\text{bits}, & \alpha = 1. \end{cases}$$

$v \colon (0,\infty) \to (0, \log 2)$ is monotonically increasing, continuous, and

$$\lim_{\alpha \downarrow 0} v(\alpha) = 0, \qquad \lim_{\alpha \to \infty} v(\alpha) = \log 2 \text{ (1 bit)}.$$

Plot

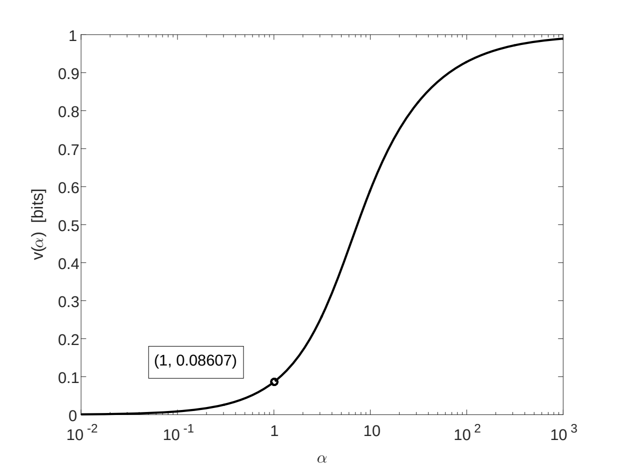

Figure: A plot of $v(\alpha)$ as a function of $\alpha > 0$.

Upper Bounding $\max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big)$

Let

- X be a discrete RV with PMF P_X, which takes n possible values;
- $f \in \mathcal{F}_{n,m}$;
- Q_X be the PMF of $f(X)$;

Then, $P_X \prec Q_X$.

Upper Bounding $\max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big)$ (Cont.)

Since the Rényi entropy is Schur-concave, it follows that

$$H_\alpha(X) = H_\alpha(P_X) \geq H_\alpha(Q_X) = H_\alpha\big(f(X)\big).$$

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Let $\mathcal{P}_n$ $(n \geq 2)$ be the set of PMFs on $\mathcal{X}$ $(|\mathcal{X}| = n)$. Since the PMF of $f(X)$ majorizes the PMF of X, then

$$\max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big) \leq \max_{Q \in \mathcal{P}_m : P_X \prec Q} H_\alpha(Q).$$

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- The max. in the right side of the last ineq. is explicitly calculated.
- A function $f^* \in \mathcal{F}_{n,m}$ is efficiently constructed such that

$$H_\alpha\big(f^*(X)\big) \geq \max_{Q \in \mathcal{P}_m : P_X \prec Q} H_\alpha(Q) - v(\alpha) \geq \max_{f \in \mathcal{F}_{n,m}} H_\alpha\big(f(X)\big) - v(\alpha).$$

Optimization Problem:

$$\max_{Q \in \mathcal{P}_m \colon P_X \prec Q} H_\alpha(Q)$$

with $X \in \{1, \ldots, n\}$, $m < n$, and $\alpha > 0$.

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Solution: $Q = R_m(P_X)$

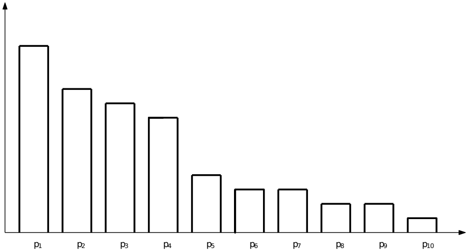
- If $P_X(1) < \frac{1}{m}$, then $R_m(P_X)$ is the equiprobable dist. on $\{1, \dots, m\}$;
- Otherwise, $R_m(P_X) := Q_X \in \mathcal{P}_m$ with

$$Q_X(i) = \begin{cases} P_X(i), & i \in \{1, \dots, n^*\}, \\ \dfrac{1}{m - n^*} \displaystyle\sum_{j=n^*+1}^{n} P_X(j), & i \in \{n^* + 1, \dots, m\}, \end{cases}$$

where n^* is the max. integer i s.t. $P_X(i) \geq \frac{1}{m-i} \sum_{j=i+1}^{n} P_X(j)$.

Intuitively

p



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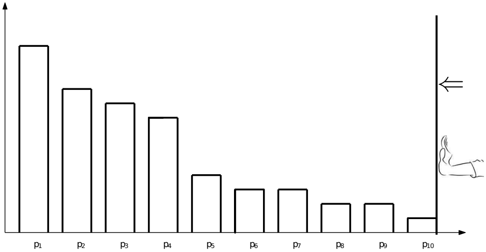
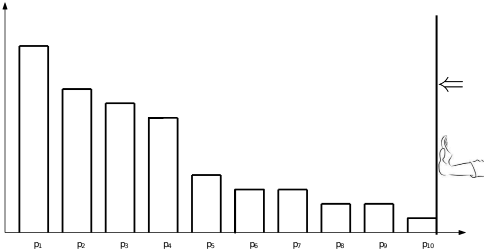
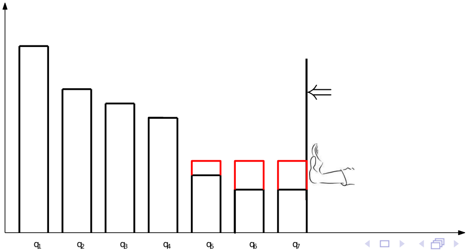


Intuitively

$\mathbf{p}$



$R(\mathbf{p})$



The Algorithm by Huffman Coding

1. Start from the PMF $P_X \in \mathcal{P}_n$ with $P_X(1) \geq \ldots \geq P_X(n)$;
2. Merge successively pairs of probability masses by applying the Huffman algorithm;
3. Stop the process in Step 2 when a PMF $Q \in \mathcal{P}_m$ is obtained (with $Q(1) \geq \ldots \geq Q(m)$);
4. Construct the deterministic function $f^* \in \mathcal{F}_{n,m}$ by setting $f^*(k) = j \in \{1, \ldots, m\}$ for all probability masses $P_X(k)$, with $k \in \{1, \ldots, n\}$, being merged in Steps 2–3 into the node of $Q(j)$.

Property - Link to the Next Auxiliary Result

Let $i \in \{0, \ldots, m-1\}$ be the maximal index such that $P_X(j) = Q(j)$ for all $j \leq i$. Then,

$$Q(i+1) \leq 2\, Q(m).$$

Notation for an Auxiliary Result

Let $\mathcal{P}_n(\rho)$, for $\rho \geq 1$ and integer $n \geq 2$, be the subset of all $P \in \mathcal{P}_n$ s.t.

$$\frac{p_{\max}}{p_{\min}} \leq \rho.$$

Theorem (an Auxiliary Result)

For $\rho > 1$ and $\alpha > 0$, let

$$c_\alpha^{(n)}(\rho) := \log n - \min_{P \in \mathcal{P}_n(\rho)} H_\alpha(P), \quad n = 2, 3, \ldots$$

with $c_\alpha^{(1)}(\rho) := 0$. Then, for every $n \in \mathbb{N}$,

$$0 \leq c_\alpha^{(n)}(\rho) \leq \log \rho,$$
$$c_\alpha^{(n)}(\rho) \leq c_\alpha^{(2n)}(\rho),$$

and $c_\alpha^{(n)}(\rho)$ is monotonically increasing in $\alpha \in [0, \infty]$.

The following limit exists for all $\alpha > 0$ and $\rho > 1$:

$$c_\alpha^{(\infty)}(\rho) := \lim_{n \to \infty} c_\alpha^{(n)}(\rho).$$

Theorem (Cont.)

Let $\rho > 1$, $\alpha > 0$, $n \geq 2$ be an integer, and $\beta \in \left[\frac{1}{1+(n-1)\rho}, \frac{1}{n}\right] := \Gamma_\rho^{(n)}$.

Let $Q_\beta \in \mathcal{P}_n(\rho)$ be defined on $\mathcal{X} = \{1, \ldots, n\}$ as follows:

$$Q_\beta(j) = \begin{cases} \rho\beta, & j \in \{1, \ldots, i_\beta\}, \\ 1 - (n + i_\beta\, \rho - i_\beta - 1)\beta, & j = i_\beta + 1, \\ \beta, & j \in \{i_\beta + 2, \ldots, n\} \end{cases}$$

where $i_\beta := \left\lfloor \frac{1-n\beta}{(\rho-1)\beta} \right\rfloor$. Then, for every $\alpha > 0$,

$$c_\alpha^{(n)}(\rho) = \log n - \min_{\beta \in \Gamma_\rho^{(n)}} H_\alpha(Q_\beta).$$

Theorem (Cont.)

1. If $\alpha \in (0,1) \cup (1,\infty)$, then

$$c_\alpha^{(\infty)}(\rho) = \frac{1}{\alpha-1} \log\left(1 + \frac{1+\alpha(\rho-1)-\rho^\alpha}{(1-\alpha)(\rho-1)}\right) - \frac{\alpha}{\alpha-1} \log\left(1 + \frac{1+\alpha(\rho-1)-\rho^\alpha}{(1-\alpha)(\rho^\alpha-1)}\right),$$

   and

$$\lim_{\alpha\to\infty} c_\alpha^{(\infty)}(\rho) = \log\rho, \quad v(\alpha) = c_\alpha^{(\infty)}(2)$$

2. If $\alpha = 1$, then

$$c_1^{(\infty)}(\rho) = \lim_{\alpha\to 1} c_\alpha^{(\infty)}(\rho) = \frac{\rho\log\rho}{\rho-1} - \log\left(\frac{\mathrm{e}\rho\log_{\mathrm{e}}\rho}{\rho-1}\right).$$

Plot

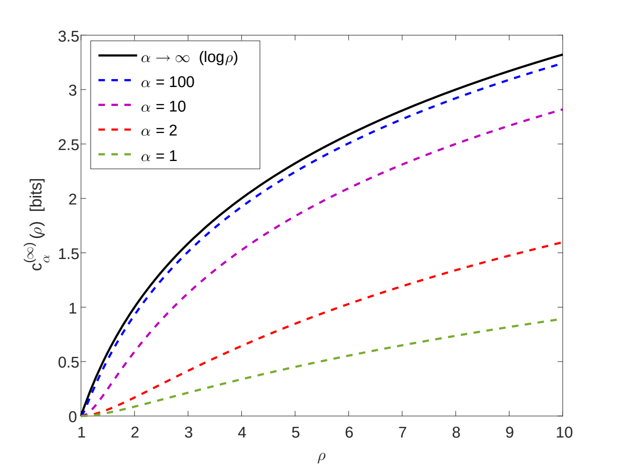


Figure: A plot of $c_\alpha^{(\infty)}(\rho)$ as a function of ρ.

Plot

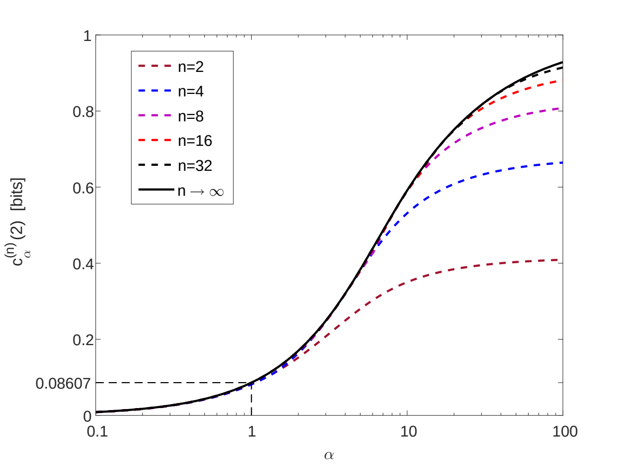


Figure: A plot of $c_\alpha^{(n)}(2)$ (log as a function of $\alpha > 0$, for several values of $n \geq 2$.

Application I: Guessing

The problem of guessing discrete random variables has found a variety of applications in

- Shannon theory,
- coding theory,
- cryptography,
- searching and sorting algorithms,

etc.

The central object of interest:

The distribution of the number of guesses required to identify a realization of a random variable, taking values on a finite or countably infinite set.

Guessing and Ranking functions

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- A guessing function is a 1-to-1 function $g \colon \mathcal{X} \to \mathcal{X}$ where the number of guesses is equal to $g(x)$ if $X = x \in \mathcal{X}$.
- For $\rho > 0$, $\mathbb{E}[g^\rho(X)]$ is minimized by selecting g to be a ranking function g_X, for which $g_X(x) = k$ if $P_X(x)$ is the k-th largest mass.

$H_\alpha(X)$ and Guessing Moments

Theorem (Arikan '96)

Let X be a discrete random variable taking values on $\mathcal{X} = \{1, \ldots, M\}$. Let $g_X(\cdot)$ be a ranking function of X. Then, for $\rho > 0$,

$$\frac{1}{\rho} \log \mathbb{E}\big[g_X^\rho(X)\big] \geq H_{\frac{1}{1+\rho}}(X) - \log(1 + \log_{\mathrm{e}} M),$$

$$\frac{1}{\rho} \log \mathbb{E}\big[g_X^\rho(X)\big] \leq H_{\frac{1}{1+\rho}}(X).$$

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$$\frac{1}{\rho} \log \mathbb{E}\big[g_X^\rho(X)\big] \leq H_{\frac{1}{1+\rho}}(X).$$

Arikan's result yields an asymptotically tight error exponent:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\big[g_{X^n}^\rho(X^n)\big] = \rho H_{\frac{1}{1+\rho}}(X), \quad \forall\, \rho > 0$$

when $X_1, \ldots, X_n$ are i.i.d. $\quad [X^n := (X_1, \ldots, X_n)]$.

Reminder: $R_m(P_X)$

Let $X \sim P_X$ take n possible values, and let $m \in \{2, \ldots, n-1\}$.

- If $P_X(1) < \frac{1}{m}$, then $R_m(P_X)$ is the equiprobable dist. on $\{1, \ldots, m\}$;
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Theorem: Guessing Moments

Let

- $\{X_i\}_{i=1}^k$ be i.i.d. with $X_1 \sim P_X$ taking values on a set $\mathcal{X}$, $|\mathcal{X}| = n$;
- $Y_i = f(X_i)$, for every $i \in \{1, \dots, k\}$, where $f \in \mathcal{F}_{n,m}$ is a deterministic function with $m < n$;
- $$g_{X^k} \colon \mathcal{X}^k \to \{1, \dots, n^k\}, \quad g_{Y^k} \colon \mathcal{Y}^k \to \{1, \dots, m^k\}$$

  be, respectively, ranking functions of the random vectors

  $$X^k := (X_1, \dots, X_k), \quad Y^k := (Y_1, \dots, Y_k).$$

Theorem: Guessing Moments (Cont.)

1. For every deterministic function $f \in \mathcal{F}_{n,m}$, and for all $\rho > 0$,

$$\frac{1}{k} \log \frac{\mathbb{E}\big[g_{X^k}^{\rho}(X^k)\big]}{\mathbb{E}\big[g_{Y^k}^{\rho}(Y^k)\big]} \geq \rho \left[H_{\frac{1}{1+\rho}}(X) - H_{\frac{1}{1+\rho}}(\widetilde{X}_m) \right] - \frac{\rho \log(1 + k \ln n)}{k}.$$

Theorem: Guessing Moments (Cont.)

1. For every deterministic function $f \in \mathcal{F}_{n,m}$, and for all $\rho > 0$,

$$\frac{1}{k} \log \frac{\mathbb{E}\big[g^{\rho}_{X^k}(X^k)\big]}{\mathbb{E}\big[g^{\rho}_{Y^k}(Y^k)\big]} \geq \rho \left[H_{\frac{1}{1+\rho}}(X) - H_{\frac{1}{1+\rho}}(\widetilde{X}_m) \right] - \frac{\rho \log(1 + k \ln n)}{k}.$$

2. For the deterministic function $f^* \in \mathcal{F}_{n,m}$, constructed by Huffman algorithm, with $Y_i = f^*(X_i)$ for all $i \in \{1, \ldots, k\}$, we have

$$I\big(X;\, f^*(X)\big) \geq \max_{f \in \mathcal{F}_{n,m}} I\big(X;\, f(X)\big) - 0.08607 \text{ bits},$$

and, for all $\rho > 0$,

$$\frac{1}{k} \log \frac{\mathbb{E}\big[g^{\rho}_{X^k}(X^k)\big]}{\mathbb{E}\big[g^{\rho}_{Y^k}(Y^k)\big]} \leq \rho \left[H_{\frac{1}{1+\rho}}(X) - H_{\frac{1}{1+\rho}}(\widetilde{X}_m) + v\left(\frac{1}{1+\rho}\right) \right] + \frac{\rho \log(1 + k \ln m)}{k}.$$

Theorem: Guessing Moments (Cont.)

For every $\rho > 0$,

3. The gap between the universal lower bound on $\frac{1}{k} \log \frac{\mathbb{E}\left[g^{\rho}_{X^k}(X^k)\right]}{\mathbb{E}\left[g^{\rho}_{Y^k}(Y^k)\right]}$, for all $f \in \mathcal{F}_{n,m}$ (with $Y_i = f(X_i)$), and the upper bound with the specific function $f = f^* \in \mathcal{F}_{n,m}$, is at most

$$\rho\, v\left(\frac{1}{1+\rho}\right) + \frac{2\rho \log(1 + k \log_{\mathrm{e}} n)}{k} \approx \frac{0.08607\, \rho}{1+\rho} + O\left(\frac{\log k}{k}\right) \text{ bits}$$

while $f = f^*$ also almost achieves the maximal mutual information of $I\big(X; f(X)\big)$ up to a difference of 0.08607 bits.

Letting $k \to \infty$, the gap in the normalized ratio of the ρ-th guessing moments is less than 0.08607 bits for all $\rho > 0$, and the construction of the function $f^* \in \mathcal{F}_{n,m}$ does not depend on ρ.

Application II:

Non-Asymptotic Bounds for Optimal Fixed-to-Variable Lossless Compression Codes

We rely on Campbell's work (1965), providing bounds on the cumulant generating function which are expressed in terms of Rényi entropies.

The cost function is exponential in the length of the codeword, penalizing for long sequences.

The details here are skipped, and appear in the journal paper.

Journal Paper

I. Sason, "Tight bounds on the Rényi entropy via majorization with applications to guessing and compression," *Entropy*, vol. 20, no. 12, paper 896, pp. 1–25, November 2018.

Journal Paper

I. Sason, "Tight bounds on the Rényi entropy via majorization with applications to guessing and compression," *Entropy*, vol. 20, no. 12, paper 896, pp. 1–25, November 2018.

A Forthcoming Journal Paper

I. Sason, "On data-processing and majorization inequalities for f-divergences with applications," to appear in the *Entropy* journal, end of October 2019.

Summary

- Tight bounds on the Rényi entropy of a function of a random variable with finite number of possible values where the function is not 1-to-1, and the cardinality of the image of the function is fixed.
- A tight lower bound on the Rényi entropy of a discrete random variable with a finite support is derived as a function of the size of the support, and the ratio of the maximal to minimal probability masses.
- Inspired by the recent work by Cicalese *et al.*, which was focused on the Shannon entropy, we strengthen and generalize their results to Rényi entropies of arbitrary positive orders.
- In view of the generalized bounds and the works by Arikan and Campbell, non-asymptotic bounds are derived for guessing moments and lossless data compression of discrete memoryless sources.