Improved Upper Bounds on the Ensemble Performance of ML Decoded Low Density Parity Check Codes

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***Abstract*—In this letter, we study improved upper bounds on the performance of low density parity check codes over binary-input additive white Gaussian noise channels, assuming that the codes are maximum-likelihood decoded. Our results demonstrate the phenomenal performance of the low density parity check codes.**

***Index Terms*—AWGN channel, distance spectrum, error bounds, error floor, low density parity check codes, maximum-likelihood decoding, Shannon capacity, turbo codes.**

I. Introduction

LOW DENSITY parity check (LDPC) codes were rediscovered by MacKay and Neal [4], [5], after being first introduced by Gallager in his seminal work [2], [3]. The experiments with LDPC codes indicate that like turbo codes [1], they exhibit low bit error rates at low signal-to-noise ratios. The performance of LDPC codes was extensively investigated since and their exceptional performance with the suboptimal and practical iterative decoding algorithm was reported in the literature, as evidenced in [2]–[11] and references therein.

In this paper, the phenomenal performance of LDPC codes with large block length and maximum-likelihood (ML) decoding is demonstrated by the tangential sphere bound. This improved upper bound on the block error probability for a binary-input additive white Gaussian noise (AWGN) channel [8], [9] depends on the distance spectrum of the code, and it is not subject to problems typical for the more popular union bounds in regions of low signal-to-noise ratios. The technique is applied in [12]–[14] for upper bounding the performance of uniformly interleaved turbo and turbo-block codes with structured or random component codes, operating over an AWGN channel and ML decoded. These upper bounds are also compared in [12]–[14] to simulation results of the "Log-MAP" iterative decoding algorithm, indicating in various cases the suboptimality of the iterative decoding algorithm at low values of energy per bit-to-noise spectral density (E_b/N_0), as compared to the optimal ML decoding algorithm. The use of the tangential sphere bound to upper bound the performance of uniformly interleaved turbo codes was limited to interleavers of length not exceeding some few thousands of bits because of inherent overflow limitations associated with the numerical calculation of the code distance spectrum.

This difficulty is circumvented in the case of upper bounding the ensemble performance of LDPC codes even with *large* block length due to the existing *exponential* upper bound on the ensemble distance spectrum of (n, j, k) binary LDPC codes, derived by Gallager [3].

Hence, for these ensembles of (n, j, k) LDPC codes, we combine here Gallager's upper bound on the code distance spectrum with the tangential sphere bounding technique, yielding a tight upper bound on the block error probability for these codes, ML decoded and coherently detected. For simplicity, we assume here that the signaling is antipodal (although it is not necessary) and that the modulated signals possess the same energy for each one of the codewords.

The letter is organized as follows: in Section II, we define briefly the ensemble of LDPC codes, adhering to Gallager [3]. In Section III, our main results are presented followed by a summary and conclusions in Section IV.

II. Preliminaries

The considered ensembles of (n, j, k) binary LDPC codes which are presented by Gallager in [3, Sec. 2.B], are specified by a sparse parity check matrix containing mostly 0's and only a relatively small number of 1's. An (n, j, k) LDPC code is a block code of length n with a parity check matrix H, where each column contains a small fixed number j $(j \geq 3)$ of 1's and each row contains a small fixed number k of 1's (where $k > j$) [2]. It follows easily from this definition that the rate R of an (n, j, k) LDPC code satisfies the inequality $R \geq 1 - j/k$ [3].

Following Gallager [2], [3], we will restrict our attention to the following ensemble of (n, j, k) LDPC codes which we define via their parity check matrices in the following way: Divide the parity check matrix H into j submatrices, each containing a single 1 in each column. The first of these submatrices contains all its 1's in descending order: that is, the ith row contains 1's in columns $(i-1)k+1$ to ik. The other submatrices are column permutations of the first. The ensemble of (n, j, k) LDPC codes discussed here is then the ensemble of all codes resulting from random permutations of the columns of each of the bottom $j-1$ submatrices in the matrix H, with equal probability. Assuming that all rows of the parity check matrix are independent, the rate of an (n, j, k) code is equal to $1 - j/k$. In general the rate is always lower bounded by $1 - j/k$ and it will typically be very

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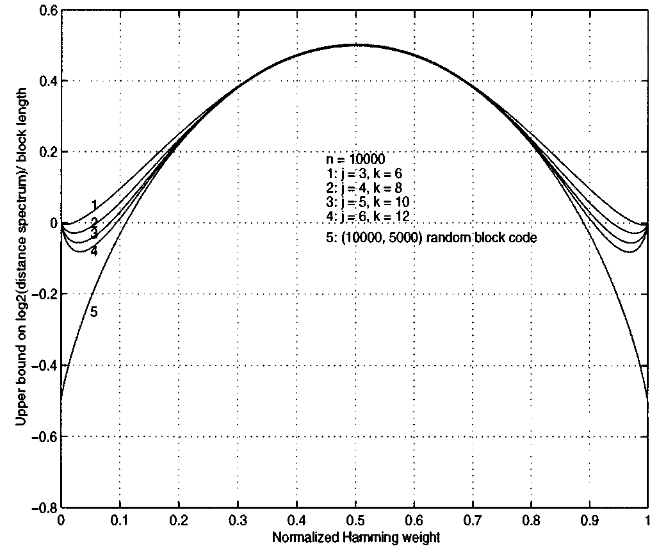


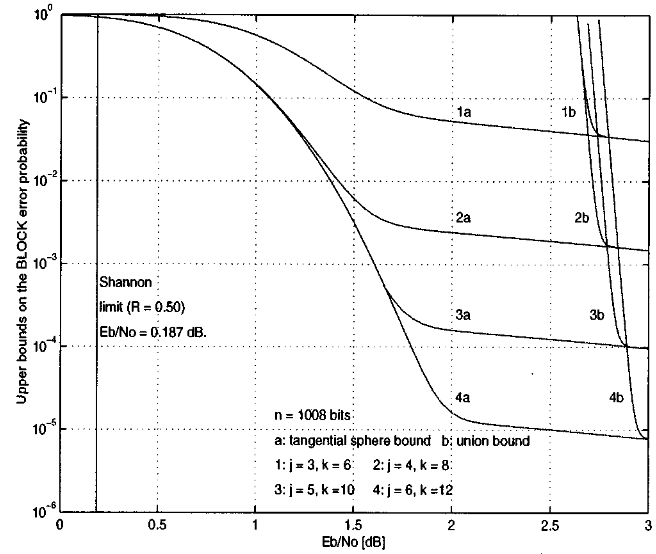
Fig. 1. The normalized logarithm (base 2) of Gallager's upper bound on the ensemble distance spectrum of (n, j, k) LDPC codes (normalized by the block length n), versus the normalized Hamming weights of the codewords, for n = 10 000, j = 3, 4, 5, 6, and $k = 2j$. The code rate of the ensembles of LDPC codes approaches one half. It is also compared to the distance spectrum of the ensemble of fully random block codes with rate $1/2$ and length 10 000.

close to this quantity for large values of n and relatively small values of j and k. Gallager's upper bound on the ensemble distance spectrum of (n, j, k) binary LDPC codes is derived in [3, Sec. 2.2].

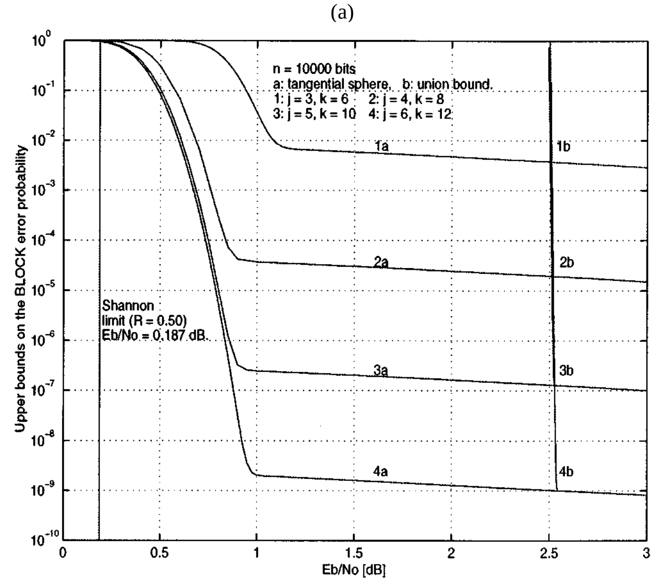
III. RESULTS

Gallager's upper bound on the ensemble distance spectrum of (n, j, k) LDPC codes is presented in Fig. 1 for some ensembles with rate one half and a block length of 10 000 coded bits. The shaping of the distance spectrum of these ensembles of codes is rather typical also for larger block lengths and other rates. However, for every pair of (j, k) values, there is an effective minimal Hamming weight increasing linearly with n for large block length. This observation is discussed in [3, Sec. 2.2] and is also demonstrated by Fig. 1. Moreover, it is demonstrated in Fig. 1 that for relatively small values of j, k, increasing the value of j and k such that the ratio of j/k is kept fixed (i.e., keeping the code rate constant), reduces the number of codewords of low Hamming weights, thus improving the error floor for these ensembles of LDPC codes, as is also indicated in Fig. 2(a) and (b) here. Increasing the value of j and k (as above), for a fixed block length n, affects therefore the upper bound on the distance spectrum in similar manner to "spectral thinning" [7], which characterizes the influence of increasing the interleaver length on the distance spectrum of turbo codes. That justifies the improvement in the ensemble performance of (n, j, k) LDPC codes for fixed n, as is also reflected in Fig. 2(a) and (b).

Upper bounds on the performance of ML decoding for some ensembles of (n, j, k) LDPC codes with rate one half are presented in Fig. 2(a) and (b), and the effect of the block length n on the closeness to the Shannon capacity limit is also demonstrated. The error bounds are based on Gallager's upper bound on the ensemble distance spectrum of LDPC codes combined



(a)



(b)

Fig. 2. Upper bounds on the block error probability of ML decoding for ensembles of LDPC codes in a binary-input AWGN channel where $j = 3, 4, 5, 6$, and $k = 2j$ (rate of one half). The upper bounds are based on Gallager's upper bound on the ensemble distance spectrum of (n, j, k) LDPC codes and the tangential sphere bound. Union bounds in Q-form appear for comparison: (a) n= 1008 coded bits and (b) n = 10 000 coded bits.

with two examined upper bounds on the ML decoding error probability: the tangential sphere bound which is an improved upper bound and the ubiquitous union bound in Q-form. The two error bounds are also compared in Fig. 2(a) and (b), demonstrating the fact that for large block lengths, the union bounds are essentially useless at rates above the cutoff rate (for a binary-input AWGN channel and a rate one half, the cutoff rate corresponds to $(E_b/N_0) = 2.45$ dB). In Fig. 2(a) and (b), the same values of (j, k) for ensembles of (n, j, k) LDPC codes are compared, for block lengths of $n = 1000$ coded bits, reflecting the dramatic improvement that results by increasing n by a factor of 10.

The considerable advantage of the tangential sphere bounding technique over the union bounds (especially for large

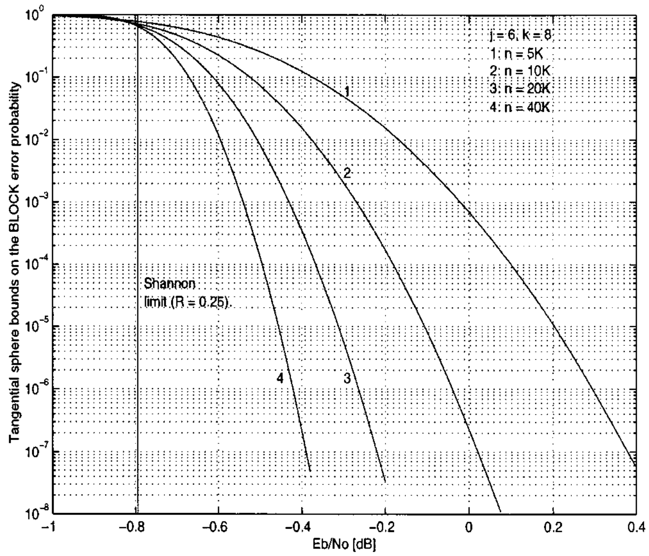


Fig. 3. Upper bounds on the block error probability of ML decoding for some ensembles of $(n,\, j,\, k)$ LDPC codes of rate 1/4 in a binary-input AWGN channel. The upper bounds are based on Gallager's upper bound on the ensemble distance spectrum of $(n,\, j,\, k)$ LDPC codes and the tangential sphere bound.

TABLE I
VALUE OF ENERGY PER BIT–TO–NOISE SPECTRAL DENSITY E_b/N_0 REQUIRED FOR AN UPPER BOUND ON THE BLOCK ERROR PROBABILITY OF 10^{-5} WITH THE ML DECODING FOR THE ENSEMBLE OF $(n,\, j,\, k)$ LDPC CODES WHERE $j = 6$. BOUNDS ARE BASED ON THE TANGENTIAL SPHERE UPPER BOUND FOR EQUI-ENERGY SIGNALS OF AN ANTIPODAL MODULATION. THE GAP (IN DECIBELS) BETWEEN VALUES OF E_b/N_0 ACHIEVING UPPER BOUND ON THE BLOCK ERROR PROBABILITY OF 10^{-5} (FOR ML DECODING) TO THE SHANNON CAPACITY OF A BINARY-INPUT AWGN CHANNEL ARE IN PARENTHESES

The number of ones (k) in each row of the parity matrix H of the ensemble of codes and a tight lower bound on the rate (R).	$k = 8$ $R = 0.250$	$k = 10$ $R = 0.400$	$k = 12$ $R = 0.500$	$k = 18$ $R = 0.667$	$k = 24$ $R = 0.750$
The block length n	The value of $\frac{E_b}{N_0}$ (and the gap from Shannon capacity)				
$n = 5$K bits	0.20 dB (0.99 dB)	0.59 dB (0.83 dB)	1.00 dB (0.81 dB)	1.93 dB (0.87 dB)	2.57 dB (0.94 dB)
$n = 10$K bits	-0.11 dB (0.68 dB)	0.37 dB (0.61 dB)	0.79 dB (0.60 dB)	1.74 dB (0.68 dB)	2.39 dB (0.76 dB)
$n = 20$K bits	-0.31 dB (0.48 dB)	0.20 dB (0.44 dB)	0.64 dB (0.45 dB)	1.61 dB (0.55 dB)	2.26 dB (0.63 dB)
$n = 40$K bits	-0.46 dB (0.33 dB)	0.09 dB (0.33 dB)	0.54 dB (0.35 dB)	1.52 dB (0.47 dB)	2.18 dB (0.55 dB)

block lengths) is demonstrated in Fig. 2(a) and (b) (see also [12]–[14]). For example, the gain achieved for a block error probability of 10^{-5} is 1.72 dB in the case that $n = 10$ K, $j = 6$, and $k = 12$. As exhibited in Fig. 2(b), there is only a slight improvement in the ensemble performance of $(n,\, j,\, k)$ LDPC codes, by increasing the value of j above 6 (while keeping $k = 2j$ for maintaining the code rate fixed). Therefore, the ML block error probabilities of ensembles of $(n,\, j,\, k)$ LDPC codes, are investigated here for $j = 6$ and a variety of rates and block lengths of $n = 5$, 10, 20, and 40 K (see Table I). This comparison demonstrates the impressive potential performance of ensembles of LDPC codes of length in the range 5–40 K. Fig. 3 presents results for ensembles of $(n,\, j,\, k)$ LDPC codes of rate 0.250. For example in the case of $j = 6$, $k = 8$, and $n = 40$ K, an upper bound on the value of E_b/N_0 required to achieve a block error probability of 10^{-5} with ML decoding is -0.46 dB, that is only 0.33 dB away from the channel capacity (see also Table I).

IV. SUMMARY

We have studied the ensemble performance of ML decoded LDPC codes using upper bounds for a binary-input AWGN channel. The bounds are based on Gallager's upper bound on the ensemble distance spectrum of LDPC codes [3] (see Fig. 1), combined with Poltyrev's tangential sphere upper bound on the block error probability [7]. The influence of the parameters of these ensembles of LDPC codes on their performance is investigated and their closeness to Shannon capacity is demonstrated for LDPC codes of length 5000–40 000 (coded bits) with a variety of rates (see Figs. 2, 3, and Table I). Simulation results of the iterative decoding algorithm for turbo and LDPC codes [1], [4]–[6], [8], [9] and also upper bounds on the ML decoding error probability of these codes [12]–[14], substantiates the conclusion that LDPC codes are strong alternatives for high performance communication systems, striving to approach the ultimate limit of channel capacity.

REFERENCES

[1] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error correcting coding and decoding: Turbo codes," in *Proc. 1993 IEEE Int. Conf. on Communications*, Geneva, Switzerland, 1993, pp. 1064–1070.
[2] R. G. Gallager, "Low density parity check codes," *IRE Trans. Inform. Theory*, vol. IT-8, pp. 21–28, Jan. 1962.
[3] ——, *Low Density Parity Check Codes*. Cambridge, MA: M.I.T Press, 1963.
[4] D. J. C. Mackay and R. M. Neal, "Near Shannon limit performance of low density parity check codes," *Electron. Lett.*, vol. 32, pp. 1645–1646, Aug. 1996.
[5] ——, "Near Shannon limit performance of low density parity check codes," *Electron. Lett.*, vol. 33, pp. 457–458, Mar. 1997, Reprint..
[6] D. J. C. MacKay, "Good error-correcting codes based on very sparse matrices," *IEEE Trans. Inform. Theory*, vol. 5, pp. 399–431, Mar. 1999.
[7] L. C. Perez, J. Seghers, and D. J. Costello, "A distance spectrum interpretation of turbo codes," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1698–1709, Nov. 1996.
[8] G. Poltyrev, "Bounds on the decoding error probability of binary linear codes via their spectra," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1284–1292, July 1994.
[9] H. Herzberg and G. Poltyrev, "The error probability of M-ary PSK block coded modulation schemes," *IEEE Trans. Commun.*, vol. 44, pp. 427–433, Apr. 1996.
[10] T. Richardson, A. Shokorollahi, and R. Urbanke. (1999, Apr.) Design of provably good low-density parity check codes. Bell Labs., Lucent Technol., Tech. Rep.. [Online] Preprint available: http://carwww.epfl.ch/~ruediger/publications.html
[11] ——, "Design of provably good low-density parity check codes," IEEE Trans. Inform. Theory, submitted for publication.
[12] I. Sason and S. Shamai (Shitz), "Improved upper bounds on the decoding error probability of parallel and serial concatenated turbo codes via their ensemble distance spectrum," *IEEE Trans. Inform. Theory*, vol. 46, pp. 24–47, Jan. 2000.
[13] ——, "Improved upper bounds on the decoding error probability of parallel and serial concatenated turbo codes via their ensemble distance spectrum," in *Proc. 1998 IEEE Int. Symp. on Information Theory (ISIT'98)*. Cambridge, MA, Aug. 16–21, 1998, p. 30.
[14] ——, "Bounds on the error probability of ML decoding for block and turbo-block codes," *Annales Telecommun.*, vol. 54, no. 3/4, pp. 183–200, Mar./Apr. 1999.