On universal LDPC code ensembles over memoryless symmetric channels

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Outline

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Motivation

- Numerical methods (such as density evolution) enable to design capacity-approaching degree distributions of LDPC code ensembles for a particular channel.

Question:

How to design LDPC code ensembles that will **provably** *operate reliably over a range of channels?*

Selected Previous Works on Universal LDPC Codes

- M. Franceschini, G. Ferrari, and R. Raheli, *"Does the performance of LDPC codes depend on the channel?,"* IEEE Trans. on COM., vol. 54, no. 12, pp. 2129-2132, December 2006.
- F. Peng, W. E. Ryan and R. D. Wesel, *"Surrogate-channel design of universal LDPC codes,"* IEEE COM. Letters, vol. 10, no. 6, pp. 480–482, June 2006.
- A. Sanaei, M. Ramezani and M. Ardakani, *"Identical-capacity channel decomposition for design of universal LDPC codes,"* IEEE Trans. on COM., vol. 57, no. 7, pp. 1972–1981, July 2009.

The works above approach the problem of universality from a numerical standpoint. In the present work the approach is analytical, albeit may not achieve the best results numerically.

Preliminaries

- We consider communication over various families of memoryless, binary-input, output-symmetric (MBIOS) channels.
- In this work, we consider two decoding methods for LDPC codes:
  - **Belief Propagation** (BP) decoding, which is a suboptimal, iterative, decoding technique.
  - **Maximum Likelihood** (ML) decoding, which is an optimal decoder, yet it is prohibitively complex.

Density Evolution

- Density evolution is a tool to assess the asymptotic performance of an LDPC codes ensemble under BP decoding for MBIOS channels.
- It tracks the evolution of message densities during BP decoding.
- Multi-dimensional equation $\rightarrow$ difficult to analyze in general.
- A fruitful approach (introduced by Burshtein and Miller[1]) is to provide bounds on a reduction of density-evolution to a single parameter.
- In this work, we follow this approach, with bounds on the B-parameter.

[1] D. Burshtein and G. Miller, "Bounds on the performance of belief propagation decoding," *IEEE Trans. on I-T*, vol. 48, no. 1, pp. 112–122, January 2002

Useful Functionals

For an MBIOS channel with L-density a,

- *The capacity functional*:

$$C(a) = \int_{-\infty}^{\infty} a(x)\big(1 - \log_2(1 + e^{-x})\big)\,\mathrm{d}x.$$

- *The Bhattacharyya functional*:

$$\mathcal{B}(a) = \int_{-\infty}^{\infty} a(x)e^{-\frac{x}{2}}\,\mathrm{d}x.$$

- *The uncoded bit error probability functional*:

$$\mathcal{E}(a) = \frac{1}{2}\int_{-\infty}^{\infty} a(x)\,e^{-(|\frac{x}{2}|+\frac{x}{2})}\,\mathrm{d}x.$$

A Useful Relationship

- The following relationship between the Bhattacharyya and bit error probability functionals can be shown:

$$2\mathcal{E}(a) \leq \mathcal{B}(a) \leq 2\sqrt{\mathcal{E}(a)\big(1-\mathcal{E}(a)\big)}.$$

- In particular,

$$\mathcal{E}(a) \to 0 \Leftrightarrow \mathcal{B}(a) \to 0.$$

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A Condition for Convergence

- Consider an arbitrary MBIOS channel with L-density a_0, and LDPC code ensemble (λ, ρ).
- Denote by a_l the pdf of the left-to-right message in the l-th iteration of BP decoding.
- Denote $x_l = \mathcal{B}(a_l)$. It follows from density evolution that

$$x_l \leq \mathcal{B}(a_0)\,\lambda\big(1 - \rho(1 - x_{l-1})\big), \quad l = 1, 2, \ldots$$

- Since $\mathcal{E}(a) \to 0 \Leftrightarrow \mathcal{B}(a) \to 0$, the bit error probability vanishes as the number of iterations grow iff $x_l \to 0$.

General Approach for Universal Achievability

- Consider an arbitrary **set** of MBIOS channels and let $\mathcal{A}$ designate the corresponding set of its L-densities.
- **Goal:** design an LDPC code ensemble with degree distributions (λ, ρ) that asymptotically achieves bit-error probability $\rightarrow 0$ over $\mathcal{A}$.
- Let

$$B \triangleq \max_{a \in \mathcal{A}} \mathcal{B}(a).$$

- Consider the recursive equation

$$y_l = B\,\lambda\big(1 - \rho(1 - y_{l-1})\big), \quad l = 1, 2, \ldots$$

with initial condition $y_0 = B$.
  - This refers to the density evolution of a BEC with erasure probability B.

General Approach for Universal Achievability – cont.

- Compare:

$$x_0 = \mathcal{B}(a_0); \quad x_l \le \mathcal{B}(a_0)\,\lambda\big(1-\rho(1-x_{l-1})\big), \quad l = 1, 2, \ldots$$
$$y_0 = B; \qquad y_l = B\,\lambda\big(1-\rho(1-y_{l-1})\big), \qquad l = 1, 2, \ldots$$

- Since by definition, $\mathcal{B}(a_0) \le B$, it follows by induction that for every $l \ge 0$ and $a \in \mathcal{A}$,

$$0 \le x_l \le y_l.$$

General Approach for Universal Achievability – cont.

- Compare:

$$x_0 = \mathcal{B}(a_0); \quad x_l \leq \mathcal{B}(a_0)\,\lambda\big(1-\rho(1-x_{l-1})\big), \quad l = 1, 2, \ldots$$
$$y_0 = B; \quad y_l = B\,\lambda\big(1-\rho(1-y_{l-1})\big), \quad l = 1, 2, \ldots$$

- Since by definition, $\mathcal{B}(a_0) \leq B$, it follows by induction that for every $l \geq 0$ and $a \in \mathcal{A}$,

$$0 \leq x_l \leq y_l.$$

- Recall: Bit error probability $\to 0 \Leftrightarrow x_l \to 0$.

General Approach for Universal Achievability – cont.

- Compare:

$$\begin{aligned} x_0 &= \mathcal{B}(a_0); \quad x_l \le \mathcal{B}(a_0)\,\lambda\big(1-\rho(1-x_{l-1})\big), \quad l=1,2,\dots \\ y_0 &= B; \qquad\;\; y_l = B\,\lambda\big(1-\rho(1-y_{l-1})\big), \qquad\quad l=1,2,\dots \end{aligned}$$

- Since by definition, $\mathcal{B}(a_0) \le B$, it follows by induction that for every $l \ge 0$ and $a \in \mathcal{A}$,

$$0 \le x_l \le y_l.$$

- Recall: Bit error probability $\to 0 \Leftrightarrow x_l \to 0$.
- Therefore, selecting a pair of degree distributions (λ, ρ) such that $y_l \to 0$ yields $x_l \to 0$ for *every* MBIOS channel from the set $\mathcal{A}$.

General Approach for Universal Achievability – cont.

- Compare:

$$x_0 = \mathcal{B}(a_0); \quad x_l \le \mathcal{B}(a_0)\,\lambda\big(1-\rho(1-x_{l-1})\big), \quad l = 1, 2, \ldots$$
$$y_0 = B; \qquad y_l = B\,\lambda\big(1-\rho(1-y_{l-1})\big), \qquad l = 1, 2, \ldots$$

- Since by definition, $\mathcal{B}(a_0) \le B$, it follows by induction that for every $l \ge 0$ and $a \in \mathcal{A}$,

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- Recall: Bit error probability $\to 0 \Leftrightarrow x_l \to 0$.
- Therefore, selecting a pair of degree distributions (λ, ρ) such that $y_l \to 0$ yields $x_l \to 0$ for *every* MBIOS channel from the set $\mathcal{A}$.

The LDPC code ensemble (λ, ρ) is universal over the entire set $\mathcal{A}$.

General Approach for Universal Achievability – Summary

To construct an LDPC code ensemble that is universal over a set $\mathcal{A}$ of MBIOS channels:

- Find
$$B = \max_{a \in \mathcal{A}} \mathcal{B}(a).$$
- Construct a capacity-achieving LDPC code ensemble for BEC(B) that achieves vanishing bit error probability under BP decoding.
    - For this code,
$$R_{\mathrm{d}} = 1 - B.$$
- This ensemble will achieve bit error probability $\to 0$ under BP decoding for **all** MBIOS channels in the set $\mathcal{A}$.

$\Longrightarrow$ This code is universal over the set.

Theorem (Universality of LDPC Codes under BP Decoding for Equi-Capacity MBIOS Channels)

- *Let* $\mathcal{A}$ *be a set of MBIOS channels that exhibit a given capacity* C*, and let* $B = \max_{a \in \mathcal{A}} \mathcal{B}(a)$*.*
- *Let* $\{(n, \lambda, \rho)\}$ *form a capacity-achieving sequence of LDPC code ensembles for BEC*(B)*, achieving vanishing bit erasure probability under BP decoding.*
- *Then, this sequence universally achieves vanishing bit error probability under BP decoding for the entire set* $\mathcal{A}$*, and the design rate of this sequence forms a fraction that is at least* $\frac{1-B}{C}$ *of the channel capacity.*

Application of the Theorem

- *The family of equi-capacity MBIOS channels with capacity C*
  - The BSC exhibits the maximal B-parameter.
  - The asymptotic achievable fraction of channel capacity is

$$\mu_1(C) \triangleq \frac{R_{\mathrm{d}}}{C} = \frac{1 - \sqrt{4h_2^{-1}(1-C)\left(1 - h_2^{-1}(1-C)\right)}}{C}.$$

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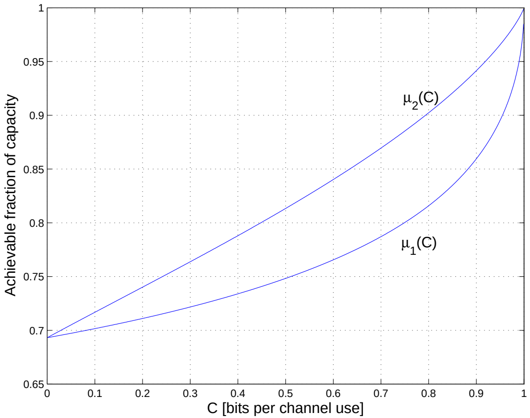
- *BEC and BIAWGNC with capacity C*
  - The BIAWGNC exhibits the maximal B-parameter, B (computed numerically).
  - The asymptotic achievable fraction of channel capacity is

$$\mu_2(C) = \frac{1-B}{C}.$$

Universal achievable fraction of capacity under BP decoding



μ_1: The entire set of equi-capacity MBIOS channels.

μ_2: BEC and BIAWGNC with capacity C bits per channel use.

Outline

LP Bounds on the Achievable Rate of LDPC Code Ensembles

- Let $\mathcal{A}$ be a family of equi-capacity MBIOS channels.
- Consider a right-regular LDPC code ensemble with left-degree distribution λ and right-degree $d_{\rm c}$ operating over any channel from $\mathcal{A}$.
- Design rate:

$$R_{\rm d} = 1 - \frac{1}{d_{\rm c} \sum_{i=2}^{d_{\rm v}^{\max}} \lambda_i / i}$$

  Therefore, maximization of $R_{\rm d} \iff$ maximization of $\sum_{i=2}^{d_{\rm v}^{\max}} \lambda_i / i$.
- A suitable LP on λ will provide an upper bound on achievable rate of right-regular LDPC code ensembles over any channel from $\mathcal{A}$ $\implies$ lower bound on gap to capacity, $\varepsilon = 1 - R_{\rm d}/C$.

LP1 Bound

Find λ that maximizes $\sum_{i=2}^{d_{\mathrm{v}}^{\max}} \lambda_i/i$ subject to:

- Necesseary conditions for achieving vanishing bit error probability under BP decoding over $\mathcal{A}$.
- Constraints on λ to form a valid probability distribution.
- Universal inequality constraints on the design rate to achieve bit error probability $\to 0$ over $\mathcal{A}$.

Necesseary conditions for vanishing bit error probability under BP decoding – 1

Theorem (A Necessary Condition for Universality of LDPC Code Ensembles under BP Decoding)

Let $\{(n, \lambda, \rho)\}$ *be a* **right-regular** *sequence of LDPC code ensembles, universally achieving vanishing bit error probability under BP decoding for a set of MBIOS channels* $\mathcal{A}$. *Then, the following condition holds*

$$B\lambda\left(\sqrt{1-\rho(1-x^2)}\right) < x, \quad \forall\, x \in (0, B]$$

where B *designates the maximal Bhattacharyya parameter over the set* $\mathcal{A}$.

Necesseary conditions for vanishing bit error probability under BP decoding – 2

- Another necessary condition for asymptotically achieving vanishing bit error probability under BP decoding is the stability condition,

$$\mathcal{B}(a)\lambda'(0)\rho'(1) < 1.$$

- It should be satisfied for all channels in $\mathcal{A}$.

Necesseary conditions for vanishing bit error probability under BP decoding – 2

- Another necessary condition for asymptotically achieving vanishing bit error probability under BP decoding is the stability condition,

$$\mathcal{B}(a)\lambda'(0)\rho'(1) < 1.$$

- Denote

$$B = \max_{a \in \mathcal{A}} \mathcal{B}(a).$$

- If

$$B\lambda'(0)\rho'(1) \leq 1$$

then the stability condition is satisfied for every channel in $\mathcal{A}$.

Universal inequality constraints on the design rate

- An LDPC code that achieves bit error probability $\rightarrow 0$ over an MBIOS channel with capacity C must satisfy:[2]

$$0 \le R_{\mathrm{d}} \le 1 - \frac{1-C}{h_2\left(\frac{1-C^{\frac{a_{\mathrm{R}}}{2}}}{2}\right)}.$$

- Using $a_{\mathrm{R}} = d_{\mathrm{c}}$ for a right-regular ensemble and the expression for R_{d} we obtain:

$$\frac{1}{d_{\mathrm{c}}} \le \sum_{i=2}^{d_{\mathrm{v}}^{\max}} \frac{\lambda_i}{i} \le \frac{1}{(1-C)d_{\mathrm{c}}} \cdot h_2\left(\frac{1-C^{\frac{d_{\mathrm{c}}}{2}}}{2}\right).$$

[2] I. Sason, *"On universal properties of capacity-approaching LDPC code ensembles,"* IEEE Trans. on I-T, vol. 55, no. 7, pp. 2956 – 2990, July 2009.

LP1 Bound

Find λ that maximizes $\sum_{i=2}^{d_{\mathrm{v}}^{\max}} \lambda_i/i$ subject to:

- Necesseary conditions for achieving vanishing bit error probability under BP decoding over a set $\mathcal{A}$ of MBIOS channels with maximal B-parameter B:
  - $B\lambda\left(\sqrt{1-\rho(1-x^2)}\right) < x, \quad \forall\, x \in (0, B]$
  - $B\lambda_2\rho'(1) \leq 1$
- Constraints on λ to form a valid probability distribution:
  - $\lambda_i \geq 0 \quad i = 2, 3, \ldots$
  - $\sum_{i=2}^{d_{\mathrm{v}}^{\max}} \lambda_i = 1$
- Universal inequality constraints on the design rate to achieve bit error probability $\to 0$ over $\mathcal{A}$.

LP2 Bound

- A possible improvement of the bound is obtained by considering the case where the BEC is in the set of equi-capacity MBIOS channels $\mathcal{A}$.
- *New necessary condition:* for a BEC, the condition for achieving vanishing bit erasure probability under BP decoding is:

$$(1-C)\lambda(1-\rho(1-x)) < x, \quad \forall\, 0 < x \leq 1-C.$$

- Other conditions:
  - The stability condition is satisfied for the entire set $\mathcal{A}$.
  - Conditions on λ to form a valid probability distribution.
  - Universal inequality constraints on the design rate to achieve bit error probability $\to 0$ over $\mathcal{A}$.

LP Bound Results

- Tables show lower bounds on $\varepsilon = 1 - R_{\mathrm{d}}/C$.
- LP1 bound results:

Capacity	Set of all Equi-Capacity Channels			BEC + BIAWGNC		
(C)	$d_c = 8$	$d_c = 10$	$d_c = 12$	$d_c = 8$	$d_c = 10$	$d_c = 12$
$\frac{1}{2}$	$2.83 \cdot 10^{-3}$	$7.05 \cdot 10^{-4}$	$1.76 \cdot 10^{-4}$	$2.83 \cdot 10^{-3}$	$7.05 \cdot 10^{-4}$	$1.76 \cdot 10^{-4}$
$\frac{3}{4}$	$9.09 \cdot 10^{-2}$	$1.79 \cdot 10^{-2}$	$7.84 \cdot 10^{-3}$	$7.90 \cdot 10^{-2}$	$1.43 \cdot 10^{-2}$	$7.84 \cdot 10^{-3}$
$\frac{9}{10}$	$2.06 \cdot 10^{-1}$	$1.57 \cdot 10^{-1}$	$1.20 \cdot 10^{-1}$	$1.73 \cdot 10^{-1}$	$1.33 \cdot 10^{-1}$	$1.03 \cdot 10^{-1}$

- Results for LP2 bound, in which BEC $\in \mathcal{A}$:

Capacity	Set of all Equi-Capacity Channels			BEC + BIAWGNC		
(C)	$d_c = 8$	$d_c = 10$	$d_c = 12$	$d_c = 8$	$d_c = 10$	$d_c = 12$
$\frac{1}{2}$	$1.50 \cdot 10^{-2}$	$9.01 \cdot 10^{-3}$	$1.94 \cdot 10^{-2}$	$1.25 \cdot 10^{-2}$	$6.76 \cdot 10^{-3}$	$1.73 \cdot 10^{-2}$
$\frac{3}{4}$	$9.09 \cdot 10^{-2}$	$4.24 \cdot 10^{-2}$	$2.75 \cdot 10^{-2}$	$7.90 \cdot 10^{-2}$	$3.99 \cdot 10^{-2}$	$2.42 \cdot 10^{-2}$
$\frac{9}{10}$	$2.06 \cdot 10^{-1}$	$1.57 \cdot 10^{-1}$	$1.20 \cdot 10^{-1}$	$1.73 \cdot 10^{-1}$	$1.33 \cdot 10^{-1}$	$1.03 \cdot 10^{-1}$

- Note: columns differ in maximal B-parameter over the set.

Outline

Theorem (Universal Conditions on the Bhattacharyya parameter for Good/ Bad Communications under BP Decoding)

Let $\{(n, \lambda, \rho)\}$ be a sequence of LDPC code ensembles whose block lengths $\rightarrow \infty$. The following universal properties hold under BP decoding:

- *This sequence achieves bit error probability $\rightarrow 0$ under BP decoding for every MBIOS channel whose B-parameter is less than*

$$B_0(\lambda, \rho) \triangleq \inf_{x \in (0,1]} \frac{x}{\lambda\big(1 - \rho(1-x)\big)}.$$

- *For a right-regular sequence, it does not achieve reliable communications over any MBIOS channel whose B-parameter is greater than*

$$B_1(\lambda, \rho) \triangleq \inf_{x \in (0,1]} \frac{x}{\lambda\big(\sqrt{1 - \rho(1-x^2)}\big)}.$$

Theorem (cont.)

For every MBIOS channel whose B-parameter B *satisfies* $B > B_1(\lambda, \rho)$*, BP decoding is not reliable in the sense that the left-to-right message error probability (i.e., the average probability of error for a message emanating from a variable node to a parity-check node) is greater than the positive value*

$$\left(\frac{1}{2}\max\left\{x \in (0,1] : \frac{x}{\lambda\left(\sqrt{1-\rho(1-x^2)}\right)} \leq B\right\}\right)^2$$

irrespective of the number of iterations performed by the BP decoder.

A proof of this theorem follows from fixed-point analyses based on the general approach presented above and the necessary condition shown in the previous section.

If all we know is that $B > B_1(\lambda, \rho)$ we can still lower-bound the left-to-right message error probability:

Corollary (1)

For every MBIOS channel with B-parameter $B > B_1(\lambda, \rho)$, define

$$x(B) \triangleq \max\left\{x \in (0,1] :\ \frac{x}{\lambda\left(\sqrt{1-\rho(1-x^2)}\right)} \leq B\right\}.$$

Then the (average) left-to-right message error probability is bounded away from zero by the universal bound

$$\eta \triangleq \lim_{B \to B_1(\lambda,\rho)^+} \left(\frac{x(B)}{2}\right)^2$$

irrespective of the number of iterations of the BP decoder.

Corollary (2)

The left-to-right message error probability stays bounded away from zero under BP decoding for every MBIOS channel whose B-parameter is greater than

$$B_2(\lambda, \rho) \triangleq \min\left\{B_1(\lambda, \rho), \frac{1}{\lambda'(0)\rho'(1)}, \sqrt{1 - R_d^2}\right\}$$

where R_d is the design rate of the ensemble.

Results for some LDPC ensembles with $R_{\rm d} = 1/2$

$\lambda(x) = \sum_i \lambda_i x^{i-1}$	$\rho(x) = \sum_i \rho_i x^{i-1}$	B_0	B_2	B_1	η
$\lambda_3 = 1$	$\rho_6 = 1$	0.4294	0.6553	0.6553	$6.50 \cdot 10^{-2}$
$\lambda_2 = 0.4127,\ \lambda_3 = 0.1762,\ \lambda_4 = 0.1177$ $\lambda_7 = 0.1202,\ \lambda_8 = 0.1731$	$\rho_6 = 1$	0.4816	0.4846	0.7066	$8.45 \cdot 10^{-2}$
$\lambda_4 = 1$	$\rho_8 = 1$	0.3834	0.6192	0.6192	$6.58 \cdot 10^{-2}$
$\lambda_2 = 0.2879,\ \lambda_3 = 0.1222,\ \lambda_4 = 0.0905$ $\lambda_6 = 0.1174,\ \lambda_7 = 0.0300,\ \lambda_{12} = 0.0807$ $\lambda_{13} = 0.0831,\ \lambda_{32} = 0.0050,\ \lambda_{33} = 0.1831$	$\rho_8 = 1$	0.4962	0.4962	0.7146	$1.02 \cdot 10^{-1}$
$\lambda_5 = 1$	$\rho_{10} = 1$	0.3416	0.5884	0.5884	$6.18 \cdot 10^{-2}$
$\lambda_2 = 0.2226,\ \lambda_3 = 0.1013,\ \lambda_4 = 0.0504,\ \lambda_5 = 0.0646$ $\lambda_6 = 0.0445,\ \lambda_{10} = 0.1219,\ \lambda_{11} = 0.0117$ $\lambda_{24} = 0.0903,\ \lambda_{25} = 0.0678,\ \lambda_{100} = 0.2248$	$\rho_{10} = 1$	0.4988	0.4992	0.7123	$1.08 \cdot 10^{-1}$

- If $B < B_0$, then $\mathcal{E}(a_l) \to 0$ under BP decoding.
- If $B > B_2$, then $\mathcal{E}(a_l) > 0$ under BP decoding.
- If $B > B_1$, then $\mathcal{E}(a_l) > \eta$ under BP decoding.

Results for some LDPC ensembles with $R_{\mathrm{d}} = 1/2$

$\lambda(x) = \sum_i \lambda_i x^{i-1}$	$\rho(x) = \sum_i \rho_i x^{i-1}$	B_0	B_2	B_1	η
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- In general, there is a gap between B_0 and B_2.
- However, in some cases, they can be very close and sometimes even coincide.

Outline

Extensions and Remarks

- Under certain conditions on λ (e.g., $\lambda_2 = 0$),[3] vanishing **bit** error probability $\Longrightarrow$ vanishing **block** error probability. Thus, the results can be extended to universality in terms of vanishing block error probability.
- The results under BP decoding can be extended to other families of codes defined on graphs that can be analyzed via density-evolution equations (e.g. IRA codes).
- The general approach for universal achievability can be applied to other families of MBIOS channels as well (e.g., the family of MBIOS channels with equal B-parameter).

[3] H. Jin and T. J. Richardson, *"Block error iterative decoding capacity for LDPC codes,"* Proc. ISIT 2005, pp. 52–56, Adelaide, Australia, September 2005.

Outline

Universality under ML decoding

- The results for universality under BP decoding automatically extend to the ML decoding case.
- However, under ML decoding stronger results are possible:
  - Capacity can be approached **arbitrarily closely** for the entire set of equi-capacity MBIOS channels.
  - ML decoding can achieve vanishing **block** error probability.

Universality under ML decoding (Cont.)

Theorem

Under ML decoding, Gallager's regular LDPC code ensembles can be made universal for the set $\mathcal{A}$ of MBIOS channels that exhibit a given capacity C. More explicitly,

- *For any $\varepsilon > 0$, there exists a sequence of these code ensembles whose design rate forms at least a fraction $1 - \varepsilon$ of the channel capacity with vanishing block error probability for the entire set $\mathcal{A}$.*
- *The right degree of this sequence scales like $\log \frac{1}{\varepsilon}$.*

Proof Outline

- Consider an arbitrary MBIOS channel with channel capacity C.
- Utilize upper bounds on the decoding error probability of a sequence of LDPC code ensembles under ML decoding that rely on the weight distribution of the code.[4]
- Using tight bounds[5] on the average weight distribution of Gallager's ensemble, determine the parameters of a sequence of Gallager's ensembles to achieve **block** error probability $\to 0$ with $R_\mathrm{d} = (1-\varepsilon)C$.

[4] G. Miller and D. Burshtein, *"Bounds on the maximum-likelihood decoding error probability of LDPC codes,"* IEEE Trans. on I-T, vol. 47, no. 7, pp. 4793 – 4821, Nov. 2001.

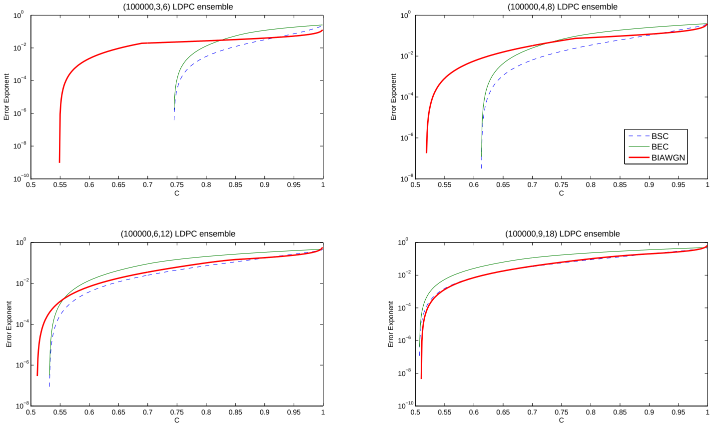
[5] I. Sason and R. Urbanke, *"Parity-check density versus performance of binary linear block codes over memoryless symmetric channels,"* IEEE Trans. on I-T, vol. 49, no. 7, pp. 1611 – 1635, July 2003.

Proof Outline

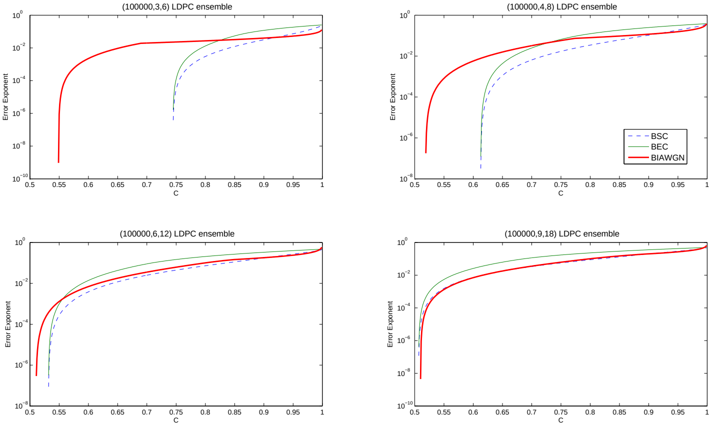
- Consider an arbitrary MBIOS channel with channel capacity C.
- Utilize upper bounds on the decoding error probability of a sequence of LDPC code ensembles under ML decoding that rely on the weight distribution of the code.[4]
- Using tight bounds[5] on the average weight distribution of Gallager's ensemble, determine the parameters of a sequence of Gallager's ensembles to achieve **block** error probability $\rightarrow 0$ with $R_{\mathrm{d}} = (1-\varepsilon)C$.
- The analysis in [5] depends *solely on the channel capacity* $\Longrightarrow$ the result is universal for the entire set of equi-capacity MBIOS channels.

[4] G. Miller and D. Burshtein, *"Bounds on the maximum-likelihood decoding error probability of LDPC codes,"* IEEE Trans. on I-T, vol. 47, no. 7, pp. 4793 – 4821, Nov. 2001.

[5] I. Sason and R. Urbanke, *"Parity-check density versus performance of binary linear block codes over memoryless symmetric channels,"* IEEE Trans. on I-T, vol. 49, no. 7, pp. 1611 – 1635, July 2003.



- Lower bounds on the error exponent for expurgated Gallager's regular LDPC code ensembles over various MBIOS channels.
- The codes have $R_{\mathrm{d}} = 1/2$ and block length $n = 100000$, with increasing variable and check node degrees.



- As the node degrees increase, the vanishing point of the error exponent approaches the channel capacity **regardless of the MBIOS channel**.

Universality under Random Puncturing

- Puncturing a linear block code generates a sequence of new codes with possibly higher rate.
- Random puncturing of nq bits of a mother code of length n and rate R yields a punctured code of length $n(1-q)$ and rate at most $R/(1-q)$.
- Rate reduction occurs when two different codewords of the mother code are mapped to the same codeword after puncturing.
- It is possible to provide conditions on the mother ensemble[6] such that Gallager's regular LDPC code ensembles suffer no rate reduction under random puncturing.

[6] C. Hsu and A. Anastasopoulos, *"Capacity achieving LDPC codes through puncturing,"* IEEE Trans. on I-T, vol. 54, no. 10, pp. 4698 – 4706, October 2008.

Universality under Random Puncturing

- Consider a mother code sequence with design rate R_d. Under the conditions for zero rate reduction, the punctured sequence has design rate:

$$R'_\mathrm{d} = \frac{R_\mathrm{d}}{1-q}.$$

- From [6], if the mother code sequence achieves a fraction $(1-\varepsilon)$ of capacity C, then the punctured sequence achieves a fraction $(1-\varepsilon)$ of

$$C' = \frac{C}{1-q}.$$

- The analysis in [6] relies only on the capacity of the MBIOS channel and the condition for zero rate reduction $\Longrightarrow$ the result is universal for the entire set of equi-capacity MBIOS channels.

Universality under Random Puncturing – cont.

Theorem

Let $\varepsilon > 0$ and consider a sequence of regular (n, j, k) LDPC code ensembles with design rate $R_d \geq (1 - \varepsilon)C$ that achieves block error probability $\to 0$ under ML decoding over the entire set of MBIOS channels with capacity C.

Random puncturing of a fraction q of code bits from this sequence produces a new sequence of punctured code ensembles with any desired design rate $R'_d > R_d$ such that:

- *It achieves block error probability $\to 0$ under ML decoding over the entire set of MBIOS channels with capacity $C' = \frac{C}{1-q}$.*
- *It achieves a fraction of at least $1 - \varepsilon$ of the capacity C'.*

Note: The design rate of the mother sequence must be low enough to satisfy the condition for zero rate reduction.

Outline

Summary

- An analytical method was derived for the design of universal LDPC code ensembles over an arbitrary set of MBIOS channels. These ensembles achieve vanishing bit error probability under BP decoding.
- The method was analyzed for families of MBIOS channels with a fixed capacity/B-parameter.
- These universal ensembles are easy to calculate by an analytical approach, but are not capacity-achieving under BP decoding.
- We presented LP upper bounds on the achievable rate of universal LDPC code ensembles over a set of equi-capacity MBIOS channels.

Summary – cont.

- We derived universal conditions on the B-parameter for reliable communications under BP decoding. The conditions were presented in an easy-to-compute, closed, form.
- These conditions were computed for several LDPC code ensembles.
- Under ML decoding, Gallager's regular LDPC code ensembles can be made universally capacity achieving with vanishing block error probability over the entire set of equi-capacity MBIOS channels.
- Furthermore, randomly punctured LDPC code ensembles can also be made universal under ML decoding.

Further Reading

This talk is based on the paper:

I. Sason and B. Shuval, *"On Universal LDPC Code Ensembles Over Memoryless Symmetric Channels,"* IEEE Trans. on Information Theory, submitted in April 2010 and revised in January 2011.

Thank you for your attention!