On Strongly Regular Graphs, the Friendship Theorem, Lovász Function, and Shannon Capacity of Graphs

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Graph Spectrum

Throughout this presentation,

- $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ is a finite, undirected, and simple graph of order $|\mathsf{V}(\mathsf{G})| = n$ and size $|\mathsf{E}(\mathsf{G})| = m$.
- $\mathbf{A} = \mathbf{A}(\mathsf{G})$ is the *adjacency matrix* of the graph.
- The eigenvalues of $\mathbf{A}$ are given in decreasing order by

$$\lambda_{\max}(\mathsf{G}) = \lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}) = \lambda_{\min}(\mathsf{G}). \qquad (1.1)$$

- The *spectrum* of G is a multiset that consists of all the eigenvalues of $\mathbf{A}$, including their multiplicities.

Orthogonal Representation of Graphs

Definition 1.1

Let G be a finite, undirected and simple graph.
An orthogonal representation of G in $\mathbb{R}^d$

$$i \in \mathsf{V}(\mathsf{G}) \;\mapsto\; \mathbf{u}_i \in \mathbb{R}^d$$

such that

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0, \quad \forall \{i, j\} \notin \mathsf{E}(\mathsf{G}).$$

An orthonormal representation of G: $\|\mathbf{u}_i\| = 1$ for all $i \in \mathsf{V}(\mathsf{G})$.

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In an orthogonal representation of a graph G:

- non-adjacent vertices: mapped to orthogonal vectors;
- adjacent vertices: not necessarily mapped to non-orthogonal vectors.

Lovász ϑ-function

Let G be a finite, undirected and simple graph.

The Lovász ϑ-function of G is defined as

$$\vartheta(\mathsf{G}) \triangleq \min_{\mathbf{u},\mathbf{c}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right)^2}, \tag{1.2}$$

where the minimum is taken over

- all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G, and
- all unit vectors $\mathbf{c}$.

The unit vector $\mathbf{c}$ is called the *handle* of the orthonormal representation.

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$$\left|\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right| \leq \|\mathbf{c}\| \, \|\mathbf{u}_i\| = 1 \implies \vartheta(\mathsf{G}) \geq 1,$$

with equality if and only if G is a complete graph.

An Orthonormal Representation of a Pentagon

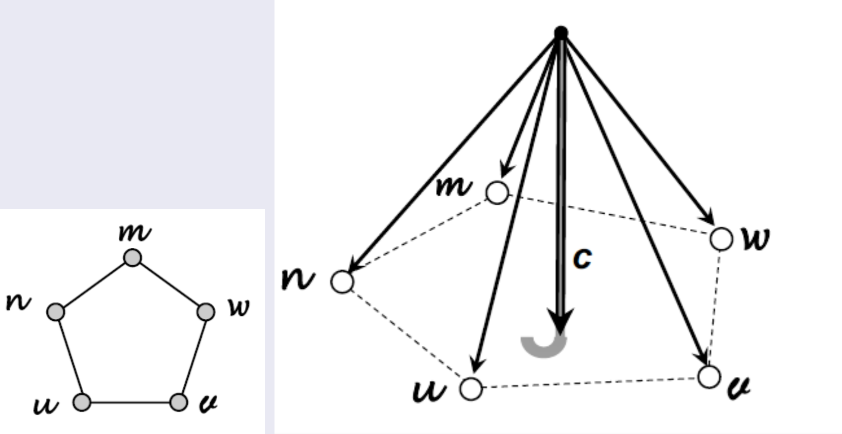

Figure 1: A 5-cycle graph and its orthonormal representation (also known as Lovász umbrella). Calculation shows that $\vartheta(\mathrm{C}_5) = \sqrt{5}$ (Lovász, 1979).

Lovász ϑ-function (Cont.)

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\vartheta(\mathsf{G})$:

$$\begin{array}{l} \text{maximize } \; \text{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \quad \text{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \; \Rightarrow \; B_{i,j} = 0, \quad i,j \in [n]. \end{cases} \end{array}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Lovász ϑ-function (Cont.)

Let $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ denote the independence number, clique number, and chromatic number of a graph G. Then,

1. Sandwich theorem:

$$\alpha(\mathsf{G}) \leq \vartheta(\mathsf{G}) \leq \chi(\overline{\mathsf{G}}), \tag{1.3}$$

$$\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G}). \tag{1.4}$$

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2. Computational complexity:
   - $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems.
   - However, the numerical computation of $\vartheta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

Hoffman-Lovasz bound and edge-transitive regular graphs

Theorem 1.2 (Lovász, 1979)

Let G be d-regular of order n. Then,

$$\vartheta(\mathsf{G}) \leq -\frac{n\,\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}, \tag{1.5}$$

with equality if G is edge-transitive.

Strongly Regular Graphs

Let G be a d-regular graph of order n. It is a *strongly regular* graph (SRG) if there exist nonnegative integers λ and μ such that

- Every pair of adjacent vertices have exactly λ common neighbors;
- Every pair of distinct and non-adjacent vertices have exactly μ common neighbors.

Such a strongly regular graph is said to belong to the family $\text{srg}(n, d, \lambda, \mu)$.

Theorem: Adjacency Spectrum of Strongly Regular Graphs

Let G be a connected strongly regular graph in the family srg(n, d, λ, μ) (i.e., $\mu > 0$). Then, its adjacency spectrum consists of three distinct eigenvalues, where the largest eigenvalue is given by $\lambda_1(\mathsf{G}) = d$ with multiplicity 1, and the other two distinct eigenvalues of its adjacency matrix are given by

$$p_{1,2} = \tfrac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(d-\mu)}\right), \tag{1.6}$$

with the respective multiplicities

$$m_{1,2} = \tfrac{1}{2}\left(n - 1 \mp \frac{2d + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(d-\mu)}}\right). \tag{1.7}$$

Theorem 1.3 (Bounds on Lovász function of Regular Graphs, I.S., '23)

Let G be a d-regular graph of order n, which is a non-complete and non-empty graph. Then, the following bounds hold for the Lovász ϑ-function of G and its complement $\overline{\mathsf{G}}$:

1)

$$\frac{n-d+\lambda_2(\mathsf{G})}{1+\lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{n\lambda_n(\mathsf{G})}{d-\lambda_n(\mathsf{G})}. \tag{1.8}$$

- Equality holds in the leftmost inequality if $\overline{\mathsf{G}}$ is both vertex-transitive and edge-transitive, or if G is a strongly regular graph;
- Equality holds in the rightmost inequality if G is edge-transitive, or if G is a strongly regular graph.

Cont. of Theorem 1.3

2)

$$1 - \frac{d}{\lambda_n(\mathsf{G})} \leq \vartheta(\overline{\mathsf{G}}) \leq \frac{n\bigl(1+\lambda_2(\mathsf{G})\bigr)}{n-d+\lambda_2(\mathsf{G})}. \tag{1.9}$$

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A Common Sufficient Condition

All inequalities hold with equality if G is strongly regular. (Recall that the graph G is strongly regular if and only if $\overline{\mathsf{G}}$ is so).

Lovász Function of Strongly Regular Graphs (I.S., '23)

Let G be a strongly regular graph in the family srg(n, d, λ, μ). Then,

$$\vartheta(\mathsf{G}) = \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda}, \tag{1.10}$$

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{2d}{t+\mu-\lambda}, \tag{1.11}$$

where

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.12}$$

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New Relation for Strongly Regular Graphs

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = n, \tag{1.13}$$

holding not only for all vertex-transitive graphs (Lovász '79), but also for all strongly regular graphs (that are not necessarily vertex-transitive).

In general, we have $\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) \geq n$ (Lovász, 1979).

Corollary 1.4 (Lovász ϑ-Function of SRGs (I.S., '23))

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Example: Chang Graphs

- Chang graphs are three non-isomorphic strongly regular graphs with parameters $\text{srg}(28, 12, 6, 4)$.
- These graphs are not vertex-transitive and also not edge-transitive.
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- These graphs are not vertex-transitive and also not edge-transitive.
- The clique numbers of these 3 graphs are 5, 6, 6.
- Nevertheless, they have the same Lovász ϑ-function, being equal to 4. The Lovász ϑ-function of the complements, all srg$(28, 15, 6, 10)$, is 7.
- Note that, indeed, $\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = 28$ for these three graphs, although they are not vertex-transitive (but SRGs).

Strongly regular graphs that belong to the same family srg(n, d, λ, μ), where $\mu > 0$, are connected and cospectral graphs. Although these graphs are not necessarily isomorphic, their Lovász ϑ-functions are identical.

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Question

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The next result gives a positive answer to this question.

Theorem 1.5 (A result with an explicit construction (I.S, 2024))

For every even integer $n \geq 14$, it is constructively proven that there exist connected, irregular, cospectral, and nonisomorphic graphs on n vertices, with the following properties:

- They are jointly cospectral with respect to their adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices.
- They share identical independence, clique, and chromatic numbers.
- They are distinguished by their Lovász ϑ-functions.

We next provide an original proof of the following celebrated theorem by Erdös, Rényi and Sós (1966), based on our expression for the Lovász ϑ-function of strongly regular graphs.

Theorem 1.6 (Friendship Theorem)

Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex adjacent to every other vertex.

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Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex adjacent to every other vertex.

A Human Interpretation of Theorem 1.6

Assume there is a party with n people, where every two people have precisely one common friend in that party. Theorem 1.6 asserts that one of these people is everybody's friend.

Remark 1 (On the Friendship Theorem - Theorem 1.6)

- The windmill graph (see Figure 2) has the desired property, and it turns out to be the only one graph with that property.
- The friendship theorem does not hold for infinite graphs.

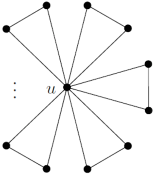

Figure 2: Windmill graph.

Alternative Proof of Theorem 1.6 (I.S., 25)

Suppose the assertion is false, and G is a counterexample — a finite graph in which any two distinct vertices have a single common neighbor, yet no vertex in G is adjacent to all other vertices. A contradiction is obtained by the following proof outline:

- It is shown that the graph is regular.
- It is then shown that the graph is strongly regular srg$(n, k, 1, 1)$.
- By assumption, $k = 1$ is excluded.
- If $k = 0$ or $k = 2$, then $\mathsf{G} = \mathsf{K}_1$ or $\mathsf{G} = \mathsf{K}_3$, respectively, which satisfy the assertion of the theorem. Hence, next assume that $k \geq 3$.
- By the theorem hypothesis, it follows that $\omega(\mathsf{G}) = \chi(\mathsf{G}) = 3$.
- By the sandwich theorem $\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G})$, so $\vartheta(\overline{\mathsf{G}}) = 3$.
- Based on the expression for the Lovász ϑ-function $\vartheta(\overline{\mathsf{G}}) = 1 + \frac{k}{\sqrt{k-1}}$.
- This leads to a contradiction for all $k \geq 3$.

The sandwich theorem for the Lovász ϑ-function applied to strongly regular graphs gives the following result.

Corollary 1.7 (Bounds on Parameters of SRGs)

Let G be a strongly regular graph in the family $\text{srg}(n, d, \lambda, \mu)$. Then,

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rfloor \tag{1.14}$$

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{2d}{t+\mu-\lambda} \right\rfloor, \tag{1.15}$$

$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{2d}{t+\mu-\lambda} \right\rceil, \tag{1.16}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rceil, \tag{1.17}$$

with

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.18}$$

Examples: Bounds on Parameters of SRGs

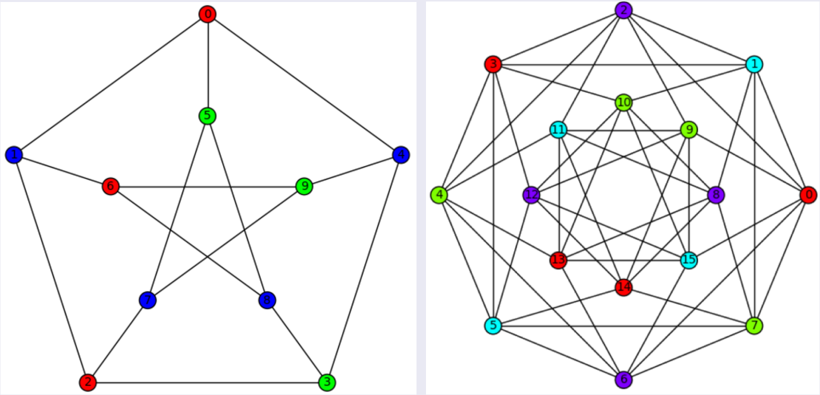

Figure 3: The Petersen graph is srg$(10, 3, 0, 1)$ (left), and the Shrikhande graph is srg$(16, 6, 2, 2)$ (right). Their chromatic numbers are 3 and 4, respectively.

Schläfli Graph

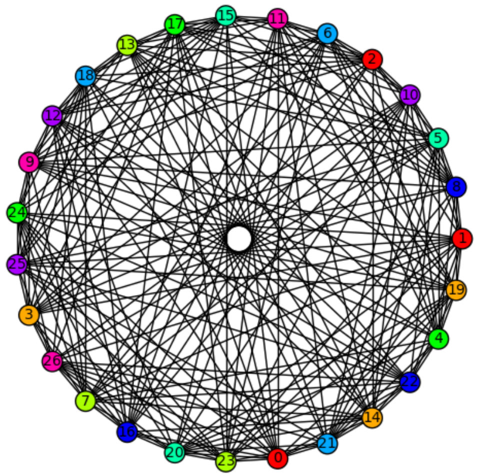

Figure 4: Schläfli graph is $\text{srg}(27, 16, 10, 8)$ with chromatic number $\chi(\mathsf{G}) = 9$.

Examples: Bounds on Parameters of SRGs (Cont.)

1. Let G_1 be the Petersen graph. Then, the bounds on the independence, clique, and chromatic numbers of G are tight:
$$\alpha(\mathsf{G}_1) = 4, \quad \omega(\mathsf{G}_1) = 2, \quad \chi(\mathsf{G}_1) = 3. \tag{1.19}$$
2. The bounds on the chromatic numbers of the Schläfli graph (G_2), Shrikhande graph (G_3) and Hall-Janko graph (G_4) are tight:
$$\chi(\mathsf{G}_2) = 9, \quad \chi(\mathsf{G}_3) = 4, \quad \chi(\mathsf{G}_4) = 10. \tag{1.20}$$
3. For the Shrikhande graph (G_3),
   - the bound on its independence number is also tight: $\alpha(\mathsf{G}_3) = 4$,
   - its upper bound on its clique number is, however, not tight (it is equal to 4, and $\omega(\mathsf{G}_3) = 3$).

Strong Product of Graphs

Let G and H be two graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with

- vertex set: $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$,
- two distinct vertices (g, h) and (g', h') in $\mathsf{G} \boxtimes \mathsf{H}$ are adjacent if one of the following three conditions hold:
  1. $g = g'$ and $\{h, h'\} \in \mathsf{E}(\mathsf{H})$,
  2. $\{g, g'\} \in \mathsf{E}(\mathsf{G})$ and $h = h'$,
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Strong products are commutative and associative.

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Strong Powers of Graphs

Let

$$\mathsf{G}^{\boxtimes k} \triangleq \underbrace{\mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}}_{\mathsf{G} \text{ appears } k \text{ times}}, \quad k \in \mathbb{N} \tag{1.21}$$

denote the k-fold strong power of a graph G.

Theorem 1.8 (chromatic number of strong product of SRGs (I.S.,'23))

Let $\mathsf{G}_1, \dots, \mathsf{G}_k$ be strongly regular graphs $\text{srg}(n_\ell, d_\ell, \lambda_\ell, \mu_\ell)$ for $\ell \in [k]$ (they need not be distinct). Then, the chromatic number of their strong product satisfies

$$\left\lceil \prod_{\ell=1}^{k} \left(1 + \frac{2d_\ell}{t_\ell + \mu_\ell - \lambda_\ell}\right) \right\rceil \leq \chi(\mathsf{G}_1 \boxtimes \ldots \boxtimes \mathsf{G}_k) \leq \prod_{\ell=1}^{k} \chi(\mathsf{G}_k), \tag{1.22}$$

where $\{t_\ell\}_{\ell=1}^k$ in the leftmost term is given by

$$t_\ell \triangleq \sqrt{(\lambda_\ell - \mu_\ell)^2 + 4(d_\ell - \mu_\ell)}, \quad \ell \in [k]. \tag{1.23}$$

The above lower bound is also larger than or equal to the product of the clique numbers of the factors $\{\mathsf{G}_\ell\}_{\ell=1}^k$.

Example: Chromatic Numbers of Strong Products

Let

$$\mathsf{G} \in \mathsf{srg}(27, 16, 10, 8), \quad \mathsf{H} \in \mathsf{srg}(16, 6, 2, 2), \quad \mathsf{J} \in \mathsf{srg}(100, 36, 14, 12)$$

be the Schläfli, Shrikhande, and Hall-Janko graphs, respectively.
The upper and lower bounds (in the previous slide) coincide here: for all integers $k_1, k_2, k_3 \geq 0$,

$$\chi(\mathsf{G}^{\boxtimes k_1} \boxtimes \mathsf{H}^{\boxtimes k_2} \boxtimes \mathsf{J}^{\boxtimes k_3}) = 9^{k_1} \, 4^{k_2} \, 10^{k_3}. \tag{1.24}$$

For comparison, the lower bound that is given by the product of the clique numbers of each factor is looser, and it is equal to $6^{k_1} 3^{k_2} 4^{k_3}$.

Shannon Capacity of a Graph

- A discrete channel consists of
  - a finite input set $\mathcal{X}$;
  - a (possibly infinite) output set $\mathcal{Y}$
  - a non-empty fan-out set $\mathcal{S}_x \subseteq \mathcal{Y}$ for every $x \in \mathcal{X}$.

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- In each channel use, a sender transmits an input $x \in \mathcal{X}$ and a receiver receives an arbitrary output in $\mathcal{S}_x$.

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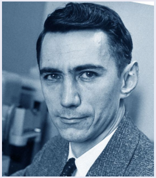
- In each channel use, a sender transmits an input $x \in \mathcal{X}$ and a receiver receives an arbitrary output in $\mathcal{S}_x$.
- Shannon (1956) initiated the study of the maximum amount (rate) of information that a channel can communicate without error.

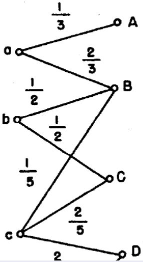


THE ZERO ERROR CAPACITY OF A NOISY CHANNEL

Claude E. Shannon
Bell Telephone Laboratories, Murray Hill, New Jersey
Massachusetts Institute of Technology, Cambridge, Mass.

Abstract

The zero error capacity C_0 of a noisy channel is defined as the least upper bound of rates at which it is possible to transmit information with zero probability of error. Various properties of C_0 are studied; upper and lower bounds and methods of evaluation of C_0 are given. Inequalities are obtained for the C_0 relating to the "sum" and "product" of two given channels. The analogous problem of zero error capacity C_{0F} for a channel with a feedback link is considered. It is shown that while the ordinary capacity of a memoryless channel with feedback is equal to that of the same channel without feedback, the zero error capacity may be greater. A solution is given to the problem of evaluating C_{0F}.



Shannon Capacity of a Graph (Cont.)

A discrete memoryless channel is represented by a *confusion graph* G:

- $V(G)$ represent the symbols of the input alphabet to that channel.

Shannon Capacity of a Graph (Cont.)

A discrete memoryless channel is represented by a *confusion graph* G:

- V(G) represent the symbols of the input alphabet to that channel.
- E(G): Two distinct vertices in G are adjacent if the corresponding two input symbols (say $x, x' \in \mathcal{X}$) are not distinguishable by the channel.

$$\begin{aligned} \mathsf{V}(\mathsf{G}) &= \mathcal{X}, \\ \mathsf{E}(\mathsf{G}) &= \bigl\{ \{x, x'\} : \; x, x' \in \mathcal{X}, \; x \neq x', \; \mathcal{S}_x \cap \mathcal{S}_{x'} \neq \varnothing \bigr\}. \end{aligned}$$

(Both distinct input symbols can result in the same output.)

Shannon Capacity of a Graph (Cont.)

The largest number of inputs a channel can communicate without error in a single use is $\alpha(\mathsf{G})$ (the independence number of G).

Shannon Capacity of a Graph (Cont.)

The largest number of inputs a channel can communicate without error in a single use is $\alpha(\mathsf{G})$ (the independence number of G):

- The sender and the receiver agree in advance on an independent set $\mathcal{I}$ of a maximum size $\alpha(\mathsf{G})$.
- The sender transmits only inputs in $\mathcal{I}$.
- Every received output is in the fan-out set of exactly one input in $\mathcal{I}$.

  $\Rightarrow$ the receiver can correctly determine the transmitted input.

Shannon Capacity of a Graph (Cont.)

- Consider a transmission of k-length strings over a channel.

Shannon Capacity of a Graph (Cont.)

- Consider a transmission of k-length strings over a channel.
  - The channel is used $k \geq 1$ times;
  - The sender transmits a sequence $x_1 \ldots x_k$;
  - The receiver receives a sequence $y_1 \ldots y_k$ of outputs, where

$$y_i \in \mathcal{S}_{x_i}, \quad i = 1, \ldots, k.$$

Shannon Capacity of a Graph (Cont.)

- Consider a transmission of k-length strings over a channel.
  - The channel is used $k \geq 1$ times;
  - The sender transmits a sequence $x_1 \dots x_k$;
  - The receiver receives a sequence $y_1 \dots y_k$ of outputs, where

$$y_i \in \mathcal{S}_{x_i}, \quad i = 1, \dots, k.$$

- k uses of the channel are viewed as a single use of a larger channel:
  - its input set is $\mathcal{X}^k$, and its output set is $\mathcal{Y}^k$.
  - The fan-out set of $(x_1, \dots, x_k) \in \mathcal{X}^k$ is the Cartesian product

$$\mathcal{S}_{x_1} \times \dots \mathcal{S}_{x_k}.$$

Shannon Capacity of a Graph (Cont.)

- The *k-th confusion graph* is the k-fold strong power of G:

$$\mathsf{G}^{\boxtimes k} \triangleq \mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}. \tag{2.1}$$

Shannon Capacity of a Graph (Cont.)

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- $\alpha(\mathsf{G}^{\boxtimes k})$ is the max. number of k-length strings at the channel input that are distinguishable by the channel (error-free communication).

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- $\alpha(\mathsf{G}^{\boxtimes k})$ is the max. number of k-length strings at the channel input that are distinguishable by the channel (error-free communication).
- The maximum *information rate per symbol* that is achievable by using input strings of length k is equal to

$$\frac{1}{k} \log \alpha(\mathsf{G}^{\boxtimes k}) = \log \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}, \quad k \in \mathbb{N}. \tag{2.2}$$

Shannon Capacity of a Graph (Cont.)

- The Shannon capacity of a graph G is defined to be the supremum of the maximum information rate over k (the length k of the input strings can be made as large as we wish):

$$\Theta(\mathsf{G}) = \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}. \tag{2.3}$$

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The last equality holds by Fekete's Lemma:
the sequence $\{\alpha(\mathsf{G}^{\boxtimes k})\}_{k=1}^{\infty}$ is super-multiplicative, i.e.,

$$\alpha(\mathsf{G}^{\boxtimes (k_1+k_2)}) \geq \alpha(\mathsf{G}^{\boxtimes k_1})\, \alpha(\mathsf{G}^{\boxtimes k_2}). \tag{2.4}$$

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Alas, the Shannon capacity can be rarely computed exactly !

On the Computability of the Shannon Capacity of Graphs

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On the Computability of the Shannon Capacity of Graphs

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  N. Alon and E. Lubetzky (IEEE T-IT, May 2006).
- However, the Lovász ϑ-function of a graph is a computable (and sometimes tight) upper bound on the Shannon capacity. ☺

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-25, NO. 1, JANUARY 1979 1

On the Shannon Capacity of a Graph

LÁSZLÓ LOVÁSZ

***Abstract*—It is proved that the Shannon zero-error capacity of the pentagon is $\sqrt{5}$. The method is then generalized to obtain upper bounds on the capacity of an arbitrary graph. A well-characterized, and in a sense easily computable, function is introduced which bounds the capacity from above and equals the capacity in a large number of cases. Several results are obtained on the capacity of special graphs; for example, the Petersen graph has capacity four and a self-complementary graph with n points and with a vertex-transitive automorphism group has capacity $\sqrt{n}$.**

A general upper bound on $\Theta(G)$ was also given in [6] (this bound was discussed in detail by Rosenfeld [5]). We assign nonnegative weights $w(x)$ to the vertices x of G such that

$$\sum_{x \in C} w(x) \leqslant 1$$

for every complete subgraph C in G; such an assignment is called a *fractional vertex packing*. The maximum of

Lovász Bound on the Shannon Capacity of Graphs (1979)

Theorem 2.1

For every finite, simple and undirected graph G,

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.5}$$

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$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.5}$$

Proof

$$\begin{aligned} \Theta(\mathsf{G}) &= \lim_{k\to\infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} && (2.6) \\ &\leq \lim_{k\to\infty} \sqrt[k]{\vartheta(\mathsf{G}^{\boxtimes k})} && (2.7) \\ &= \vartheta(\mathsf{G}) && (2.8) \end{aligned}$$

where the last equality holds since $\vartheta(\mathsf{G}^{\boxtimes k}) = \vartheta(\mathsf{G})^k$ forall $k \in \mathbb{N}$.

In some cases, the capacity of a graph can be calculated exactly :)

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Theorem 2.2 (Lovász, 1979)

Let G be a self-complementary and vertex-transitive graph on n vertices. Then,

$$\Theta(\mathsf{G}) = \sqrt{n} = \vartheta(\mathsf{G}), \tag{2.9}$$

$$\alpha(\mathsf{G} \boxtimes \mathsf{G}) = n. \tag{2.10}$$

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Theorem 2.3 (Lovász, 1979)

Let $\mathsf{G} = \mathsf{K}(n,k)$ be a non-empty Kneser graph $(n \geq 2k)$. Then,

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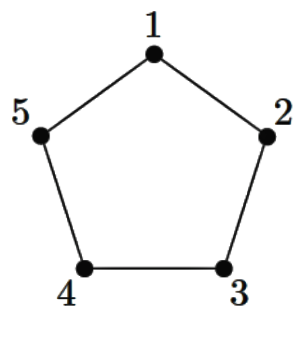
Theorem 2.4 (I.S., '24)

The same result in Theorem 2.2 for self-complementary vertex-transitive graphs also holds for self-complementary strongly regular graphs.

Example: Shannon Capacity of a 5-Cycle Graph (Lovász, 1979)

The pentagon (5-cycle) C_5 is self-complementary and vertex-transitive, so

$$\Theta(\mathsf{C}_5) = \sqrt{5} = \vartheta(\mathsf{C}_5), \quad \alpha(\mathsf{G} \boxtimes \mathsf{G}) = 5. \tag{2.12}$$



$\alpha(C_5 \boxtimes C_5) = 5$

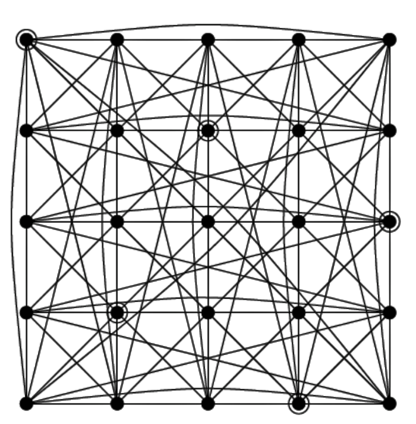


Figure 5: $C_5 \boxtimes C_5$. Independent set: $\{(1,1), (2,3), (3,5), (4,2), (5,4)\}$.

Theorem 2.5 (On the Shannon capacity of graphs, I.S. '24)

Let G be an undirected and simple graph on n vertices.

1. If G is a vertex-transitive or strongly regular graph, then
$$\alpha(\mathsf{G} \boxtimes \overline{\mathsf{G}}) = \Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}}) = \vartheta(\mathsf{G} \boxtimes \overline{\mathsf{G}}) = n. \tag{2.12}$$
2. If G is a conference graph, then $\vartheta(\mathsf{G}) = \sqrt{n}$.
3. If G is a self-complementary graph with $\alpha(\mathsf{G}) = k$, then
$$\sqrt{n} \leq \Theta(\mathsf{G}) \leq 16\, n^{\frac{k-1}{k+1}}. \tag{2.13}$$
4. If G is self-complementary and either vertex-transitive or strongly regular, then
$$\Theta(\mathsf{G}) = \sqrt{n} = \vartheta(\mathsf{G}), \qquad \sqrt{\alpha(\mathsf{G} \boxtimes \mathsf{G})} = \Theta(\mathsf{G}). \tag{2.14}$$

   Hence, the minimum Shannon capacity among all self-complementary graphs of a fixed order n is achieved by such graphs, and it is $\sqrt{n}$.

Lovász ϑ-function $\vartheta(\mathsf{G})$

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\vartheta(\mathsf{G})$:

$$\begin{array}{l} \text{maximize } \; \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \;\; \mathrm{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Schrijver's ϑ-function $\vartheta'(\mathrm{G})$

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathrm{V(G)}|)$;
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$$\begin{array}{l} \text{maximize } \; \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \;\; \mathrm{Trace}(\mathbf{B}) = 1, \\ B_{i,j} \geq 0, \quad i,j \in [n], \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i,j \in [n]. \end{cases} \end{array}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta'(\mathrm{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Theorem 2.6

For every graph G,

$$\alpha(\mathsf{G}) \le \vartheta'(\mathsf{G}) \le \vartheta(\mathsf{G}). \tag{2.15}$$

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$$\alpha(\mathsf{G}) \le \vartheta'(\mathsf{G}) \le \vartheta(\mathsf{G}). \qquad (2.15)$$

Proof

The leftmost inequality in (2.15) holds by selecting a feasible solution in the maximization problem for $\vartheta'(\mathsf{G})$ as follows.

- Let $\mathcal{I}$ be a largest independent set in G, and let $\mathcal{I} = \{i_1, \ldots, i_\ell\} \subseteq [n]$ with $\ell = \alpha(\mathsf{G})$.
- Define $\mathbf{B}$ to be the $n \times n$ symmetric matrix whose elements are given by $B_{i,j} \triangleq \frac{1}{\alpha(\mathsf{G})}$ whenever $i, j \in \mathcal{I}$, and $B_{i,j} \triangleq 0$ otherwise.
- $\Rightarrow \mathbf{B}$ is indeed a positive semidefinite matrix whose trace is equal to 1, and the objective function is then equal to $\alpha(\mathsf{G})$.

The rightmost inequality in (2.15) is due to the additional constraint in the optimization problem for $\vartheta'(\mathsf{G})$, as compared to one for $\vartheta(\mathsf{G})$.

Question

Can the upper bound on the Shannon capacity,

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G})$$

be improved to

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Our work resolves this query regarding the variant of the ϑ-function by Schrijver (1978). We show, by a counterexample, that

$$\Theta(\mathsf{G}) \not\leq \vartheta'(\mathsf{G})$$

i.e., the ϑ-function variant by Schrijver does not possess the property of the Lovász ϑ-function of forming an upper bound on the Shannon capacity of a graph.

Example 2.7 (Counterexample (I.S., '24))

Let G be the Gilbert graph on 32 vertices, where

$$\mathsf{V}(\mathsf{G}) = \{0,1\}^5, \qquad \mathsf{E}(\mathsf{G}) = \Big\{\underline{u}, \underline{v} \in \{0,1\}^5 : \; 1 \leq d_{\mathrm{H}}(\underline{u}, \underline{v}) \leq 2\Big\},$$

so, every two vertices are adjacent if and only if the Hamming distance of their corresponding 5-tuples binary vectors is either 1 or 2.

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so, every two vertices are adjacent if and only if the Hamming distance of their corresponding 5-tuples binary vectors is either 1 or 2.

- G is 15-regular, vertex-transitive, edge-transitive, distance-regular.
- The complement $\overline{\mathsf{G}}$ is 16-regular, vertex-transitive, but not edge-transitive nor distance-regular.
- $\alpha(\mathsf{G}) = 4$. An example of such a maximal independent set of G:

$$\big\{(1,0,0,1,0), \; (0,1,1,1,0), \; (0,0,0,0,1), \; (1,1,1,0,1)\big\}.$$

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- Solving the SDP problem for $\vartheta'(\mathsf{G})$ gives

$$\vartheta'(\mathsf{G}) = 4 = \alpha(\mathsf{G}).$$

Example 2.7 (cont. - I.S., '24)

- G is 15-regular and edge-transitive on 32 vertices, with $\lambda_{\min}(\mathsf{G}) = -3$, so by Theorem 1.2,

$$\vartheta(\mathsf{G}) = -\frac{n\lambda_{\min}(\mathsf{G})}{d(\mathsf{G}) - \lambda_{\min}(\mathsf{G})} = \tfrac{32\cdot 3}{15+3} = 5\tfrac{1}{3}.$$

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- Hence, for this graph,

$$4 = \alpha(\mathsf{G}) = \vartheta'(\mathsf{G}) < \vartheta(\mathsf{G}) = 5\tfrac{1}{3},$$

so $\vartheta'(\mathsf{G})$ coincides with the independence number of G, and it is strictly smaller than $\vartheta(\mathsf{G})$.

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so $\vartheta'(\mathsf{G})$ coincides with the independence number of G, and it is strictly smaller than $\vartheta(\mathsf{G})$.

- It can be verified that

$$\alpha(\mathsf{G} \boxtimes \mathsf{G}) = 20,$$

and the strong product graph $\mathsf{G} \boxtimes \mathsf{G}$ has $368,640$ such maximal independent sets of size 20.

Example 2.7 (cont. - I.S., '24)

- An example of a maximal independent set (of size 20) for $\mathrm{G} \boxtimes \mathrm{G}$:

$$
\begin{aligned}
\big\{ &((1,1,0,0,0),(1,1,1,1,1)), && ((1,0,1,0,0),(1,1,0,0,0)), \\
&((0,1,1,0,0),(0,0,1,1,0)), && ((1,1,1,0,0),(0,0,0,0,1)), \\
&((1,0,0,1,0),(0,0,1,0,1)), && ((0,1,0,1,0),(1,0,0,0,0)), \\
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&((1,0,0,0,1),(0,0,0,1,0)), && ((0,1,0,0,1),(0,1,0,0,1)), \\
&((1,1,0,0,1),(1,0,1,0,0)), && ((0,0,1,0,1),(1,0,1,0,1)), \\
&((1,0,1,0,1),(0,1,1,1,1)), && ((0,1,1,0,1),(1,1,0,1,0)), \\
&((0,0,0,1,1),(1,1,1,1,0)), && ((1,0,0,1,1),(1,1,0,0,1)), \\
&((0,1,0,1,1),(0,0,1,1,1)), && ((0,0,1,1,1),(0,0,0,0,0)) \big\}.
\end{aligned}
$$

Example 2.7 (cont. - I.S., '24)

- An example of a maximal independent set (of size 20) for $G \boxtimes G$:

$$\begin{aligned}
\{((1,1,0,0,0),(1,1,1,1,1)), \quad & ((1,0,1,0,0),(1,1,0,0,0)),\\
((0,1,1,0,0),(0,0,1,1,0)), \quad & ((1,1,1,0,0),(0,0,0,0,1)),\\
((1,0,0,1,0),(0,0,1,0,1)), \quad & ((0,1,0,1,0),(1,0,0,0,0)),\\
((1,1,0,1,0),(0,1,0,1,0)), \quad & ((0,0,1,1,0),(0,1,0,1,1)),\\
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((0,0,0,1,1),(1,1,1,1,0)), \quad & ((1,0,0,1,1),(1,1,0,0,1)),\\
((0,1,0,1,1),(0,0,1,1,1)), \quad & ((0,0,1,1,1),(0,0,0,0,0))\}.
\end{aligned}$$

- Consequently, we get

$$\Theta(G) \geq \sqrt{\alpha(G \boxtimes G)} = \sqrt{20} > 4 = \vartheta'(G).$$

Journal Papers

This talk presents in part results from our recent papers:

1. I.S., "Observations on the Lovász ϑ-function, graph capacity, eigenvalues, and strong products," *Entropy*, vol. 25, no. 1, paper 104, pp. 1–40, January 2023. https://doi.org/10.3390/e25010104
2. I.S., "Observations on graph invariants with the Lovász ϑ-function," *AIMS Mathematics*, vol. 9, pp. 15385–15468, April 2024. https://doi.org/10.3934/math.2024747
3. I.S., "On strongly regular graphs and the friendship theorem," *Mathematics*, vol. 13, paper 970, pp. 1–21, March 2025. https://doi.org/10.3390/math13060970