Generalized Nyquist Criterion and Generalized Bode Diagram for Analysis and Synthesis of Uncertain Control Systems

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Abstract— Uncertainties in control systems models often have to be taken into account in their analysis and/or design. Negligence of such uncertainties is often unjustifiable and is done only due to lack of methods to treat the uncertainties. The presented work is concerned with analysis and design of interval uncertainty control systems, with regard to clustering of poles inside a simple symmetric bounded contour \( \Gamma \). We extend the well known Nyquist and Mikhailov stability theorems to \( \Gamma \)-stability tests of uncertain systems, defined by their generalized Bode envelopes. Also, using generalized definitions and theorems we solve the design problem of a controller which ensures clustering of closed loop poles of an interval uncertain family of transfer functions inside such prescribed \( \Gamma \)-region.

Index Terms—Stability Theorems, Uncertain Systems, Bode Envelopes, Controller Synthesis.

I. INTRODUCTION

The wide exploration of frequency methods in control theory began with the seminal works of H. Nyquist [1] and H.W. Bode [2]. Nyquist developed the well-known stability criterion, called on his name. Later, in 1946, W. Frey gave the “unstable plant” version of Nyquist criterion [3]. In 1958 Mikhailov has formulated the similar criterion for polynomial stability in frequency domain. Bode, among other things, showed the usefulness of gain-phase diagrams in analysis and synthesis of control systems. Their works were followed by many researchers, and many extensions and generalizations have been found since then.

At the beginning of 70’s of past century, M. Lichtsinder published numerous papers (in Russian), for example [4], about generalization of frequency methods and Nyquist criterion for \( \Gamma \)-stability analysis. He used an alternative formulation of Nyquist criterion, that counts the number of encirclements by Nyquist plot of \((-1+0j)\) point in complex plane with the help of counting intersections of Nyquist plot with real axis ray \((-\infty,1)\). Some of these results derived independently in [5]. Lichtsinder also reformulated his criterion for using Bode diagrams (generalized Bode theorem).

In this work we will take further the research of M. Lichtsinder and will generalize some well-known stability theorems for \( \Gamma \)-stability and uncertain systems. The similar to [6] analytical form of Mikhailov criterion can be useful in \( \Gamma \)-stability analysis of uncertain systems.

In this paper we assume that given uncertain system \( G(s) \) is rational, with interval coefficients, and we can build the generalized Bode envelope for \( G(s) \), as explained in [7], [8], and in section III of this paper. We will use the generalized theorems for analysis. Finally, we will present a new technique for controller synthesis for uncertain systems. A more detailed version including the proofs will be published elsewhere.

II. BASIC DEFINITIONS

Every control system has its performance specifications. In general, these specifications are given in time domain, but in this work we will assume that it is possible to find the transformation from time specifications to some closed and bounded region \( \Gamma \) in frequency domain. In other words, if all poles in closed loop are placed inside \( \Gamma \) region, then all stability and performance demands are satisfied.

Let’s define \( \Gamma \) as arbitrary, bounded, simple connected, and symmetrical (with respect to real axis) region in complex plane. The boundary of this region is complex function \( \Gamma(\theta) = \partial \Gamma \) (see Fig. 1).

The parameterization of \( \Gamma(\theta) : [0,2\pi) \rightarrow \mathbb{C} \) is given by

\[
\Gamma(\theta) = R(\theta)e^{i\theta} - \lambda
\]

where \( \theta \) is generalized frequency argument, \( R(\theta) \in \mathbb{R}^+ \), and \(-\lambda \in \mathbb{R}\) is a real point inside the \( \Gamma \) region.

Additional definitions:
1. Generalized frequency response is defined by complex function \( G(\Gamma(\theta)) \), where \( G(s) \) is a transfer function of linear time invariant (LTI) system in open loop; \( \Gamma(\theta) \) is
2. Generalized phase response is defined by
\[ \phi(\theta) = \angle G(\Gamma(\theta)). \]

3. Generalized logarithmic amplitude response is defined by
\[ A(\theta)[\text{dB}] = 20 \log \left| G(\Gamma(\theta)) \right|. \]

4. The plot of generalized frequency response in complex plane is called Generalized Nyquist diagram, and the plot of generalized phase and logarithmic amplitude is called Generalized Bode diagram (GBD).

5. The G(s) is F-stable if all its poles placed inside the given \( \Gamma \) region.

6. \# \text{P}_{\text{in(out)}} - The number of G(s) poles inside (outside) the \( \Gamma \) region.

7. \# \text{Q}_{\text{in(out)}} - The number of G(s)/\{1+G(s)\} poles inside (outside) the \( \Gamma \) region.

8. \# \text{Z}_{\text{in(out)}} - The number of G(s) zeros inside (outside) the \( \Gamma \) region.

9. The set \( \Theta \) is defined by
\[ \Theta = \{ \phi(\theta) | k \in \mathbb{Z}, \phi(\theta) = 180^{\circ} + 360^{\circ}k \}, \]
that is if \( \phi(\theta) \in \Theta \), then the function \( \phi(\theta) \) has an intersection with one of horizontal lines \( 180^{\circ} + 360^{\circ}k ; k \in \mathbb{Z} \) at \( \theta \) point.

10. The positive (negative) intersection of \( 180^{\circ} + 360^{\circ}k ; k \in \mathbb{Z} \) lines with \( \phi(\theta) \) is such intersection point \( \theta \), the phase \( \phi(\theta) \) is rising (falling) from below upwards (from above downwards), as the frequency argument increases from 0 to \( \pi \).

11. The crossing index is defined by:
\[ ci(\theta) = \begin{cases} 0.5 & \text{if } \theta = 0 \text{ or } \theta = \pi \text{ and (raising } \phi) \text{ and } \phi(\theta) \in \Theta \ \
-0.5 & \text{if } \theta = 0 \text{ or } \theta = \pi \text{ and (falling } \phi) \text{ and } \phi(\theta) \in \Theta \\ 1 & \text{if } 0 < \theta < \pi \text{ and (raising } \phi) \text{ and } \phi(\theta) \in \Theta \\ -1 & \text{if } 0 < \theta < \pi \text{ and (falling } \phi) \text{ and } \phi(\theta) \in \Theta \\ 0 & \text{else} & \end{cases} \]

12. The G(s) crossing indexes sum is defined by:
\[ \text{cis}(G(s)) = \sum_{\theta} ci(\theta) \]
for all intersection points \( \theta \).

Remarks

- Due to the symmetry of the \( \Gamma \) region, it is sufficient to use the frequency argument \( \theta \in [0, \pi] \) and not \( \theta \in [0, 2\pi] \) in GBD.
- For sake of simplicity, in further discussion we will use the words “gain” and “phase” instead of “generalized logarithmic amplitude response” and “generalized phase response”.
- The frequency argument \( \theta \) is measured in radian, and phase is measured in degrees.
- G(s) is assumed to be in minimal form, without poles/zeros cancellations.

### III. Generalized Bode Envelopes

For left hand plane and unit circle the analytical techniques for building Bode envelopes described in [7] and [8]. Unfortunately, it is not easy to generalize these techniques for arbitrary \( \Gamma \) region. Nevertheless, it is possible to use similar technique with grating (by choosing dense enough grid for \( \theta \)) to get an approximation to generalized Bode envelope (GBE).

Let’s define the polynomial interval family
\[ D(s,a) = a_0s^0 + a_1s^{-1} + \ldots + a_n \]
where \( a \) is a coefficients vector, and \( \forall i : a_i \leq \bar{a}_i \). Suppose there is no degree reduction, that is \( 0 \notin [a_n, \bar{a}_n] \) (if degree reduction is present, then we will test 3 different cases: \( a_n = 0, a_n > 0 \) and \( a_n < 0 \). The GBE is the envelope that includes all 3 cases).

For any complex \( s_1 \), the value set of \( D(s_1,a) \) is a convex polygon. Its vertices are given by affine mapping of \( D(s_1,a) \) for extreme values of coefficients in vector \( a \) [9]. From geometric point of view, the maximal distance from the origin, the minimal and maximal phase is turn out to be in vertices. The minimal distance can be measured to vertices, or to closest to origin polygon side, or it can be 0 (see Fig. 2).

![Fig. 1. Definition of \( \Gamma \) contour in complex plane.](image1)

![Fig. 2. Polygonal value set of polynomial, and extreme distance and angle computation from the origin.](image2)
\[ \alpha = (a_1, \ldots, a_n), \quad \beta = (b_1, \ldots, b_n) \quad \text{and} \quad \forall i: a_i \leq a \leq \bar{a}_i; \quad b_i \leq b \leq \bar{b}_i. \] We assume that \( \alpha \) and \( \beta \) vectors are independent, so we can build the GBE of \( N(s, \beta) \) and \( D(s, \alpha) \) independently. From those two diagrams we can build the GBE of \( G(s, \alpha, \beta) \) [7]:

\[ \begin{align*}
\phi_{\text{max}}[G] &= \phi_{\text{min}}[N] - \phi_{\text{min}}[D] \\
\phi_{\text{min}}[G] &= \phi_{\text{min}}[N] - \phi_{\text{max}}[D] \\
A_{\text{max}}[G][dB] &= A_{\text{max}}[N][dB] - A_{\text{min}}[D][dB] \\
A_{\text{min}}[G][dB] &= A_{\text{min}}[N][dB] - A_{\text{max}}[D][dB]
\end{align*} \] (2)

where \( \phi \) is phase, and \( A \) is gain.

IV. GENERALIZATION OF MIKHAILOV CRITERION

Mikhailov criterion gives the necessary and sufficient conditions for stability without explicit solution of characteristic equation, by examination of frequency response of (characteristic polynomial) plot in complex plane. The extended and analytical version of this theorem can be found in [6]. In this section we will generalize Mikhailov theorem for \( \Gamma \)-stability case and for systems with uncertainty.

Theorem 1

Let \( D(s) = a_1s^n + a_{n-1}s^{n-1} + \cdots + a_0 \) be real polynomial with \( a_n \neq 0 \), and let \( \phi(\theta) = \angle D(\Gamma(\theta)) \) be continuous plot of \( D(\Gamma(\theta)) \) phase (\( \theta \in [0, \pi] \)). Denote by \( \phi(\theta) = \angle D(\Gamma(\theta)) \) where the roots of equations system \( \begin{cases} \text{Im}[D(\Gamma(\theta))] = 0 \\ \text{Re}[D(\Gamma(\theta))] < 0 \end{cases} \) (or, equivalently, the roots of \( \phi(\theta) \in \Theta \) and \( \theta_l = 0 ; \theta_u = \pi \).

If \( D(s) \) has no roots on the boundary \( \Gamma(\theta) = \Gamma^* \), then:

\[ \# z_m = (\phi(\pi) - \phi(0))/180' = 2\sum_{i=1}^{\infty} \text{ci}(\theta_i) \] (4)

where \( \# z_m \) is number of \( D(s) \) zeros inside the \( \Gamma \) region.

In particular case, when \( \# z_m = n \), the polynomial is \( \Gamma \)-stable.

We may expand the previous result for uncertain polynomials described by GBE.

Theorem 2

Suppose the uncertain family of polynomials of order \( n \) is given by GBE.

1. If the value set of polynomial family excludes the origin point for all \( \theta \in [0, \pi] \), then the whole family has an identical number of roots inside \( \Gamma \) and also \( \phi_{\text{min}}(0) = \phi_{\text{min}}(0) \) and \( \phi_{\text{max}}(\pi) = \phi_{\text{min}}(\pi) \). In this case, the identical number of roots equals \( (\phi_{\text{max}}(\pi) - \phi_{\text{min}}(0))/180' \) (or \( (\phi_{\text{min}}(\pi) - \phi_{\text{min}}(0))/180' \)).

2. The family is \( \Gamma \)-stable iff the value set of polynomial family excludes the origin for all \( \theta \in [0, \pi] \) and \( \phi_{\text{max}}(\pi) - \phi_{\text{max}}(0) = 180' n \) or \( \phi_{\text{min}}(\pi) - \phi_{\text{min}}(0) = 180' n \).

Remark

The condition that the value set of uncertain polynomial family excludes the origin point for all \( \theta \in [0, \pi] \) is equivalent to bounded gain envelope, and equivalent to condition of no zeros on the boundary \( \partial \Gamma \) for the whole family.

V. GENERALIZATION OF NYQUIST THEOREM

For the arbitrary open loop transfer function \( G(s) \) (not necessarily strictly proper or proper), we define the closed loop with unity feedback, so the closed loop transfer function is \( \frac{G(s)}{1+G(s)} \). Our goal is to find the number of closed loop poles outside the given \( \Gamma \) region, from the knowledge of \# \( p_m^{\text{OL}} \) (or \# \( z_m^{\text{OL}} \)) and GBD of \( G(s) \) in open loop.

Theorem 3 (Generalized Bode Theorem)

If \( G(s) = \frac{N(s)}{D(s)} = \frac{b_1s^n + b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \) transfer function has no zeros and poles on the boundary \( \partial \Gamma \), then:

\[ \# p_m^{\text{OL}} = \max(m, n) - \max(1/G) - \# z_m^{\text{OL}} = \max(m, n) - \max(1/G) - n + \# p_m^{\text{OL}} = \max(m, n) - \max(1/G) - m + \# z_m^{\text{OL}} \]

where \( \max(\ldots) \) is the crossing indexes sum for transfer function \( \ldots \).

In particular case, when \( m \leq n \), the system is \( \Gamma \)-stable in closed loop iff \( \# p_m^{\text{OL}} = 2\max(1/G) + \# z_m^{\text{OL}} \).

Now we will determine the number of closed loop poles of interval uncertain family, given by GBE.

Theorem 4

Suppose the gain envelope of given interval family (of order \( n \)) is bounded, that is \( \exists M < \infty, \forall \theta \in [0, \pi] \):

\[ -M < A_{\text{min}}(\theta) \leq A_{\text{max}}(\theta) < M \]

(180 \mod 360°) lines. In other words,

\[ \{ \theta: A_{\text{max}}(\theta) \geq 0[\text{dB}] \} \cap \{ \theta: A_{\text{min}}(\theta) \leq 0[\text{dB}] \} = \emptyset \]

If the condition from the above satisfied, then the whole family has the same cis, and the number of closed loop poles inside \( \Gamma \) region can be calculated with the help of theorem 3. If the number of poles equals \( n \), then the whole family is \( \Gamma \)-stable in closed loop.

VI. CONTROLLER DESIGN

In previous sections we have described the techniques for analysis of open and closed loop by its GBD or GBE. Now we need to design suitable serial controller to get all closed loop
poles inside the given $\Gamma$ region. From previous sections we know, that all we need is to get $2\cos$ equals to number of open loop poles outside $\Gamma$. The plots of generalized phase and gain of controller and plant are added, like in regular Bode diagrams, so we need to choose controller in such a way that the sum diagram (plant + controller) will lead to needed cis. To find such controller some graphical methods can be explored. For example, we may choose the needed plant+controller GBD, and then will find the controller diagram by subtraction (of plant diagram). From GBD of controller we may approximate its transfer function by numerical methods. This experimental method may require much iteration, before desirable satisfactory controller is found, so in this work we will present more systematic way for controller design.

The plant is given by $G(s)$ transfer function, or by GBD: 
\[
\{gain A(\theta),\ phase \ \phi(\theta)\}.
\]
If the gain of $G(s)$ is bounded, then it is always possible to get the positive gain (in dB) for plant+controller, by choosing high enough controllers gain. The plant+controller that have positive gain for all $\theta \in [0, \pi]$ is defined as class $I$ system. The controller phase denoted by $\Psi(\theta)$, and $m$ is a number of open loop poles outside $\Gamma$, divided by 2. By theorem 4, for class I controller design is enough to concentrate on two end-points $\phi(0)$ and $\phi(\pi)$.

**Theorem 5**

If the controller fulfills the following condition:
\[
\Psi(\pi) = 360^\circ m + \phi(0) - \phi(\pi) + \Psi(0) \tag{8}
\]
or
\[
\text{# controller} - \text{# controller} = 2m + (\phi(0) - \phi(\pi))/180^\circ \tag{9}
\]
(where # controller and # controller is the number of controller zeros and poles inside $\Gamma$), then the system with controller is $\Gamma$-stable in closed loop (with unity feedback) for controllers gain high enough (that gives class I system).

If for some engineering reason the provided solution is not satisfactory, or not applicable, then we suggest using class II or class III design, described below.

**Class II** system is described by plant+controller gain that begins above the 0dB (for $\theta = 0$), and ends below 0dB (for $\theta = \pi$) with single crossover point at $\theta_1$.

**Class III** system is described by plant+controller gain that begins below the 0dB (for $\theta = 0$), and ends above 0dB (for $\theta = \pi$) with single crossover point at $\theta_1$.

**Theorem 6**

1. The class II controller has to satisfy the following conditions:
   - If $\phi(0) = 0^\circ$ and $m \in Z$, then $\Psi(0) = 0^\circ$.
   - If $\phi(0) = 180^\circ$ and $m \notin Z$, then $\Psi(0) = 180^\circ$.
   - If $\phi(0) = 180^\circ$ and $m \in Z$, then $\Psi(0) = 180^\circ$.
   - If $\phi(0) = 180^\circ$ and $m \notin Z$, then $\Psi(0) = 0^\circ$.
\[
360^\circ m - 180^\circ + \phi(0) + \Psi(0) - \phi(\pi)_2 < \Psi(\pi)_2 < 360^\circ m + 180^\circ + \phi(0) + \Psi(0) - \phi(\pi)_2 \tag{10}
\]
2. The class III controller has to satisfy the following conditions:
   - If $\phi(\pi) \notin \Theta$ and $m \in Z$, then $\Psi(\pi) = 360^\circ k, k \in Z$.
   - If $\phi(\pi) \notin \Theta$ and $m \notin Z$, then $\Psi(\pi) = 180^\circ + 360^\circ k, k \in Z$.
   - If $\phi(\pi) \notin \Theta$ and $m \in Z$, then $\Psi(\pi) = 180^\circ + 360^\circ k, k \in Z$.

**Remark**

If all conditions of previous theorems are satisfied by controller, then the controllers gain has to be chosen to get a correct class.

Generalizing previous theorems to uncertain systems case is straightforward.

**Theorem 7**

Let define an interval family of order $n$ by GBE with bounded gain (the same phase end-points $\phi(0)$ and $\phi(\pi)$) \[ \{\ A_{max}(\theta)[dB], A_{min}(\theta)[dB], A_{max}(\theta), A_{min}(\theta)\} \] and let define the controller by its GBD \[ \{A_{controller}(\theta)[dB], \Psi(\theta)\} \].

1. Class I controller satisfies:
   \[ \Psi(\pi) = 360^\circ m + \phi(0) - \phi(\pi) + \Psi(0) \]
   where $\phi(0) \leq \phi_{\min}(0) - \phi_{\max}(0)$ and $\phi(\pi) \geq \phi_{\min}(0) = \phi_{\max}(0)$.

Also, it satisfies:
\[
\{\theta : A_{max} + A_{controller} \geq 0[\text{dB}] \text{ and } A_{max} + A_{controller} \leq 0[\text{dB}]\} \cap \{\theta : 180^\circ \text{ and } 180^\circ \} \in \text{[a, b] + [c, d]} = \emptyset \tag{14}
\]

The (9) equation is also true.

2. Class II controller satisfies (14),(10) and
\[
360^\circ m - 180^\circ + \phi(0) + \Psi(0) - \phi_{\min}(\pi) < \Psi(\pi) < 360^\circ m + 180^\circ + \phi(0) + \Psi(0) - \phi_{\max}(\pi) \tag{13}
\]

3. Class III controller satisfies (14),(12) and
\[
360^\circ m - 180^\circ + \phi(0) + \Psi(0) - \phi_{\min}(\pi) < \Psi(\pi) < 360^\circ m + 180^\circ + \phi(0) + \Psi(0) - \phi_{\max}(\pi) \tag{13}
\]

**Example**

The transfer functions interval family is given by
\[
G(s) = \frac{s^2 + [10,11]s + [40,41]}{s^3 + [15,16]s^2 + [63,65]s + [50,51]}, \text{ and the } \Gamma \text{ region is given by disk with radius } R=20 \text{ and with } \lambda = 23 \text{ displacement to the left from the origin } (\Gamma(\theta)=20e^{j\theta})^{-23}.
\]

The GBE of $G(s)$ denominator can be seen in Fig. 3. The gain is bounded, so, by theorem 2, we know the number of $G(s)$ open loop poles inside $\Gamma$ is the same for the whole family, and this number equals \[ \phi_{\min}(\pi)-\phi_{\max}(0)/180^\circ = (540^\circ - 180^\circ )/180^\circ = 2 \].

The number of poles outside $\Gamma$ is 1, so we need $m=\text{cis}=0.5$ to get $\Gamma$-stable system in closed loop.

The GBE of $G(s)$ is given in Fig. 4. It can be seen that gain envelope is always negative, so we have cis($G(\theta))=0$. Let’s use the theorem 5 to design the controller. We choose $\Psi(0) = 0^\circ$, then $\Psi(\pi) = 360^\circ m + \phi(0) - \phi(\pi) + \Psi(0) = 180^\circ$. The suitable controller with such phase characteristics is a PD controller.
For example $H(s) = 2(s+20)$ gives the GBE with needed $\text{cis}\{HG\} = 0.5$ (see Fig. 5).

![Fig. 3. Generalized Bode envelope of $s^3 + [15.16]s^2 + [63.65]s + [50.51]$](image)

![Fig. 4. Generalized Bode envelope of $G(s)$](image)

![Fig. 5. Generalized Bode envelope of $H(s)G(s)$ – Type I design](image)

**VII. CONCLUSION**

This work gives a new important way of looking at robust control systems analysis and design. The technique for controller design avoids the need in phase or gain margin usage and can assure the stability and step response specifications at the same time for the whole family, defined by GBE. The future research can continue in many ways, to include MIMO systems, systems with delays, or with other types of uncertainty.

**REFERENCES**