Optimal Prefix Codes for Sources with Two-Sided Geometric Distributions*

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Abstract

A complete characterization of optimal prefix codes for off-centered, two-sided geometric distributions of the integers is presented. These distributions are often encountered in lossless image compression applications, as probabilistic models for image prediction residuals. The family of optimal codes described is an extension of the Golomb codes, which are optimal for one-sided geometric distributions. The new family of codes allows for encoding of prediction residuals at a complexity similar to that of Golomb codes, without recourse to the heuristic approximations frequently used when modifying a code designed for non-negative integers so as to apply to the encoding of any integer. Optimal decision rules for choosing among a lower complexity subset of the optimal codes, given the distribution parameters, are also investigated, and the relative redundancy of the subset with respect to the full family of optimal codes is bounded.

Index Terms: Lossless image compression, Huffman code, infinite alphabet, geometric distribution, exponential distribution, Golomb codes, prediction residual.

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1 Introduction

Predictive coding techniques [1] have become very widespread in lossless image compression, due to their usefulness in capturing expected relations (e.g., smoothness) between adjacent pixels. It has been observed [2] that a good probabilistic model for image prediction errors is given by a *two-sided geometric distribution* (TSGD) centered at zero. Namely, the probability of an integer error value x is proportional to $\theta^{|x|}$, where $\theta \in (0,1)$ is a scalar parameter that controls the two-sided exponential decay rate. We assume in the sequel that prediction errors can take on any integer value, an assumption that, in the context of exponential distributions, is well approximated in practice by the use of large symbol alphabets (e.g., 8 bits per pixel).

Although the centered TSGD is an appropriate model for memoryless image compression schemes, it has been observed [3, 4] that prediction errors in *context-based* schemes [5, 6, 3, 4] exhibit a DC offset, and a more appropriate model is given by an *off-centered* TSGD. This model is also useful for better capturing the two adjacent modes often observed in empirical context-dependent histograms of prediction errors. More specifically, in this paper we consider integer distributions of the form

$$P_{(\theta,d)}(x) = C(\theta,d)\theta^{|x+d|}, \qquad x = 0, \pm 1, \pm 2, ..., \tag{1}$$

where $0 < \theta < 1$, $0 \leq d \leq \frac{1}{2}$, and $C(\theta,d)$ is a normalization factor given by

$$C(\theta,d) = \frac{1-\theta}{\theta^{1-d} + \theta^d}\,. \tag{2}$$

The parameter θ determines the rate of decay of the distribution, while d determines the offset of its center. The restriction on the range of d is justified through straightforward application of appropriate translations and reflections of the real line. In practice, the unit interval containing the center of the distribution can be located by a suitable adaptive predictor with an error feedback loop [3, 4]. The TSGD centered at zero corresponds to $d = 0$, and, when $d = \frac{1}{2}$, $P_{(\theta,d)}$ is a bi-modal distribution with equal peaks at -1 and 0.[1]

The TSGD model is attractive in practical context-conditioned image compression schemes since the distribution in each context is determined by just two parameters (rate of decay and

[1] The preference of -1 over $+1$ here is arbitrary.

offset), despite the source alphabet being, in principle, infinite (and, in practice, finite but quite large). This allows for the utilization of a fairly large number of contexts, while keeping the total number of parameters in the system at a moderate level. This is particularly important in an adaptive setting, where the statistics are "learned" from the data, and the code length includes a *model cost* term proportional to the total number of free statistical parameters [7]. Adaptive strategies for TSGDs based on symbol-by-symbol prefix codes, as well as universal schemes based on arithmetic coding, are discussed in the companion paper [8].

It is readily verified that the TSGD (1) has a finite entropy rate, given by

$$H(\theta, d) = \frac{h(\theta)}{1-\theta} + h(\rho)\,, \tag{3}$$

where $h(u) \triangleq -u\log u - (1-u)\log(1-u)$ is the binary entropy function ($\log u$ denotes the logarithm to base two of u), and

$$\rho = \frac{\theta^d}{\theta^{1-d} + \theta^d} \tag{4}$$

is the probability that a random variable drawn according to the distribution (1) be non-negative. By [9], the finiteness of $H(\theta, d)$ guarantees that a minimum expected-length prefix code exists and can be obtained by a sequence of Huffman-like procedures (however, this general result is non-constructive). Infinite entropy distributions are addressed in [10].

The main result of this paper is a complete characterization of optimal prefix codes for the TSGD (1). The family of optimal codes will be an extension of the *Golomb codes* [11], which are optimal for *one-sided* geometric distributions (OSGDs) of non-negative integers [12]. The optimal codes for the TSGD preserve the simplicity of the Golomb code, which enables simple calculation of the codeword of every given source symbol, without recourse to the storage of code tables for large alphabets. This property makes the family attractive for use in adaptive schemes [13, 14, 3], since it avoids the need to dynamically update code tables as in traditional adaptive Huffman coding (see, e.g., [15]). Thus, the economy of parameters of the TSGD is reflected in the simplicity of the codes, and only a small number of variables need to be updated, and simple rules applied, to adaptively select a code for each sample. The optimal family of prefix codes derived here enables the adaptive strategies for the TSGD studied in [8] and also in [16].

Previous approaches to finding efficient prefix codes for TSGDs have focused mainly on the case $d = 0$. A popular approach [13] is to encode an integer by applying a Golomb code to its index in the sequence $0, -1, +1, -2, +2, -3, +3, \ldots$. Notice that with $d \leq \frac{1}{2}$, this "folding" of the negative values into the positive ones ranks the integers in non-increasing probability order. A different heuristic approach, based on encoding the absolute value with a Golomb code and appending a sign bit for nonzero values, was proposed in [17]. As shown in Section 2, these strategies are sub-optimal for some ranges of the parameters (θ, d), even when restricted to the line $d = 0$. Some partial answers to the question of optimal codes for $d = 0$ can also be found in [18].

The remainder of the paper is organized as follows: In Section 2, we present our main result, characterizing the optimal prefix code for a TSGD given its parameters θ, d. As it turns out, the two-sided nature of the distribution, and the two-dimensionality of the parameter space add surprising complexity to the characterization, as compared to the one-sided case. The parameter space of (θ, d) will be divided into four types of regions, with a different optimal code construction applying to each type. The codes for two of the region types are, in general, fairly non-intuitive variants of Golomb codes, which had not been previously described in the literature. The section includes a general discussion of the method of proof of optimality, and insight into the origins of the fairly intricate partition of the (θ, d) plane. Once the codes and regions are appropriately "guessed", the actual proof, which uses a technique from [12], involves relatively tedious calculations, and is deferred to the Appendix. In Section 3 we derive the average code lengths attained by the optimal codes over the parameter space of (θ, d), and investigate their redundancy. Finally, in Section 4, we consider a simplified, sub-optimal subset of codes used in practice [13, 3]. We present optimal criteria to choose among these codes for given values of θ and d. These criteria extend results in [13] and [14], and admit efficient approximation in an adaptive setting, which is explored in more detail in [8]. Moreover, we bound the relative redundancy of the reduced family with respect to the full family of optimal codes, thus providing formal proof of a fact that had been observed in the literature (see [16], and [13] for OSGDs).

2 Optimal prefix codes for TSGDs

In this section, we develop a complete characterization of minimum expected-length prefix codes for the TSGD (1). To this end, we will partition the parameter space of (θ, d), $0 < \theta < 1$, $0 \le d \le \frac{1}{2}$, into regions, each region corresponding to a variant of a basic code construction device. In the next few definitions and lemmas, we describe the partition and some of its basic properties.

For a given value of d, define $\delta \triangleq \min\{d, \frac{1}{2} - d\}$. Clearly, $\delta \le \frac{1}{4}$. For every positive integer ℓ and every pair of model parameters (θ, d), define the functions

$$r_0(\ell, \theta, d) = \theta^{2\ell-1}(1 + \theta^{-2\delta}) + \theta^{\ell-1} - 1, \tag{5}$$

$$r_1(\ell, \theta, d) = \theta^{2\ell-1}(1 + \theta^{2\delta}) + \theta^{\ell} - 1, \tag{6}$$

$$r_2(\ell, \theta, d) = \theta^{\ell}(1 + \theta^{-2\delta}) - 1, \tag{7}$$

and

$$r_3(\ell, \theta, d) = \theta^{\ell}(1 + \theta^{2\delta}) - 1. \tag{8}$$

Lemma 1 *(i) Given $\ell > 1$ and d, r_0 has a unique root $\theta_0(\ell, d) \in (0,1)$. Similarly, for $\ell \ge 1$, r_1, r_2, and r_3 have unique roots in $(0,1)$, denoted, respectively, $\theta_1(\ell, d), \theta_2(\ell, d)$, and $\theta_3(\ell, d)$.*

(ii) For $\theta \in (0,1)$ and $0 \le i \le 3$, we have $\theta \le \theta_i(\ell, d)$ if and only if $r_i(\ell, \theta, d) \le 0$.

(iii) For $\ell \ge 1$, we have

$$\theta_0(\ell, d) < \theta_1(\ell, d) \le \theta_2(\ell, d) \le \theta_3(\ell, d) \le \theta_0(\ell+1, d)\,,$$

where we define $\theta_0(1, d) = 0$. Moreover, equality between $\theta_1(\ell, d)$ and $\theta_2(\ell, d)$, and between $\theta_3(\ell, d)$ and $\theta_0(\ell+1, d)$ occurs only at $d = \frac{1}{4}$, while equality between $\theta_2(\ell, d)$ and $\theta_3(\ell, d)$ occurs only at $d \in \{0, \frac{1}{2}\}$. Therefore, $\theta_1(\ell, d) < \theta_0(\ell+1, d)$.

Proof. (i) The existence and uniqueness of a root $\theta_i(\ell, d) \in (0,1)$ of r_i, $0 \le i \le 3$, is established by observing that, for fixed ℓ and d in the appropriate ranges, r_i is a continuous function of θ in $(0,1)$, $r_i(\ell, \theta, d) \to -1$ as $\theta \to 0^+$, $r_i(\ell, \theta, d)$ has a positive limit as $\theta \to 1^-$, and $\partial r_i / \partial \theta > 0$, $\theta \in (0,1)$. The monotonicity of r_i also yields part (ii) of the lemma. Notice that $r_0(1, \theta, d) \to 0$ as $\theta \to 0^+$, justifying the definition of $\theta_0(1, d) = 0$.

As for part (iii), we first observe that

$$r_1(\ell,\theta,d) - r_0(\ell,\theta,d) = \theta^{2\ell-1}(\theta^{2\delta} - \theta^{-2\delta}) + (\theta^{\ell} - \theta^{\ell-1}) < 0,$$

where the last inequality follows from $\theta < 1$, $\delta \geq 0$, and $\ell \geq 1$. Thus, due to the strict monotonicity of r_0 and r_1, we must have $\theta_0(\ell,d) < \theta_1(\ell,d)$. We now compare $\theta_2(\ell,d)$ with $\theta_1(\ell,d)$. For clutter reduction, we omit the arguments (ℓ,d) of the θ_i when they are clear from the context. It follows from the definition of θ_2 that

$$\theta_2^{\ell} = \frac{1}{1+\theta_2^{-2\delta}}.$$

Substituting θ_2 for θ in definition (6), we obtain

$$r_1(\ell,\theta_2,d) = \frac{\theta_2^{-1}}{(1+\theta_2^{-2\delta})^2}(1+\theta_2^{2\delta}) + \frac{1}{1+\theta_2^{-2\delta}} - 1 = \frac{\theta_2^{4\delta-1}-1}{1+\theta_2^{2\delta}} \geq 0,$$

where the last inequality follows from $\delta \leq \frac{1}{4}$. Thus, by part (ii), $\theta_2(\ell,d) \geq \theta_1(\ell,d)$, with equality occurring at $d = \delta = \frac{1}{4}$.

Next, definitions (7) and (8) imply $r_2(\ell,\theta,d) \geq r_3(\ell,\theta,d)$ for $\theta \in (0,1)$. Thus, we must have $\theta_2 \leq \theta_3$ by parts (i) and (ii) of the lemma. Equality occurs at $d \in \{0, \frac{1}{2}\}$, in which case $\delta = 0$, and $r_2(\ell,\theta,d) = r_3(\ell,\theta,d)$. Also, we have

$$\theta_3^{\ell} = \frac{1}{1+\theta_3^{2\delta}}.$$

Substituting θ_3 for θ in the expression for $r_0(\ell+1,\theta,d)$ derived from definition (5), we obtain

$$r_0(\ell+1,\theta_3,d) = \frac{\theta_3}{(1+\theta_3^{2\delta})^2}(1+\theta_3^{-2\delta}) + \frac{1}{1+\theta_3^{2\delta}} - 1 = \frac{\theta_3^{1-2\delta}-\theta_3^{2\delta}}{1+\theta_3^{2\delta}} \leq 0.$$

Thus, $\theta_3(\ell,d) \leq \theta_0(\ell+1,d)$, with equality at $d = \delta = \frac{1}{4}$. □

It follows from Lemma 1 that, for a given value of d, the functions r_i define a partition of the interval $(0,1)$ into sub-intervals, with boundaries given by the values $\theta_i(\ell,d)$, ordered as follows:

$$\begin{aligned} 0 = \theta_0(1,d) < \theta_1(1,d) \;&\leq\; \theta_2(1,d) \leq \theta_3(1,d) \leq \theta_0(2,d) < \theta_1(2,d) \leq \cdots \\ \cdots \;&\leq\; \theta_0(\ell,d) < \theta_1(\ell,d) \leq \theta_2(\ell,d) \leq \theta_3(\ell,d) \leq \cdots < 1. \end{aligned} \tag{9}$$

Moreover, it is easy to see from definition of θ_0 and from (5) that $\theta_0(\ell,d) \to 1$ as $\ell \to \infty$.

The different intervals defined by the boundaries $\theta_i(\ell, d)$ become two-dimensional regions once the dependence on d is taken into account. Each pair of model parameters (θ, d) falls in a region characterized by an integer parameter $\ell(\theta, d)$, and by one of four sub-intervals associated with ℓ, and determined by $\theta_i(\ell, d)$, $i = 0, 1, 2, 3$. By Lemma 1, part (ii), the parameter $\ell(\theta, d)$ is given by

$$\ell(\theta, d) = \max_{\ell \geq 1}\{\ell \,|\, r_0(\ell, \theta, d) > 0\}. \tag{10}$$

Since $\lim_{\ell \to \infty} \theta_0(\ell, d) = 1$, $\ell(\theta, d)$ is well defined for all θ and d in the range of interest. In fact, $\ell(\theta, d)$ can be explicitly computed by setting $z = \theta^\ell$ and solving the quadratic equation

$$z^2(1 + \theta^{-2\delta}) + z - \theta = 0,$$

which has a unique solution z_0 in the open interval $(0, 1)$. Then,

$$\ell(\theta, d) = \left\lfloor \frac{\log z_0}{\log \theta} \right\rfloor .$$

For ease of reference, we label the regions defined by the partition in (9), for each value of ℓ, as follows:

Region I: $\theta_0(\ell, d) < \theta \leq \theta_1(\ell, d)$,

Region II: $\theta_1(\ell, d) < \theta \leq \theta_2(\ell, d)$, $d \leq \frac{1}{4}$,

Region II': $\theta_1(\ell, d) < \theta \leq \theta_2(\ell, d)$, $d > \frac{1}{4}$,

Region III': $\theta_2(\ell, d) < \theta \leq \theta_3(\ell, d)$,

Region IV: $\theta_3(\ell, d) < \theta \leq \theta_0(\ell + 1, d)$, $d \leq \frac{1}{4}$,

Region IV': $\theta_3(\ell, d) < \theta \leq \theta_0(\ell + 1, d)$, $d > \frac{1}{4}$.

We define *Region III* as the union of regions II', III' and IV'. The various two-dimensional regions for $\ell = 1, 2$ are illustrated in Figure 1. Notice the symmetry around $d = \frac{1}{4}$.

We now turn to the basic building blocks of our code construction. For any integer x, define

$$M(x) = \begin{cases} 2x & x \geq 0, \\ 2|x| - 1 & x < 0. \end{cases} \tag{11}$$

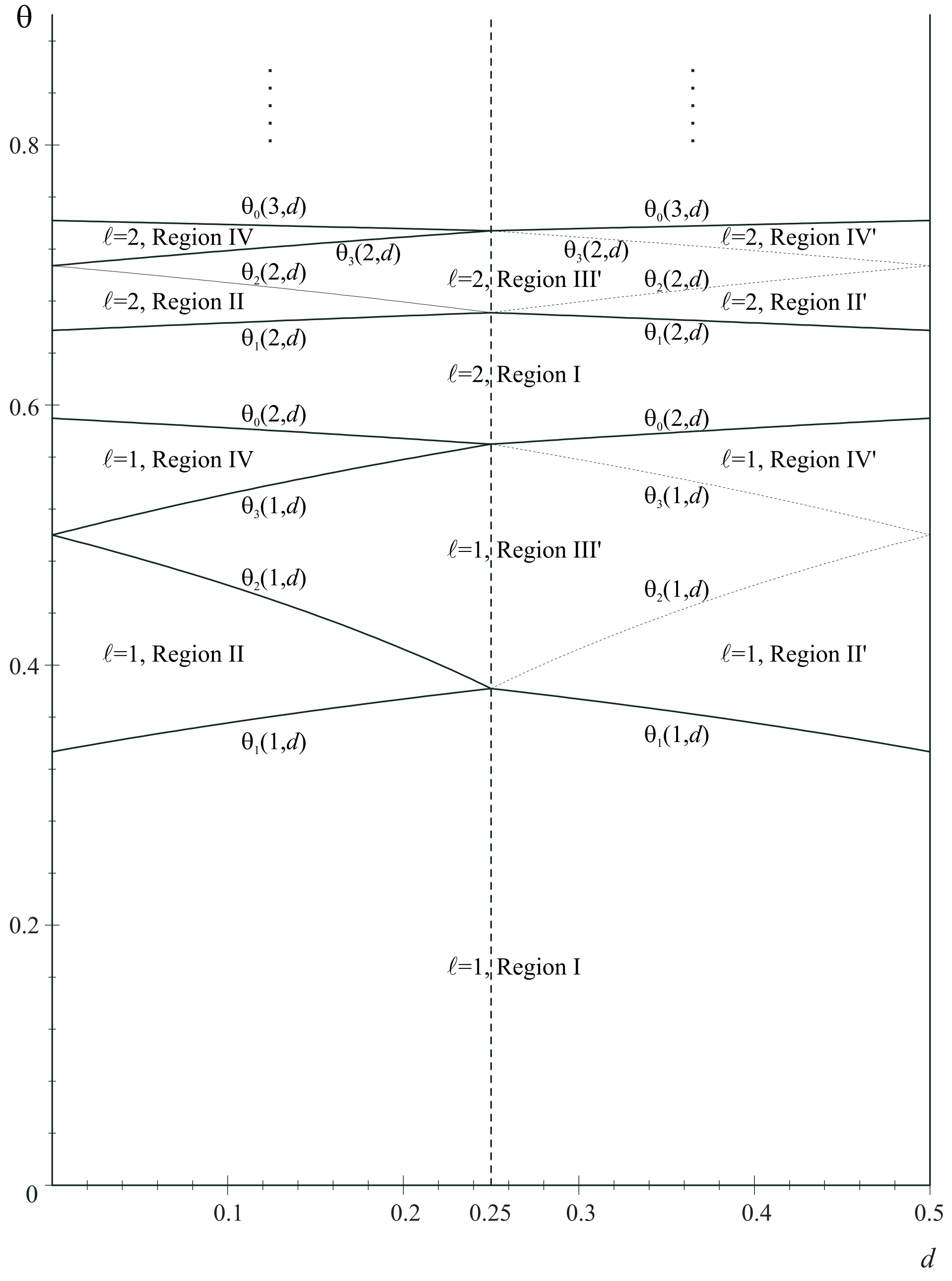

Figure 1: Parameter regions. Region III is defined as the union of regions II', III', and IV'.

For non-negative integers i, the inverse function $\mu(i)$ of M is given by

$$\mu(i) = (-1)^i \left\lceil \frac{i}{2} \right\rceil .$$

Since $0 \leq d \leq \frac{1}{2}$, the integers are ranked in decreasing probability order by

$$P_{(\theta,d)}(0) \geq P_{(\theta,d)}(-1) \geq P_{(\theta,d)}(1) \geq P_{(\theta,d)}(-2) \geq P_{(\theta,d)}(2) \geq \cdots \tag{12}$$

Thus, $M(x)$ is the index of x in the probability ranking, starting with index 0 and with ties, if any, broken according to the order in (12). Conversely, $\mu(i)$ is the symbol with the ith highest probability.

For any positive integer L, let G_L denote the Golomb code [11] of order L, which encodes a non-negative integer u into a binary codeword $G_L(u)$ consisting of two parts: (a) an *adjusted binary* representation of $u' = u \bmod L$, using $\lfloor \log L \rfloor$ bits if $u' < 2^{\lceil \log L \rceil} - L$, or $\lceil \log L \rceil$ bits otherwise, and (b) a *unary* representation of $q = \lfloor u/L \rfloor$, using $q+1$ bits. Here, $a \bmod b$ denotes the least non-negative residue of a modulo b. We will denote by $G_L(u)b$ the binary string resulting from appending $b \in \{0, 1\}$ to $G_L(u)$.

We are now ready to state the main result of the paper.

Theorem 1 *Let x denote an integer-valued random variable distributed according to the TSGD (1) for a given pair of model parameters (θ, d), $0 < \theta < 1$, $0 \leq d \leq \frac{1}{2}$, and let $\ell = \ell(\theta, d)$ as defined in (10). Then, an optimal prefix code for x is constructed as follows:*

Region I. *If $\theta_0(\ell, d) < \theta \leq \theta_1(\ell, d)$, encode x using $G_{2\ell-1}(M(x))$.*

Region II. *If $d \leq \frac{1}{4}$ and $\theta_1(\ell, d) < \theta \leq \theta_2(\ell, d)$, encode $|x|$ using the code $G_\ell(\chi_\ell(|x|))$, where the mapping $\chi_\ell(|x|)$ is defined below, and append a sign bit whenever $x \neq 0$. Let r be the integer satisfying $2^{r-1} \leq \ell < 2^r$, and let $s = 2^r - \ell$. Define*

$$\chi_\ell(|x|) = \begin{cases} s, & |x| = 0 \textit{ and } s \neq \ell, \\ 0, & |x| = s \textit{ and } s \neq \ell, \\ |x|, & \textit{otherwise.} \end{cases}$$

Region III. *If $d \leq \frac{1}{4}$ and $\theta_2(\ell, d) < \theta \leq \theta_3(\ell, d)$, or $d > \frac{1}{4}$ and $\theta_1(\ell, d) < \theta \leq \theta_0(\ell + 1, d)$, encode x using $G_{2\ell}(M(x))$.*

Region IV. *If $d \leq \frac{1}{4}$ and $\theta_3(\ell, d) < \theta \leq \theta_0(\ell+1, d)$, define s as in Region II, encode $|x|$ using $J_\ell(|x|)$ defined below, and append a sign bit whenever $x \neq 0$.*

$$J_\ell(|x|) = \begin{cases} G_\ell(|x|-1), & |x| > s, \\ G_\ell(|x|), & 1 \leq |x| < s, \\ G_\ell(0)0, & x = 0, \\ G_\ell(0)1, & |x| = s. \end{cases}$$

□

Discussion.

Relation to prior work. Theorem 1 includes the main result of [12] as a special case when $d = \frac{1}{4}$. In this case, the distribution (1), after reordering of the integers in decreasing probability order, is equivalent to an OSGD with parameter $\phi = \sqrt{\theta}$. As shown in [12], the optimality transition for such a distribution between the L-th order Golomb code and the $L+1$st, $L \geq 1$, occurs at the (unique) value $\phi \in (0,1)$ such that $g_L(\phi) = \phi^L + \phi^{L-1} - 1 = 0$. It can readily be verified that $r_0(\ell, \phi^2, \frac{1}{4}) = 0$ if and only if $g_{2\ell-1}(\phi) = 0$, and $r_1(\ell, \phi^2, \frac{1}{4}) = 0$ if and only if $g_{2\ell}(\phi) = 0$, $\ell \geq 1$.

Notice that the optimal codes for regions I and III are *asymmetric*, in that they assign different code lengths to x and $-x$ for some values of x. In contrast, the codes for regions II and IV are symmetric. The mapping (11) was first employed in [13] to encode TSGDs centered at zero by applying a Golomb code to $M(x)$. Theorem 1 shows that this strategy (which was also used in [3] and always produces asymmetric codes) is optimal for values of θ and d corresponding to regions I and III, but is not so for regions II and IV. In fact, both [13] and [3] actually use a *sub-family* of the Golomb codes, for which the code parameter is a power of two, making the encoding and decoding procedures extremely simple. This sub-family is further investigated in Section 4. A different heuristic approach, based on encoding the absolute value with a Golomb code and appending a sign bit for nonzero values, was proposed in [17]. Theorem 1 shows that this heuristic (which always produces symmetric codes) is optimal only in Region II, and then only when ℓ is a power of two, in which case $\chi_\ell(\cdot)$ is the identity mapping.

Method of the proof. In the proof of Theorem 1, we will borrow the concept of a *reduced source*, used in [12] to prove the optimality of Golomb codes for OSGDs. Reduced sources were also applied in [19] to construct optimal prefix codes for distributions whose tails decay faster than a geometric rate with ratio $(\sqrt{5}-1)/2$, e.g., Poisson distributions. The concept is generalized in [9] and shown to be applicable to all finite entropy distributions of the integers, albeit in a non-constructive fashion.

Here, for each of the regions defined for (θ, d), and each integer $m \geq 0$, we will define a finite mth order reduced source $\mathcal{R}_{L,m}$ as a multiset containing the first $2m - b$ probabilities in the ranking (12), where $b \in \{0,1\}$ depends on the region, and a finite set of *super-symbol* probabilities, some of which represent infinite "tails" of the remaining integers. The index L also expresses region dependence, and it satisfies $L = 2\ell - 1$ for Region I and $L = 2\ell$ otherwise, where $\ell = \ell(\theta, d)$.

We will use Huffman's algorithm to construct an optimal prefix code for $\mathcal{R}_{L,m}$, and will then let m tend to infinity, thus obtaining a code for the integers. The code length assigned by our construction to an arbitrary integer x will be the one assigned by the optimal prefix code for $\mathcal{R}_{L,m}$, for the least m such that $2m - b \geq M(x)$. By the nature of the construction, this code length will remain unchanged for larger values of m. The formal argument validating the limiting step, and why it yields an optimal prefix code for the original infinite source, is given in [12] and it carries to our construction. The exact definition of the reduced sources used, and the way the Huffman construction on a reduced source proceeds, will vary according to the region the parameter pair (θ, d) falls into, thus leading to different code structures for the different regions.

It turns out that the two-sided nature of the distribution, and the two-dimensionality of the parameter space add surprising complexity to the characterization, as compared to the one-sided case. This complexity is evidenced by the variety of regions and codes in Theorem 1 (in fact, much of the intricate structure exists even in the simpler, one-dimensional case $d = 0$). The codes for regions II and IV had not been described in the literature, except for the special case mentioned above in connection with the heuristic in [17]. Examples of optimal code trees for regions II and IV, with $\ell = 3$ $(s = 1)$, are shown in Figure 2, together with the tree of G_3 for ease of reference. In either region, the tree is a fairly non-intuitive transformation of the one resulting from applying G_3 to $|x|$ and appending a sign bit when $x \neq 0$. In the case of Region II, the nodes for the symbol 0 and the pair $\pm s = \pm 1$ (regarded as one symbol) have

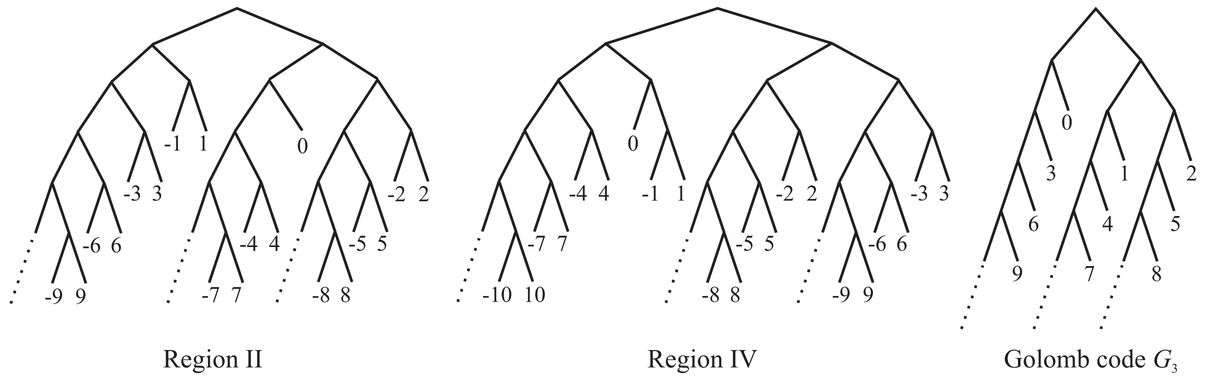

Figure 2: Coding trees for regions II and IV, $\ell = 3$ ($s = 1$).

switched places relative to the locations of 0 and 1, respectively, in the tree for G_3. This is due to the action of χ_3. In the case of Region IV, the original node for 0 in G_3 has been split to accommodate 0 and ± 1, and all other nodes have been "promoted", i.e. the pair $\pm j$ is at the location of $j-1$ in G_3, $j > 1$. This corresponds to the application of $J_3(|x|)$, noting that in this case, there are no integers $|x|$ in the range $1 \leq |x| < s$.

We now offer some insight into how the various parameter regions (and hence the above mentioned complexity) arise. The functions $r_0(\ell, \theta, d)$ and $r_1(\ell, \theta, d)$ determine the positive integer parameter L characterizing a basic property of Golomb-type codes: Starting from some codeword length Λ, the code contains exactly L codewords of length $\Lambda + i$ for all $i \geq 0$ (for the codes of Theorem 1, Λ is at most $\lambda + 2$, where λ is the minimal codeword length). The lines $r_1(\ell, \theta, d) = 0$ mark the transition from regions with $L = 2\ell - 1$ to regions with $L = 2\ell$, $\ell \geq 1$, while the lines $r_0(\ell, \theta, d) = 0$ mark the transition from $L = 2\ell$ to $L = 2\ell + 1$. The role of the functions r_0 and r_1 is, therefore, analogous to that of the function g_L determining the code transitions in [12].

The lines $r_2(\ell, \theta, d) = 0$ and $r_3(\ell, \theta, d) = 0$, in turn, determine how the optimal code construction handles "natural pairs" of symbols in regions with $L = 2\ell$. These are pairs of symbols that are close in probability, i.e., $\{x, -x\}$ for $d \leq \frac{1}{4}$ and $\{x-1, -x\}$ for $d > \frac{1}{4}$, where x is a positive integer. Focusing first on the case $d \leq \frac{1}{4}$, and assuming x is sufficiently large, we observe that if the optimal code tree construction merges x and $-x$ (i.e., makes them sibling leaves), then by the constraints imposed by r_0 and r_1 in determining the value of L, the resulting probability $\sigma = P_{(\theta,d)}(x) + P_{(\theta,d)}(-x)$ must fall in the proximity of the interval $[P_{(\theta,d)}(x-\ell), P_{(\theta,d)}(-(x-\ell))]$ on the real line. It turns out that the regions for $L = 2\ell$ are determined by whether σ is to the *left* (Region II), *inside* (Region III'), or to the *right* (Region IV) of the interval. When σ falls inside the interval, merging of x and $-x$ in the optimal tree construction would prevent the translated natural pair $\{x-\ell, -(x-\ell)\}$ from merging. Because of the self-similar character of the distribution, this condition applies to all x, and it results, in general, in a construction that does not merge natural pairs (e.g., the asymmetric codes of Region III'). On the other hand, when σ falls outside the interval between the probabilities of a natural pair, these pairs tend to merge, resulting in the symmetric codes of regions II and IV.

A similar situation exists for $d > \frac{1}{4}$, but with a twist. There, again, the optimal construction will not merge natural pairs in Region III', and it will in regions II' and IV'. Nevertheless, regardless of whether they are merged or not, symbols in a natural pair end up with equal code lengths in the three sub-regions of Region III. This is due to the fact that the optimal code is a Golomb code of even parameter, and that every integer belongs to a natural pair when $d > \frac{1}{4}$. Thus, while the three sub-regions comprising Region III for a given ℓ admit the same optimal prefix code, the iterations leading to the optimal length distribution, and their underlying trees are different: the tree constructed in regions II' and IV' corresponds to that of G_ℓ, but with each leaf split into two, while the tree for Region III' corresponds directly to that of G_L, $L = 2\ell$. The two tree configurations are illustrated in Figure 3, for $\ell = 1$. A discussion of the number of different coding trees that can be optimal for a given distribution, including some infinite alphabet cases, can be found in [20].

The proof of Theorem 1 is deferred to the Appendix.

3 Code length and redundancy

We refer to the prefix codes defined in Theorem 1 for regions I, II, III, and IV as codes of *type I, II, III, and IV*, respectively. The expected code lengths for these codes when applied to the TSGD (1) are derived from their definitions in Theorem 1, and from the length distribution of the Golomb code, which follows directly from its definition. The resulting average code lengths, summarized in the following lemma, are computed by applying straightforward geometric sums,

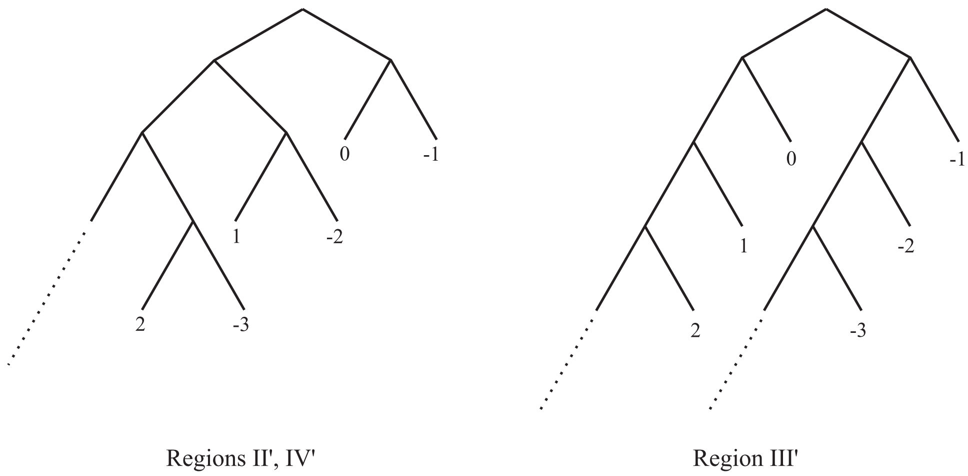

Figure 3: Coding trees for Region III, $\ell = 1$.

and derived sums of the general form $\sum_i i\alpha^i$. Notice that the expected code lengths apply to all allowable parameter values (θ, d), and not just to the region for which a code is optimal.

Lemma 2 *Let ℓ be an arbitrary positive integer, and let $\overline{\lambda}_{X,\ell}(\theta, d)$ denote the average code length for a code of type X (X=I,II,III,IV), for the given value of ℓ, when applied to a TSGD with parameters (θ, d). Let r and s be defined as in Theorem 1, and let $s' = s \bmod 2^{r-1}$. Then, we have*

$$\overline{\lambda}_{\mathrm{I},\ell}(\theta, d) = 1 + \lfloor \log(2\ell - 1) \rfloor + \frac{\theta^{s'}}{1 - \theta^{2\ell-1}}(1 - P_{(\theta,d)}(0) + \theta^{\ell}),$$

$$\overline{\lambda}_{\mathrm{II},\ell}(\theta, d) = 1 + \lceil \log \ell \rceil + (1 - P_{(\theta,d)}(0))\theta^{s'} \left(1 + \frac{\theta^{\ell-1}}{1 - \theta^{\ell}}\right),$$

$$\overline{\lambda}_{\mathrm{III},\ell}(\theta, d) = 1 + \lfloor \log(2\ell) \rfloor + \frac{\theta^{s}}{1 - \theta^{\ell}},$$

$$\overline{\lambda}_{\mathrm{IV},\ell}(\theta, d) = 2 + \lfloor \log \ell \rfloor + (1 - P_{(\theta,d)}(0))\theta^{s-1} \left(1 + \frac{\theta^{\ell+1}}{1 - \theta^{\ell}}\right).$$

□

Let $\overline{\lambda}_{\mathrm{opt}}(\theta, d)$ denote the expected code length for the optimal prefix code corresponding to the parameters (θ, d). Define the *redundancy* of the family of optimal prefix codes as

$$R(\theta, d) = \overline{\lambda}_{\mathrm{opt}}(\theta, d) - H(\theta, d),$$

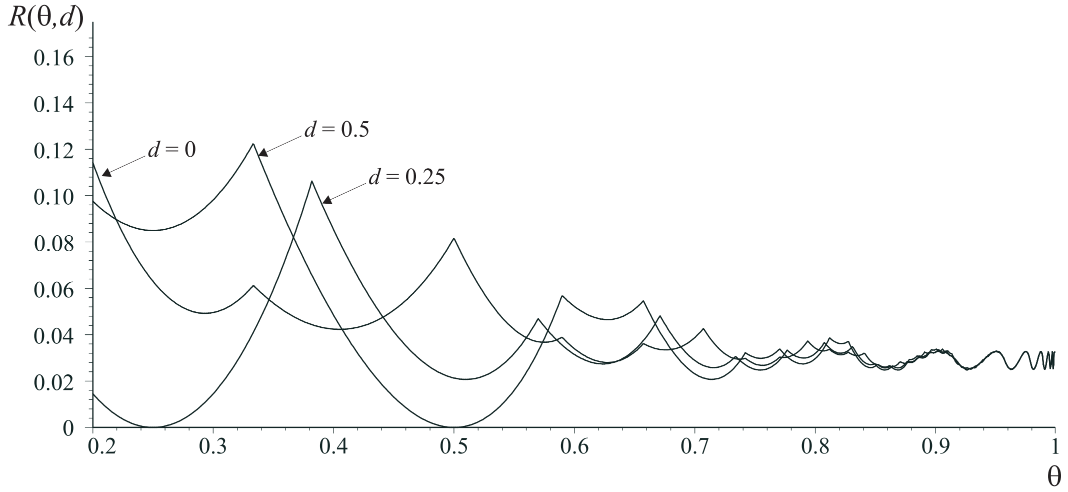

Figure 4: Redundancy of optimal prefix codes (in bits/symbol).

where $H(\theta, d)$ is the entropy rate given in (3).

Figure 4 plots the redundancy as a function of θ for $d = 0, \frac{1}{4}$ and $\frac{1}{2}$, respectively. The region near $\theta = 0$ is omitted, to allow appropriate scaling of the rest of the plot. The redundancy is highest in that region, and its behavior is as follows: When $d < \frac{1}{2}$, the entropy rate tends to zero as $\theta \to 0$, and the expected code length tends to 1 bit/symbol. When $d = \frac{1}{2}$, the entropy rate tends to one as $\theta \to 0$, since the TSGD degenerates, in the limit, into a bi-modal distribution with all the probability mass distributed equally between $x = 0$ and $x = -1$. The expected code length of the optimal code, in turn, tends to 1.5 bits/symbol, yielding a limit of 0.5 bits/symbol for the redundancy as θ approaches zero. However, for values of $\theta \geq 0.5$, which are most likely to occur in practical applications, the *relative* redundancy remains below 3%.

It is apparent from the plots in Figure 4 that the redundancy vanishes at the points $(\theta, d) = (\frac{1}{2}, \frac{1}{2})$ and $(\theta, d) = (\frac{1}{4}, \frac{1}{4})$. It is well known that the redundancy of the Huffman code for a discrete distribution is zero if and only if the distribution is *dyadic*, i.e., all its symbol probabilities are powers of two. The following lemma shows that the TSGD (1) is indeed dyadic for the mentioned two points in the parameter space, and nowhere else.

Lemma 3 *The TSGD (1) is dyadic if and only if* $(\theta, d) \in \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4})\}$.

Proof. Assume the distribution is dyadic. Then, all ratios between symbol probabilities are powers of two. In particular, we can write $\theta = 2^{-u}$ and $\theta^{2d} = 2^{-v}$ for integers $0 \leq v \leq u$, where at least one of the inequalities is strict. Consider the probability $P_{(\theta,d)}(0)$. We have

$$P_{(\theta,d)}(0) = C(\theta,d)\theta^d = \frac{1-\theta}{\theta^{1-2d}+1} = \frac{1-2^{-u}}{2^{v-u}+1} = 2^{-v}\frac{2^u-1}{2^{u-v}+1}\,.$$

Clearly, this probability is dyadic if and only if $(2^u-1)/(2^{u-v}+1)$ is a power of two, which holds if and only if either $u = v = 1$ or $u = 2$ and $v = 1$. These cases correspond to $(\theta,d) = (\frac{1}{2},\frac{1}{2})$ and $(\theta,d) = (\frac{1}{4},\frac{1}{4})$, respectively. It is readily verified that the corresponding dyadic distributions are $P_{(\theta,d)}(x) = 2^{-|x+\frac{1}{2}|-\frac{3}{2}}$ and $P_{(\theta,d)}(x) = 2^{-(M(x)+1)}$, respectively. The first distribution corresponds to either tree in Figure 3, while the second corresponds to the tree for a unary code. $\square$

The plots of Figure 4 are analogous to the redundancy plot in [12] for the OSGD case. In fact, the plot for $d = \frac{1}{4}$ is equivalent to that in [12], except for the fact that a point with abscissa θ in the figure corresponds to one with abscissa $\sqrt{\theta}$ in [12]. Furthermore, the dependency of the redundancy on d vanishes as θ approaches 1. This is evidenced by the three curves merging in the figure, and can be formally verified by inspecting the expression for the entropy in (3) and the code lengths in Lemma 2. Therefore, the oscillatory pattern of the redundancy near $\theta = 1$, characterized in [12] for Golomb codes, applies to the codes of Theorem 1 as well.

4 Low complexity code sub-families

The family of optimal prefix codes presented in Section 2 makes use of Golomb codes of arbitrary order $L \geq 1$. Practical schemes such as [13] and [3] have been restricted to sub-families based on Golomb codes for which L is a power of two, which yield clear complexity advantages already recognized in [11]. Given an integer parameter $r \geq 0$, the code G_{2^r} encodes a non-negative integer u in two parts: the r least significant bits of u, followed by the number formed by the remaining higher order bits of u, in unary representation. The code length is $r + 1 + \lfloor u/2^r \rfloor$ bits. Thus, with this restriction, the division by an arbitrary positive integer L, necessary for encoding u with G_L, and deemed too complex in some applications, is trivialized. Furthermore, the modified binary encoding of the remainder $u \bmod L$ becomes an identity mapping, instead of a non-trivial, variable length mapping in the general case. Finally, as shown in [3, 16] and the companion paper [8], codes for which $L = 2^r$ admit very simple yet accurate on-line adaptation strategies, and, as we shall see in the sequel, the complexity benefits obtained from the restriction of the code family have a rather modest cost in code length.

Motivated by these considerations, we next study the family $\mathcal{C}$ of prefix codes utilized in [13] and [3]. Let $\mathcal{G}_r$, $r \geq 0$, denote a code that maps an arbitrary integer x to $G_{2^r}(M(x))$. Then, the family of interest is defined as $\mathcal{C} = \{\mathcal{G}_r \,|\, r \geq 0\}$. Notice that $\mathcal{G}_0$ is the code of type I with $\ell = 1$, while for $r \geq 1$, $\mathcal{G}_r$ is the code of type III with $\ell = 2^{r-1}$. Thus, $\mathcal{C}$ is the sub-family of asymmetric codes of Section 2 for which the parameter of the associated Golomb code is a power of two.

Lemma 4 below presents optimal decision regions to choose among codes in $\mathcal{C}$ for given values of θ and d. In the lemma, we will relax the restriction on d, and allow for values in the range $0 \leq d \leq 1$. As it turns out, this comes almost for free with the family $\mathcal{C}$. In general, the optimal code for a value of d in the range $\frac{1}{2} < d \leq 1$ is obtained by using the optimal code for $1-d$, and applying the transformation $x \to -(x+1)$ to the integer symbols to be encoded. Now, since the codes of type III in $\mathcal{C}$ assign the same code length to x and $-(x+1)$ for all integers x, one needs only be concerned with the transformation when dealing with $\mathcal{G}_0$. Lemma 4 addresses this case by denoting $\mathcal{G}_0'(x) = \mathcal{G}_0(-x-1)$, and implementing the check on the range of d as part of the classification procedure. Also, the classification will be more readily described by changing variables, and mapping the parameter pair (θ, d) to a pair (S, ρ), where

$$S = \theta/(1-\theta), \tag{13}$$

and ρ is given in (4). We recall that ρ is the probability of an integer x distributed according to $P_{(\theta,d)}$ being non-negative. With $\theta \in (0,1)$ and $d \in [0,1]$, the range of S is $(0,\infty)$, and the range of ρ is $[\theta/(1+\theta), 1/(1+\theta)]$. Also, $d = \frac{1}{2}$ corresponds to $\rho = \frac{1}{2}$, and the mapping $d \to 1-d$ corresponds to $\rho \to 1-\rho$. In adaptive schemes, the parameters S and ρ are a more natural choice for characterizing the TSGD, as they are estimated by very simple functions of the observed data [3, 8, 16].

Lemma 4 *Consider an integer-valued random variable with probability distribution $P_{(\theta,d)}$ for a given pair of model parameters (θ, d), $0 < \theta < 1$, $0 \leq d \leq 1$. Let (S, ρ) be the transformed* *parameters derived according to (13) and (4), and define* $\phi \triangleq (\sqrt{5}+1)/2$*. The following decision rules minimize the expected codeword length over the sub-family of codes* $\mathcal{C}$*.*

a. *If* $S \leq \phi$*, compare* S*,* ρ*, and* $1-\rho$*. If* S *is largest, choose code* $\mathcal{G}_1$*. Otherwise, if* ρ *is largest, choose* $\mathcal{G}_0$*. Otherwise, choose* $\mathcal{G}'_0$*.*

b. *If* $S > \phi$*, choose code* $\mathcal{G}_{r+1}$*,* $r \geq 1$ *provided that*

$$\frac{1}{\phi^{(2^{-r+1})}-1} < S \leq \frac{1}{\phi^{(2^{-r})}-1}\,. \tag{14}$$

Proof. Let $\overline{\lambda}_r(\theta,d)$ denote the average code length for code $\mathcal{G}_r$, $r \geq 0$, and let $\overline{\lambda'}_0(\theta,d)$ denote the average code length for code $\mathcal{G}'_0$. It follows from Lemma 2 applied to the code of type I and $\ell = 1$, and from the definitions of S, ρ, and $P_{(\theta,d)}$ that

$$\overline{\lambda}_0(\theta,d) = \overline{\lambda'}_0(\theta,1-d) = 2+2S-\rho\,. \tag{15}$$

The other codes in the sub-family under consideration ($r > 0$) are of type III, so we apply Lemma 2 for that type, with $\ell = 2^{r-1}$, to obtain

$$\overline{\lambda}_r(\theta,d) = \overline{\lambda}_{\mathrm{III},\ell}(\theta,d) = r+1+\frac{\theta^\ell}{1-\theta^\ell}\,. \tag{16}$$

In particular, $\overline{\lambda}_1(\theta,d) = 2+S$, which must be compared to $\overline{\lambda}_0(\theta,d)$ and $\overline{\lambda}'_0(\theta,d)$ to select among $\mathcal{G}_0$, $\mathcal{G}'_0$, and $\mathcal{G}_1$. The code selection for $r > 0$ is done according to the sign of

$$\overline{\lambda}_{r+1}(\theta,d) - \overline{\lambda}_r(\theta,d) = 1 - \frac{\theta^\ell}{1-\theta^{2\ell}}\,. \tag{17}$$

By (17), the maximum value of θ for which $\mathcal{G}_r\,, r > 0$, is the best code from $\mathcal{C}$, is such that $\theta^{2^{r-1}} = 1-\theta^{2^r}$, namely,

$$\theta = \phi^{-2^{-r+1}}\,. \tag{18}$$

Thus, the maximum value of S for which $\mathcal{G}_r\,, r > 0$, is the best code from $\mathcal{C}$, is

$$S = \frac{1}{\phi^{(2^{-r+1})}-1}\,. \tag{19}$$

The decision rule of Lemma 4 follows from equations (15), (16), and (19). □

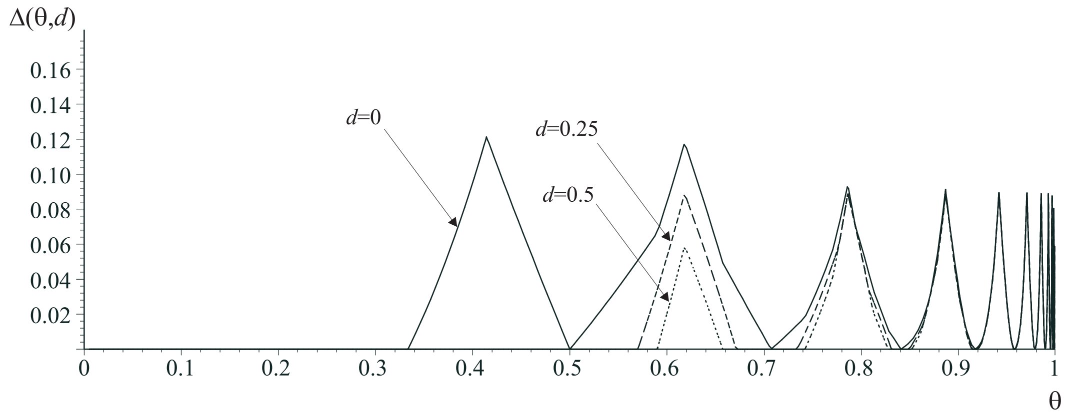

Figure 5: Code penalty of $\mathcal{C}$ vs. optimal prefix codes (in bits/symbol).

Lemma 4 extends results in [13] and [14]. The golden ratio ϕ is mentioned in connection with Golomb codes in [20], and in [14]. Also, ϕ^{-1} serves as a threshold in rate of decay for applicability of the results in [19].

Next, we study the penalty in code length incurred by restricting the code family to $\mathcal{C}$, as opposed to the full family of optimal codes. Let $\overline{\lambda}_{\mathcal{C}}(\theta, d)$ denote the expected code length of the best code from $\mathcal{C}$ for the parameters (θ, d), as selected by the decision rule of Lemma 4, and define

$$\Delta(\theta, d) = \overline{\lambda}_{\mathcal{C}}(\theta, d) - \overline{\lambda}_{\mathrm{opt}}(\theta, d)\,.$$

Figure 5 plots $\Delta(\theta, d)$ as a function of θ for $d = 0, \frac{1}{4}$, and $\frac{1}{2}$, respectively. By Lemma 2, the expected code length for codes of type III is independent of d, whereas it increases with d for the other types. In addition, it is readily verified that $\overline{\lambda}_{\mathrm{I},1}(\theta, d) - \overline{\lambda}_{\mathrm{II},1}(\theta, d)$ decreases with d. Therefore, we focus on the line $d = 0$ (represented by the upper, solid line curve in the figure), which yields the worst case penalty

$$\max_{d} \Delta(\theta, d) = \Delta(\theta, 0) \triangleq \Delta(\theta)\,.$$

This case is also special in that $\Delta(\theta, 0) > 0$ almost everywhere for $\theta > \theta_1(1, 0) = \frac{1}{3}$, since the codes $\mathcal{G}_r$, $r \geq 1$, are not optimal on the line $d = 0$, except for the discrete points $\theta{=}\theta_2(\ell, 0){=}\theta_3(\ell, 0){=}\left(\frac{1}{2}\right)^{1/\ell}$, $\ell = 2^{r-1}$ (refer to Theorem 1 and Figure 1). When $d > 0$, each $\mathcal{G}_r$, $r > 1$, is optimal for some interval of θ, of positive measure, for which $\Delta(\theta, d)$ vanishes.

Theorem 2 below formalizes the behavior observed in Figure 5. It shows that $\Delta(\theta)$ attains its largest value $\Delta_{\max} \approx 0.12$ at $\theta \approx 0.41$ and it oscillates between 0 and local maxima that, as $\theta \to 1$, approach $\Delta_{\lim} \approx 0.09$.

Theorem 2 *Define* $\Delta_{\max} = \frac{3}{\sqrt{2}} - 2$ *and* $\Delta_{\lim} = 5\phi - 8$. *Then,*

(i) $\Delta(\theta) = 0$ *for every* θ, $0 < \theta \leq \frac{1}{3}$, *and for* $\theta = \left(\frac{1}{2}\right)^{2^{-r+1}}$, *where* r *is any positive integer.*

(ii) $\Delta(\theta)$ *has local maxima at* $\theta = \sqrt{2} - 1$ *and all* θ *such that* $\theta^{2^{r-1}} = \phi^{-1}$ *for a positive integer* r. *Local maxima and zeroes alternate, and the function is monotonic between them.*

(iii) The absolute maximum is $\Delta(\sqrt{2} - 1) = \Delta_{\max}$, *and* $\limsup_{\theta \to 1} \Delta(\theta) = \Delta_{\lim}$.

Proof. Part (i) follows from the discussion preceding the theorem and the fact that $\mathcal{G}_0$ is optimal on the interval $0 < \theta \leq \theta_1(1,0) = \frac{1}{3}$. For part (ii), assume first $\theta \leq \frac{1}{2}$. The best code in $\mathcal{C}$ prescribed by Lemma 4 in the centered case is $\mathcal{G}_0$ for $\theta \leq \sqrt{2}-1$ and $\mathcal{G}_1$ for θ in the range $(\sqrt{2}-1, \frac{1}{2}]$. Direct computation of the code length penalty using Lemma 2 for the various regions prescribed by Theorem 1 reveals the claimed behavior. Furthermore,

$$\Delta(\sqrt{2}-1) = \overline{\lambda}_{\mathrm{II},1}(\sqrt{2}-1) - \overline{\lambda}_{\mathrm{I},1}(\sqrt{2}-1) = \Delta_{\max} \tag{20}$$

where here and throughout the proof we omit the offset argument d, which is understood to be zero, in the expected code lengths.

Next, given $M \triangleq 2^{r-1}$, $r \geq 1$, define

$$\ell^* \triangleq \left\lfloor \frac{\ln 2}{\ln \phi} M \right\rfloor .$$

For an integer ℓ, $M \leq \ell < 2M$, consider the interval $\mathcal{I}_\ell = [\theta_\ell, \theta_{\ell+1}]$, where the simplified notation $\theta_\ell \triangleq \left(\frac{1}{2}\right)^{1/\ell}$ is used in lieu of the equivalent expressions $\theta_2(\ell,0)$ or $\theta_3(\ell,0)$. Theorem 1 prescribes five codes for values of $\theta \in \mathcal{I}_\ell$: one for each type IV, I, II, and two of type III, one for each interval end point (see also Figure 1). By Lemma 4, the best code in $\mathcal{C}$ for $\theta \in \mathcal{I}_\ell$, in turn, is $\mathcal{G}_r$ if $\ell < \ell^*$ and $\mathcal{G}_{r+1}$ if $\ell > \ell^*$, whereas the interval $\mathcal{I}_{\ell^*}$ contains the transition point $\theta = \phi^{-\frac{1}{M}} \triangleq \phi_M$ between the regions in which $\mathcal{G}_r$ and $\mathcal{G}_{r+1}$ are optimal among codes in $\mathcal{C}$. To show that $\Delta(\theta)$ is increasing in (θ_M, ϕ_M) and decreasing in (ϕ_M, θ_{2M}), we use the following property, which can be proved using the code length expressions:

Property 1: Let C_1 and C_2 denote two of the five codes prescribed by Theorem 1 for values of $\theta \in \mathcal{I}_\ell$, with expected code lengths $\overline{\lambda}_{C_1}(\theta)$ and $\overline{\lambda}_{C_2}(\theta)$, respectively. Assume that at least one of the two codes is of type III, and that C_1 is optimal for smaller values of θ than C_2. Then, $\overline{\lambda}_{C_1}(\theta) - \overline{\lambda}_{C_2}(\theta)$ has a unique root θ_C, $\theta_C \in \mathcal{I}_\ell$, and is increasing for $\theta > \min(\theta_C, \phi_M)$.

Now, assume $\theta \in \mathcal{I}_\ell$. If $\theta<\phi_M$ and C_2 is optimal, we have

$$\Delta(\theta) = \overline{\lambda}_{\mathrm{III},M}(\theta) - \overline{\lambda}_{C_2}(\theta) = \sum_{i=M}^{\ell-1} \left[\overline{\lambda}_{\mathrm{III},i}(\theta) - \overline{\lambda}_{\mathrm{III},i+1}(\theta)\right] + \left[\overline{\lambda}_{\mathrm{III},\ell}(\theta) - \overline{\lambda}_{C_2}(\theta)\right] . \tag{21}$$

By Property 1, each difference on the right-hand side of (21) is increasing in the region of optimality of C_2. Therefore, $\Delta(\theta)$ is increasing in that region, and, consequently, in (θ_M, ϕ_M). Similarly, if $\theta>\phi_M$ and C_1 is optimal,

$$\Delta(\theta) = -\sum_{i=\ell+1}^{2M-1} \left[\overline{\lambda}_{\mathrm{III},i}(\theta) - \overline{\lambda}_{\mathrm{III},i+1}(\theta)\right] - \left[\overline{\lambda}_{C_1}(\theta) - \overline{\lambda}_{\mathrm{III},\ell+1}(\theta)\right] .$$

Again, by Property 1, $\Delta(\theta)$ is decreasing in the region of optimality of C_1 and, consequently, in (ϕ_M, θ_{2M}). Part (ii) follows from the monotonic behavior in the intervals (θ_M, ϕ_M) and (ϕ_M, θ_{2M}), and from the inequalities $\phi^{-2} < \frac{1}{2} < \phi^{-1}$.

As for part (iii), we bound the locally maximum penalties

$$\Delta(\phi_M) = \left[\overline{\lambda}_{\mathrm{III},M}(\phi_M) - \overline{\lambda}_{\mathrm{III},\ell^*}(\phi_M)\right] + \left[\overline{\lambda}_{\mathrm{III},\ell^*}(\phi_M) - \overline{\lambda}_{C_2}(\phi_M)\right] \triangleq \Delta_1 + \Delta_2 \tag{22}$$

where, again, C_2 denotes an optimal code for ϕ_M. By Lemma 2, we have

$$\Delta_1 = \phi - \frac{\phi^{-2}}{\phi^{-\frac{\ell^*}{M}}(1-\phi^{-\frac{\ell^*}{M}})} . \tag{23}$$

The right-hand side of (23) vanishes for $M=1,2$ (since $\ell^*{=}M$), whereas for $M{>}2$ it is upper-bounded by $\Delta_{\lim}$ due to the inequality $x(1-x) \le 1/4$. By Property 1, we upper-bound Δ_2 by evaluating it at $\theta_{\ell^*+1} > \phi_M$. Furthermore, since $\overline{\lambda}_{C_2}(\theta_{\ell^*+1}) \ge \overline{\lambda}_{\mathrm{III},\ell^*+1}(\theta_{\ell^*+1})$, we obtain

$$\Delta_2 \le \overline{\lambda}_{\mathrm{III},\ell^*}(\theta_{\ell^*+1}) - \overline{\lambda}_{\mathrm{III},\ell^*+1}(\theta_{\ell^*+1}) = \frac{2\theta_{\ell^*+1}^{2M-1}(1-\theta_{\ell^*+1})^2}{1-\theta_{\ell^*+1}^{\ell^*}} \tag{24}$$

where the equality follows from Lemma 2 and the definition of θ_ℓ. It is readily verified that the right-hand side of (24) is decreasing and vanishes as $M \to \infty$. For $M \ge 4$ this term is below 0.025; for $M{=}2$ the upper bound is close to 0.116, but in this case (23) vanishes. Direct computation for $M{=}1$ further yields $\Delta(\phi^{-1}) = \phi - \frac{3}{2}$. Summarizing these facts, by (22), we conclude $\Delta(\theta) \le \Delta(\sqrt{2}-1) = \Delta_{\max}$ and

$$\limsup_{\theta\to 1} \Delta(\theta) \le \Delta_{\lim} . \tag{25}$$

Finally, since the definition of ℓ^* guarantees $\frac{1}{2} \le \phi^{-\frac{\ell^*}{M}} \le \frac{1}{2}\phi^{\frac{1}{M}}$, (22) and (23) imply

$$\limsup_{\theta\to 1} \Delta(\theta) \ge \Delta_{\text{lim}}\,,$$

which together with (25) completes the proof. $\square$

Corollary 1 *The relative code length penalty* $\Delta(\theta,d)/\overline{\lambda}_{\text{opt}}(\theta,d)$ *is maximum for* $d = 0$, *in which case it vanishes at a rate* $O(1/\log{(1-\theta)}^{-1})$ *as* $\theta\to 1$ *and achieves its maximum value* $\frac{4\sqrt{2}-5}{14} \approx 4.7\%$ *at* $\theta=\sqrt{2}-1$.

Proof. Clearly, $\overline{\lambda}_{\text{opt}}(\theta,d)$ increases with θ and tends to infinity at a rate $O(\log{(1-\theta)}^{-1})$ as $\theta\to 1$. The maximum relative penalty follows from direct computation in the interval $\frac{1}{3} < \theta \le \sqrt{2}-1$. $\square$

The sub-family $\mathcal{C}$ represents a specific code length vs. complexity trade-off adopted in [13] and [3]. Similar derivations are possible with other sub-families, which may lead to reduced code length penalties at a moderate increase in the complexity of both the encoding and code selection procedures. In particular, reference [16] considers the family obtained by augmenting $\mathcal{C}$ with the symmetric codes of type II, with ℓ a power of two. It is then observed that the peak relative code length penalty drops, for the augmented code family, below 2% (notice that the code of type II with $\ell = 1$ is optimal in the region that produces the worst peak in Figure 5). However, the proof of Theorem 2 shows that, as $\theta \to 1$, the asymptotic penalty Δ_{lim} is due to the power-of-two restriction rather than the code type selection.

Appendix: Proof of Theorem 1

We first present a few definitions and a lemma that will aid in the proof of Theorem 1.

For all integers j, define

$$p_j \;=\; \begin{cases} C(\theta,d)\theta^{\lceil j/2\rceil - d}, & j \text{ odd},\\ C(\theta,d)\theta^{(j/2)+d}, & j \text{ even}. \end{cases} \tag{26}$$

Notice that when $j \ge 0$, we have $p_j = P_{(\theta,d)}(\mu(j))$, i.e., p_j is the j-th probability in the ranking (12). Let $m \ge 0$, $L > 0$, and i be integers. We define a *single tail* $f_{L,i}(m)$ as follows:

$$f_{L,i}(m) = \sum_{j=0}^{\infty} p_{2m+i+jL}. \tag{27}$$

For all integers j, let

$$\eta_j = \ p_{2j-1} + p_{2j}. \tag{28}$$

Notice that, for $j \geq 1$, we have $\eta_j = \Pr(|x| = j)$ under the TSGD $P_{(\theta,d)}$. For even values of L, we define a *symmetric double tail* $F_{L,i}(m)$ as

$$F_{L,i}(m) = f_{L,2i-1}(m) + f_{L,2i}(m) = \sum_{j=0}^{\infty} \eta_{m+i+j\frac{L}{2}}. \tag{29}$$

The claims of the following lemma follow immediately from the definitions (1), (26), (27), and (29), and from straightforward geometric sum calculations.

Lemma 5 *Let $L > 0$, $m \geq 0$, and i be integers.*
(i) For any integer $k \geq i$, we have

$$f_{L,i}(m) \geq f_{L,k}(m)\,,$$

and, for even L,

$$F_{L,i}(m) \geq F_{L,k}(m)\,.$$

(ii) For integers k and $h \geq -m$, we have

$$f_{L,i+2k}(m+h) = \theta^{k+h} f_{L,i}(m),$$

and, for even L,

$$F_{L,i+k}(m+h) = \theta^{k+h} F_{L,i}(m). \tag{30}$$

(iii) We have

$$f_{L,i}(m) = C(\theta,d)\theta^{m+1}\hat{f}_{L,i} \tag{31}$$

and

$$F_{L,i}(m) = C(\theta,d)\theta^{m+1}\hat{F}_{L,i}, \tag{32}$$

where

$$\hat{f}_{L,i} = \begin{cases} \dfrac{\theta^{j-1+d}}{1-\theta^{2\ell-1}}(1+\theta^{\ell-2d}) & L = 2\ell{-}1,\ i = 2j, \\[2ex] \dfrac{\theta^{j-d}}{1-\theta^{2\ell-1}}(1+\theta^{\ell-1+2d}) & L = 2\ell{-}1,\ i = 2j{+}1, \\[2ex] \dfrac{\theta^{j-1+d}}{1-\theta^{\ell}} & L = 2\ell,\ i = 2j, \\[2ex] \dfrac{\theta^{j-d}}{1-\theta^{\ell}} & L = 2\ell,\ i = 2j{+}1, \end{cases} \tag{33}$$

and

$$\hat{F}_{L,i} = \frac{\theta^{i-1}(\theta^{-d}+\theta^{d})}{1-\theta^{\ell}}, \quad L = 2\ell. \tag{34}$$

Let $\overline{\delta} = \frac{1}{2} - \delta$. We define the auxiliary functions $\overline{r}_i(\ell,\theta,d)$, $i = 0, 1$, by substituting $\overline{\delta}$ for δ in $r_i(\ell,\theta,d)$ as defined in (5)-(6). Since $\overline{\delta} \geq \frac{1}{4} \geq \delta$, the following inequalities hold:

$$\overline{r}_0(\ell,\theta,d) \geq r_0(\ell,\theta,d), \tag{35}$$

$$\overline{r}_1(\ell,\theta,d) \leq r_1(\ell,\theta,d), \tag{36}$$

with equalities holding only for $d = \frac{1}{4}$.

In the sequel, we will often loosely refer to the "code length assigned to probability p" rather than the more cumbersome "code length assigned to the symbol whose probability is p."

Proof of Theorem 1.

Region I. Let $L = 2\ell - 1$. We recall that Region I is characterized by the conditions $\theta_0(\ell,d) < \theta \leq \theta_1(\ell,d)$ or, equivalently $r_0(\ell,\theta,d) > 0$ and $r_1(\ell,\theta,d) \leq 0$. We refer to the latter two conditions as $\mathrm{C}_{1\mathrm{a}}$ and $\mathrm{C}_{1\mathrm{b}}$ respectively. By the inequalities (35) and (36), $\mathrm{C}_{1\mathrm{a}}$ and $\mathrm{C}_{1\mathrm{b}}$ imply the weaker conditions $\overline{r}_0(\ell,\theta,d) > 0$ and $\overline{r}_1(\ell,\theta,d) \leq 0$, which will be referred to, respectively, as $\overline{\mathrm{C}}_{1\mathrm{a}}$ and $\overline{\mathrm{C}}_{1\mathrm{b}}$.

Define an mth order reduced source $\mathcal{R}^I_{L,m}$, $m \geq 0$, as the multiset of probabilities

$$\mathcal{R}^I_{L,m} = \{p_0,\, p_1,\, \cdots,\, p_{2m-1},\, f_{L,0}(m),\, f_{L,1}(m),\, \cdots\, f_{L,L-1}(m)\}$$

(when $m = 0$, the source includes only tails). We build an optimal prefix code for $\mathcal{R}^I_{L,m}$, $m > 0$, using the usual Huffman procedure. For real numbers a, b, c, d, we use the notation $\{a,b\} \leq \{c,d\}$ to denote $\max\{a,b\} \leq \min\{c,d\}$. This notation is extended in the natural way to relations of the form $a \leq \{c,d\}$ and $a \geq \{c,d\}$. We claim that the probabilities in $\mathcal{R}^I_{L,m}$ are ordered as follows:

$$\begin{aligned} \{p_{2m-1}, f_{L,L-1}(m)\} \;&\leq\; \{p_{2m-2}, f_{L,L-2}(m)\} \leq \{p_{2m-3}, f_{L,L-3}(m)\} \leq \cdots \\ \cdots \;&\leq\; \{p_{2m-L}, f_{L,0}(m)\} \leq p_{2m-L-1} \leq \cdots \leq p_0\,. \end{aligned} \tag{37}$$

To prove the claim, it suffices to prove the two leftmost inequalities, since the remaining inequalities are scaled versions of the first two. For $L = 1$ or $m = 1$, some of the symbols involved in the two leftmost inequalities are not part of $\mathcal{R}^I_{L,m}$, but the inequalities still apply (with some negative-indexed p_j and $f_{L,i}$). To prove the leftmost inequality in (37), after applying (31), it suffices to show $\theta^{-1-d} < \hat{f}_{L,L-2}$ and $\hat{f}_{L,L-1} \leq \theta^{-2+d}$. Using the expression for $\hat{f}_{L,L-2}$ from (33), the former inequality is equivalent to the condition

$$\theta^{2\ell-1}(1+\theta^{-1+2d}) + \theta^{\ell-1} - 1 > 0,$$

which in turn is equivalent to $\overline{\mathrm{C}}_{1\mathrm{a}}$ if $d \leq \frac{1}{4}$, or to $\mathrm{C}_{1\mathrm{a}}$ otherwise. Similarly, $\hat{f}_{L,L-1} \leq \theta^{-2+d}$ is equivalent to either $\mathrm{C}_{1\mathrm{b}}$ or $\overline{\mathrm{C}}_{1\mathrm{b}}$.

For the second leftmost inequality in the chain (37), it suffices to prove $\theta^{-2+d} < \hat{f}_{L,L-3}$ and $\hat{f}_{L,L-2} \leq \theta^{-2-d}$. As before, the first inequality is equivalent to, or dominated by $\mathrm{C}_{1\mathrm{a}}$, while the second one is in the same situation with respect to $\mathrm{C}_{1\mathrm{b}}$.

It follows that the first merge of the Huffman algorithm on $\mathcal{R}^I_{L,m}$ produces the probability

$$p_{2m-1} + f_{L,L-1}(m) = p_{2m-1} + \sum_{j=0}^{\infty} p_{2m+L-1+jL} = \sum_{j=0}^{\infty} p_{2m-1+jL} = f_{L,1}(m-1).$$

Notice that $f_{L,1}(m-1) = f_{L,-1}(m)$, so after scaling by $\theta^{-\ell-1}$, the second leftmost inequality in (37) implies that the newly created probability satisfies $f_{L,1}(m-1) \geq \{p_{2m-L}, f_{L,0}(m)\}$. Hence, the next merge in the Huffman algorithm produces the probability

$$p_{2m-2} + f_{L,L-2}(m) = p_{2m-2} + \sum_{j=0}^{\infty} p_{2m+L-2+jL} = \sum_{j=0}^{\infty} p_{2m-2+jL} = f_{L,0}(m-1). \tag{38}$$

If $L=1$, (38) still applies, since the second step uses the probability $f_{L,-1}(m) = f_{L,1}(m-1)$ produced in the first step.

Also, by (27), we have $f_{L,i}(m) = f_{L,i+2}(m-1)$, $0 \leq i \leq L-3$. Thus, after two steps (one *round*) of the Huffman algorithm, $\mathcal{R}^I_{L,m}$ is transformed into $\mathcal{R}^I_{L,m-1}$. The process continues for a total of m rounds, building up the tails $f_{L,i}$, until $\mathcal{R}^I_{L,0}$ is reached. This reduced source is given, in ascending probability order, by

$$\mathcal{R}^I_{L,0} = \{\, f_{L,L-1}(0),\, f_{L,L-2}(0),\, \cdots,\, f_{L,0}(0)\,\}.$$

We say that a finite source with probabilities $\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{N-1}$ is *quasi-uniform* if either $N \leq 2$ or $\sigma_0 + \sigma_1 \geq \sigma_{N-1}$. As noted in [12], an optimal prefix code for a quasi-uniform source of N probabilities admits at most two distinct codeword lengths, and it consists of $2^{\lceil \log N \rceil} - N$ codewords of length $\lfloor \log N \rfloor$, and $2N - 2^{\lceil \log N \rceil}$ codewords of length $\lceil \log N \rceil$, the shorter codewords being assigned to the larger probabilities.

We claim that $\mathcal{R}^{I}_{L,0}$ is quasi-uniform. For $L = 1$, there is nothing to prove. Otherwise, we need to show $f_{L,L-1}(0) + f_{L,L-2}(0) - f_{L,0}(0) \geq 0$. By Lemma 5(iii), after straightforward manipulations, the latter inequality is equivalent to

$$\theta^{2\ell-1}(\theta^{-1} + \theta^{-2d}) + \theta^{\ell-1} - 1 + \theta^{\ell-2d}(\theta^{-1} - 1) \geq 0. \tag{39}$$

By definition (5), since $\theta^{-1} > 1$ and $d \geq \delta$, the left-hand side of (39) is larger than $r_0(\ell, \theta, d)$, which is positive by $\mathrm{C}_{1\mathrm{a}}$. Therefore, $\mathcal{R}^{I}_{L,0}$ is quasi-uniform, and its optimal prefix code is constructed as described above, with $N = L$.

Tracing the way $\mathcal{R}^{I}_{L,0}$ "unfolds" into $\mathcal{R}^{I}_{L,m}$, it follows that the code length assigned to p_j, $j \geq 0$, for m such that $2m-1 \geq j$, is $\lfloor j/L \rfloor + 1 + \Lambda_I(f_{L,j'})$. Here, $\Lambda_I(f)$ denotes the code length assigned by the optimal prefix code for $\mathcal{R}^{I}_{L,0}$ to f, and $j' = j \mod L$. Thus, the code length for p_j is precisely the code length assigned by G_L to j, as claimed in Theorem 1 for Region I.

Region II. Let $L = 2\ell$. In Region II we have $\delta = d \leq \frac{1}{4}$, $r_1(\ell, \theta, d) > 0$ and $r_2(\ell, \theta, d) \leq 0$. The latter two inequalities are referred to, respectively, as conditions $\mathrm{C}_{2\mathrm{a}}$ and $\mathrm{C}_{2\mathrm{b}}$.

We use a reduced source $\mathcal{R}^{II}_{L,m}$, defined by

$$\begin{aligned} \mathcal{R}^{II}_{L,m} = \ & \{ p_0, p_1, \cdots, p_{2m}, \eta_{m+1}, \eta_{m+2}, \cdots, \eta_{m+\ell-1}, \\ & \quad F_{L,1}(m+\ell-1), F_{L,2}(m+\ell-1), \cdots, F_{L,\ell}(m+\ell-1) \ \} \end{aligned}$$

(for $\ell = 1$, $\mathcal{R}^{II}_{L,m}$ contains no η_i's), and we claim that the probabilities in $\mathcal{R}^{II}_{L,m}$ are ordered as follows (listed in ascending order, with inequality signs omitted):

$$\begin{aligned} & p_{2m}, p_{2m-1}, F_{L,\ell}(m+\ell-1), \eta_{m+\ell-1}, p_{2m-2}, p_{2m-3}, F_{L,\ell-1}(m+\ell-1), \cdots \\ & \quad \cdots, \eta_{m+1}, p_{2(m-\ell+1)}, p_{2(m-\ell+1)-1}, F_{L,1}(m+\ell-1), p_{2(m-\ell)}, \cdots, p_1, p_0 \,. \end{aligned}$$

When $m < \ell$, the sequence of original probabilities p_i stops at p_0 "in the middle" of the chain (i.e., just before one of the $F_{L,j}$'s), but the order relations between all remaining symbols are still claimed to hold.

To prove the claim, it suffices to show that (a) $p_{2m-1} < F_{L,\ell}(m+\ell-1)$, (b) $F_{L,\ell}(m+\ell-1) \leq \eta_{m+\ell-1}$, and (c) $\eta_{m+\ell-1} \leq p_{2m-2}$. The rest of the chain follows by virtue of scaling. Using the expression for $F_{L,\ell}$ from (32) and (34), the definitions of r_1 and p_j, and the fact that $d = \delta$, we observe that inequality (a) is equivalent to $\mathrm{C}_{2\mathrm{a}}$. By definition (28), and after eliminating common factors, inequality (b) is equivalent to

$$2\theta^{\ell} - 1 \leq 0. \tag{40}$$

Clearly, (40) is implied by $\theta^{\ell}(1+\theta^{-2\delta}) - 1 \leq 0$, the latter inequality being $\mathrm{C}_{2\mathrm{b}}$. Inequality (c), in turn, is also equivalent to $\mathrm{C}_{2\mathrm{b}}$, as $d = \delta$.

The first step of the Huffman algorithm on $\mathcal{R}^{II}_{L,m}$ merges p_{2m} and p_{2m-1}, creating η_m. By inequality (b) above, suitably scaled by $\theta^{\ell-1}$, we have $\eta_m \geq F_{L,\ell}(m) = F_{L,1}(m+\ell-1)$. Hence, the second Huffman step joins $F_{L,\ell}(m+\ell-1)$ with $\eta_{m+\ell-1}$. Recalling the definition of $F_{L,\ell}$ in (29), we obtain

$$F_{L,\ell}(m+\ell-1) + \eta_{m+\ell-1} = \sum_{j=0}^{\infty} \eta_{m+2\ell-1+j\ell} + \eta_{m+\ell-1} = \sum_{j=0}^{\infty} \eta_{m+\ell-1+j\ell} = F_{L,1}(m+\ell-2).$$

When $\ell = 1$, the probability η_m created in the first step is used in the second one. Notice that, by (29), we can also write $F_{L,j}(m+\ell-1) = F_{L,j+1}(m+\ell-2)$ for $1 \leq j \leq \ell-1$. Hence, after two steps (one round) of the Huffman algorithm, $\mathcal{R}^{II}_{L,m}$ is transformed into $\mathcal{R}^{II}_{L,m-1}$. After m rounds, we obtain $\mathcal{R}^{II}_{L,0}$, given in ascending probability order by

$$\mathcal{R}^{II}_{L,0} = \{ p_0,\, F_{L,\ell}(\ell-1),\, \eta_{\ell-1},\, F_{L,\ell-1}(\ell-1),\, \cdots,\, F_{L,2}(\ell-1),\, \eta_1,\, F_{L,1}(\ell-1) \}. \tag{41}$$

When $\ell = 1$, (41) translates to $\mathcal{R}^{II}_{2,0} = \{ p_0,\, F_{2,1}(0) \}$.

We claim that $\mathcal{R}^{II}_{L,0}$ is quasi-uniform, i.e., $p_0 + F_{L,\ell}(\ell-1) - F_{L,1}(\ell-1) \geq 0$ whenever $\ell > 1$. By (30), (32) and (34), the required condition is equivalent to

$$\theta^d + \frac{(\theta^{\ell-1}-1)\theta^{\ell}(\theta^{-d}+\theta^d)}{1-\theta^{\ell}} \geq 0.$$

Multiplying by $(1-\theta^{\ell})$, and rearranging terms, the above inequality is equivalent to

$$\theta^d(1-2\theta^{\ell}) + \theta^{\ell-d} r_3(\ell-1, \theta, d) \geq 0.$$

By (40), and since $r_3(\ell-1,\theta,d) > 0$, as in Region II we have $\theta > \theta_1(\ell,d) > \theta_0(\ell,d) \geq \theta_3(\ell-1,\theta,d)$, it follows that $\mathcal{R}^{II}_{L,0}$ is quasi-uniform.

We now construct an optimal prefix code for $\mathcal{R}^{II}_{L,0}$, and show how it translates into an optimal code for $\mathcal{R}^{II}_{L,m}$ and, thus, for the integers under the TSGD (1). Let r be the integer satisfying $2^{r-1} \leq \ell < 2^r$, and $s = 2^r - \ell$. Assume first that $s < \ell$. Since $\mathcal{R}^{II}_{L,0}$ contains 2ℓ probabilities, an optimal prefix code for it assigns code length r to the $2s$ largest probabilities, namely,

$$\eta_s,\, F_{L,s}(\ell-1),\, \cdots,\, \eta_2,\, F_{L,2}(\ell-1),\, \eta_1,\, F_{L,1}(\ell-1)\,. \tag{42}$$

The code assigns length $r+1$ to the $2\ell-2s$ remaining probabilities, namely,

$$p_0,\, F_{L,\ell}(\ell-1),\, \eta_{\ell-1},\, \cdots,\, \eta_{s+1},\, F_{L,s+1}(\ell-1)\,. \tag{43}$$

Notice that in the iterative construction of the Huffman code for $\mathcal{R}^{II}_{L,m}$, $m \geq 1$, pairs p_{2j-1}, p_{2j}, $j \geq 1$, which correspond to integers of opposite signs, merge to form η_j. Thus, it suffices to characterize the code length assigned by the construction to the η_j, $j > 0$, and p_0. Similarly to Region I, tracing this time the way $\mathcal{R}^{II}_{L,0}$ "unfolds" into $\mathcal{R}^{II}_{L,m}$, we observe that $\eta_{i+j\ell}$, $0 \leq i < \ell$, $1 \leq j \leq m$, is a leaf in a subtree rooted at $F_{L,i+1}(\ell-1)$, j levels down from the root of the subtree. Thus, the code length assigned to $\eta_{i+j\ell}$ when m is sufficiently large is $j + \Lambda_{II}(F_{L,i+1}(\ell-1))$, $j \geq 1$. Here, $\Lambda_{II}(\cdot)$ is the code length assignment of the Huffman code for $\mathcal{R}^{II}_{L,0}$, i.e., $\Lambda_{II}(\sigma) = r$ for probabilities σ in the list (42), and $\Lambda_{II}(\sigma) = r+1$ for probabilities σ in the list (43). Code lengths for p_0 and η_i, $0 < i < \ell$, are assigned directly by Λ_{II}. The foregoing discussion is summarized in Table 1, which shows the code lengths assigned by our construction for Region II, under the assumption $s < \ell$.

For ease of comparison, Table 2 explicitly lists the code length assignments of an ℓth order Golomb code, in a purposely redundant manner to match the lines of Table 1. Comparing the code lengths in Table 1 with those of Table 2, we observe that our construction assigns to p_0 and η_i, $i > 0$, the same code lengths that G_ℓ assigns to $i = 0$ and $i > 0$, respectively, except that the code lengths for $i = 0$ and $i = s$ have been exchanged. This is due to the fact that p_0 is the smallest probability in $\mathcal{R}^{II}_{L,0}$, and thus falls into the list (43) of probabilities assigned code length $r+1$, whereas the length of $G_\ell(0)$ is r. On the other hand, because p_0 is in the list (43), η_s has a place in the list (42), and it gets assigned code length r, instead of $r+1$ for $G_\ell(s)$.

When $s = \ell$, i.e., ℓ is a power of two, all the probabilities in $\mathcal{R}^{II}_{L,0}$ are assigned the same code length r, and no swapping takes place, even though p_0 is still the lowest probability in $\mathcal{R}^{II}_{L,0}$

Table 1: Code length assignments for Region II, $s < \ell$.

Probability	:	Code Length
p_0	:	$r+1,$
$\eta_{j\ell}$	:	$r+j, \;\; j \geq 1,$
$\eta_{k+j\ell}$	:	$r+j, \;\; 1 \leq k \leq s-1, \;\; j \geq 0,$
η_s	:	$r,$
$\eta_{s+j\ell}$	:	$r+1+j, \;\; j \geq 1,$
$\eta_{h+j\ell}$	:	$r+1+j, \;\; s+1 \leq h \leq \ell-1, \;\; j \geq 0.$

Table 2: Code length assignments for the Golomb code G_ℓ.

Symbol	:	Code Length
0	:	$r,$
$j\ell$	:	$r+j, \;\; j \geq 1,$
$k+j\ell$	:	$r+j, \;\; 1 \leq k \leq s-1, \;\; j \geq 0,$
s	:	$r+1,$
$s+j\ell$	:	$r+1+j, \;\; j \geq 1,$
$h+j\ell$	:	$r+1+j, \;\; s{+}1 \leq h \leq \ell-1, \;\; j \geq 0.$

(η_s is not a member of $\mathcal{R}^{II}_{L,0}$ in this case; it gets assigned code length $r{+}1$ through $F_{L,1}(\ell{-}1)$). To complete the proof of Theorem 1 for Region II, the code for the original probabilities p_{2j-1}, p_{2j}, $j \geq 1$, is obtained by appending a sign bit to the code for the corresponding η_j.

Region IV. Let $L = 2\ell$. Region IV is characterized by $d \leq \frac{1}{4}$, and the conditions $r_3(\ell, \theta, d) > 0$ and $r_0(\ell+1, \theta, d) \leq 0$, referred to, respectively, as conditions $\mathrm{C}_{4\mathrm{a}}$ and $\mathrm{C}_{4\mathrm{b}}$. The construction in Region IV follows along similar lines as that of Region II, and it will only be outlined here.

We use a reduced source $\mathcal{R}^{IV}_{L,m}$, defined by

$$\mathcal{R}^{IV}_{L,m} = \{\, p_0,\, p_1,\, \cdots,\, p_{2m},\, \eta_{m+1},\, \eta_{m+2},\, \cdots,\, \eta_{m+\ell},\, F_{L,1}(m{+}\ell),\, F_{L,2}(m{+}\ell),\, \cdots,\, F_{L,\ell}(m{+}\ell)\,\}.$$

It follows from C_{4a} and C_{4b} that the probabilities in $\mathcal{R}^{IV}_{L,m}$ are sorted in ascending order as follows:

$$p_{2m},\, p_{2m-1},\, \eta_{m+\ell},\, F_{L,\ell}(m+\ell),\, p_{2m-2},\, p_{2m-3},\, \eta_{m+\ell-1},\, F_{L,\ell-1}(m+\ell),\, \cdots$$
$$\cdots,\, p_{2(m-\ell+1)},\, p_{2(m-\ell+1)-1},\, \eta_{m+1},\, F_{L,1}(m+\ell),\, p_{2(m-\ell)},\, \cdots,\, p_2,\, p_1,\, p_0\,.$$

Similarly to Region II, a round consisting of two steps of the Huffman algorithm on $\mathcal{R}^{IV}_{L,m}$ leads to $\mathcal{R}^{IV}_{L,m-1}$, with the pair p_{2m}, p_{2m-1} merging to form η_m, and $\eta_{m+\ell}$ merging with $F_{L,\ell}(m+\ell)$ to form $F_{L,1}(m+\ell-1)$. After m rounds, we obtain the reduced source $\mathcal{R}^{IV}_{L,0}$, which consists of the following $2\ell+1$ probabilities, in ascending order:

$$p_0,\, \eta_\ell,\, F_{L,\ell}(\ell),\, \eta_{\ell-1},\, F_{L,\ell-1}(\ell),\, \cdots,\, \eta_1,\, F_{L,1}(\ell)\,.$$

It follows from C_{4a} and C_{4b} that $\mathcal{R}^{IV}_{L,0}$ is quasi-uniform. Now, if r is the integer satisfying $2^{r-1} \le \ell < 2^r$, then r also satisfies $2^r < 2\ell+1 < 2^{r+1}$. Thus, an optimal prefix code for $\mathcal{R}^{IV}_{L,0}$ contains $2^{r+1} - 2\ell - 1 = 2s - 1$ codewords of length r, and $2\ell - 2s + 2$ codewords of length $r+1$. The list of probabilities corresponding to codewords of length r is given by

$$F_{L,s}(\ell),\, \eta_{s-1},\, F_{L,s-1}(\ell),\, \cdots,\, \eta_2,\, F_{L,2}(\ell),\, \eta_1,\, F_{L,1}(\ell),$$

while the list of probabilities corresponding to codewords of length $r+1$ is given by

$$p_0,\, \eta_\ell,\, F_{L,\ell}(\ell),\, \cdots,\, \eta_{s+1},\, F_{L,s+1}(\ell),\, \eta_s\,.$$

(If $\ell = 1$, the first list consists just of $F_{L,1}(1)$, while the second list consists of p_0 and η_1.)

Proceeding as in Region II, from the code length distribution for $\mathcal{R}^{IV}_{L,0}$, we can now derive a code length distribution for p_0 and η_j, $j \ge 0$. This distribution is summarized in Table 3.

We observe that our construction assigns to p_0 and η_i, $i > 0$, the same code length that an ℓth order Golomb code assigns to $i = 0$ and $i > 0$, respectively, except that p_0 is assigned a code one bit longer, and all $\eta_{s+j\ell}$, $j \ge 1$, are assigned a code one bit shorter. Since length transitions in G_ℓ occur precisely at integers congruent to s modulo ℓ (in that the length of $G_\ell(s+j\ell)$ is one more than the length of $G_\ell(s-1+j\ell)$), this shortening is equivalent to assigning to η_i, $i > s$, the code length that G_ℓ would assign to $i-1$. The codewords for p_0 (that needs to grow by one bit relative to the length r of $G_\ell(0)$) and η_s (that retains the length $r+1$ of $G_\ell(s)$ but just lost that

Table 3: Code length assignments for Region IV.

Probability	:	Code Length
p_0	:	$r+1,$
$\eta_{j\ell}$	:	$r+j, \;\; j \geq 1,$
$\eta_{k+j\ell}$	:	$r+j, \;\; i \leq k \leq s-1, \;\; j \geq 0,$
η_s	:	$r+1,$
$\eta_{s+j\ell}$	:	$r+j, \;\; j \geq 1,$
$\eta_{h+j\ell}$	:	$r+1+j, \;\; s+1 \leq h \leq \ell-1, \;\; j \geq 0.$

codeword to η_{s+1}) are obtained by appending a bit to $G_\ell(0)$. As in the case of Region II, the proof is completed by observing that the η_j, $j > 0$, split into original probabilities, amounting to the appendage of a sign bit to the code for η_j.

It remains to prove Theorem 1 for Region III. Recall that this region is the union of regions II', III', and IV'.

Regions II' and IV'. These regions satisfy the same conditions as regions II and IV (respectively), but are defined for $d > \frac{1}{4}$, i.e., we have $\delta = \frac{1}{2} - d$. Consider the distribution $P_{(\theta,\delta)}(\cdot)$, and let $\overline{p}_j = P_{(\theta,\delta)}(\mu(j))$ for $j \geq 0$, extending in analogy with p_j for $j < 0$. We can write

$$P_{(\theta,d)}(x) = C(\theta,d)\theta^{|x+d|} = \begin{cases} C(\theta,d)\theta^{-\frac{1}{2}}\theta^{x+1-\delta} = \gamma P_{(\theta,\delta)}(-x-1) & x \geq 0, \\ C(\theta,d)\theta^{-\frac{1}{2}}\theta^{-x+\delta} = \gamma P_{(\theta,\delta)}(-x) & x < 0, \end{cases}$$

for a constant γ. Therefore, we have $p_j = \gamma\overline{p}_{j+1}$ for all integers j, which means that once the probabilities are ordered, and ignoring scaling factors, the distribution $P_{(\theta,d)}(\cdot)$, $d > \frac{1}{4}$, "looks" exactly like the distribution $P_{(\theta,\delta)}(\cdot)$, with p_0 removed. Noting that in the constructions for reduced sources in regions II and IV, p_0 was involved only in the final stage, when an optimal code for the "core" reduced sources $\mathcal{R}^{II}_{L,0}$ or $\mathcal{R}^{IV}_{L,0}$ was determined, we conclude that the same constructions can be used for regions II' and IV', except for that final stage. The formalization of this idea is outlined next.

Let $L = 2\ell$. For all integers j, let

$$\nu_j = p_{2j-2} + p_{2j-1},$$

and, for all integers i define the *asymmetric double tail*

$$K_{L,i}(n) = \sum_{j=0}^{\infty} \nu_{n+i+j\ell}.$$

It follows from the discussion above that we can write

$$\nu_j = \gamma \overline{\eta}_j, \quad j \in \mathbf{Z},$$

and

$$K_{L,i}(n) = \gamma \overline{F}_{L,i}(n), \quad i \in \mathbf{Z},$$

where $\overline{\eta}_j$ and $\overline{F}_{L,i}$ are analogous to η_j and $F_{L,i}$, respectively, but defined for the distribution $P_{(\theta,\delta)}$. Assume (θ, d) falls in Region II', and define the reduced source

$$\begin{aligned} \mathcal{R}^{II'}_{L,m} = \;& \{ p_0, p_1, \cdots, p_{2m-1}, \nu_{m+1}, \nu_{m+2}, \cdots, \nu_{m+\ell-1}, \\ & K_{L,1}(m+\ell-1), K_{L,2}(m+\ell-1), \cdots, K_{L,\ell}(m+\ell-1) \}. \end{aligned}$$

This source is equivalent to a scaled version of $\mathcal{R}^{II}_{L,m} - \{p_0\}$, but for the distribution $P_{(\theta,\delta)}$. Therefore, the iteration leading from $\mathcal{R}^{II}_{L,m}$ to $\mathcal{R}^{II}_{L,0}$ applies, and after $2m$ steps of the Huffman algorithm, we have

$$\mathcal{R}^{II'}_{L,0} = \{ K_{L,\ell}(\ell-1),\, \nu_{\ell-1},\, K_{L,\ell-1}(\ell-1),\, \nu_{\ell-2},\, \cdots,\, K_{L,2}(\ell-1),\, \nu_1,\, K_{L,1}(\ell-1) \},$$

where the probabilities are listed in ascending order. The construction now departs from that of Region II. We observe that, without an analog of p_0 in the way, we can carry out $\ell - 1$ additional merges, which, by the definition of $K_{L,i}$, yield

$$K_{L,i}(\ell-1) + \nu_{i-1} = K_{L,i-1}(0), \quad 2 \le i \le \ell\,.$$

In addition, we have $K_{L,1}(\ell-1) = K_{L,\ell}(0)$. Thus, we obtain a further reduced source

$$\mathcal{R}^{II'}_{L,-1} = \{ K_{L,\ell}(0),\, K_{L,\ell-1}(0),\, \cdots,\, K_{L,1}(0) \}.$$

It is readily verified that $\mathcal{R}^{II'}_{L,-1}$ is quasi-uniform. Thus, the construction yields an ℓth order Golomb code for $\nu_1, \nu_2, \nu_3, \cdots$, where ν_j, $j \geq 1$, gets assigned a code length corresponding to that of $G_\ell(j-1)$. Splitting each leaf in the tree of G_ℓ to obtain codewords for the original probabilities p_{2j-2} and p_{2j-1} that formed ν_j, we obtain a code length distribution identical to that of an Lth order Golomb code for $p_0, p_1, p_2, \cdots$, as claimed by Theorem 1 for Region II' (as part of Region III). Notice however, that the coding tree for G_L is different from that of a "split" G_ℓ, as illustrated in Figure 3.

Similar considerations apply to Region IV', where we define a reduced source $\mathcal{R}^{IV'}_{L,m}$ analogous to $\mathcal{R}^{IV}_{L,m} - \{p_0\}$. This leads to a core source

$$\mathcal{R}^{IV'}_{L,0} = \{\, \nu_\ell,\, K_{L,\ell}(\ell),\, \nu_{\ell-1},\, K_{L,\ell-1}(\ell),\, \cdots,\, \nu_2,\, K_{L,2}(\ell),\, \nu_1,\, K_{L,1}(\ell)\,\},$$

which can be further reduced to obtain

$$\mathcal{R}^{IV'}_{L,-1} = \{\, K_{L,\ell}(0),\, K_{L,\ell-1}(0),\, \cdots,\, K_{L,1}(0)\,\}.$$

This source, again, leads to an ℓth order Golomb code for the ν_j, or, equivalently, an Lth order Golomb code for the original probabilities.

Region III'. Let $L = 2\ell$. Region III' is characterized by the conditions $r_2(\ell, \theta, d) > 0$ and $r_3(\ell, \theta, d) \leq 0$, referred to, respectively, as conditions $\mathrm{C}_{3\mathrm{a}}$ and $\mathrm{C}_{3\mathrm{b}}$.

Here, we define a reduced source $\mathcal{R}^{III'}_{L,m}$ given by

$$\mathcal{R}^{III'}_{L,m} = \{\, p_0,\, p_1,\, \cdots,\, p_{2m-1},\, f_{L,0}(m),\, f_{L,1}(m),\, \cdots,\, f_{L,L-1}(m)\,\}.$$

This reduced source appears to be formally identical to $\mathcal{R}^{I}_{L,m}$ used in Region I. However, inspecting (31) and (33), we note that the expressions for $f_{L,i}(m)$ when L is even are quite different from those applying when L is odd. Nevertheless, after appropriate reinterpretation of $f_{L,i}$, the order relations claimed in (37) hold for Region III', being implied this time by $\mathrm{C}_{3\mathrm{a}}$ and $\mathrm{C}_{3\mathrm{b}}$ rather than $\mathrm{C}_{1\mathrm{a}}$ and $\mathrm{C}_{1\mathrm{b}}$. Similarly, the evolution from $\mathcal{R}^{III'}_{L,m}$ to $\mathcal{R}^{III'}_{L,0}$ by way of the Huffman procedure is formally identical to that from $\mathcal{R}^{I}_{L,m}$ to $\mathcal{R}^{I}_{L,0}$. Finally, the quasi-uniformity of $\mathcal{R}^{III'}_{L,0}$ follows from $\mathrm{C}_{3\mathrm{a}}$, and thus, the construction in Region III' yields a code whose length assignment for p_i, $i \geq 0$, is identical to that of an Lth order Golomb code for $i \geq 0$.

Optimality. The optimality of the codes prescribed by Theorem 1 follows from the same argument presented in [12], applied separately to each region. The main formal requirement is the convergence of the average code length, which was established in Lemma 2. This completes the proof of Theorem 1. □

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