

Optimal Correlators for Detection and Estimation in Optical Receivers

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What is it About?

Classical signal detection/estimation problems are optimally solved using a **matched filter**, or **matched correlator**:

Signal detection:

$$\mathcal{H}_0 : y(t) = n(t), \quad 0 \leq t \leq T, \quad \{n(t)\} = \text{AWGN}$$

$$\mathcal{H}_1 : y(t) = \lambda(t) + n(t), \quad 0 \leq t \leq T$$

LRT: Compare $\int_0^T y(t)\lambda(t)dt$ to a threshold.

What is it About? (Cont'd)

Estimation of delay:

$$y(t) = \lambda(t - \tau) + n(t), \quad 0 \leq t \leq T.$$

ML estimator:

$$\hat{\tau} = \arg \max_{\theta} \int_0^T y(t) \lambda(t - \theta) dt.$$

Difficulty: In optical receivers the noise is **not Gaussian**.

Approach: Find **optimal waveforms** for correlation **accordingly**:

Detection: compare $\int_0^T y(t) \mathbf{w}_d^*(t) dt$ to a threshold.

Estimation: $\hat{\tau} = \arg \max_{\theta} \int_0^T y(t) \mathbf{w}_e^*(t - \theta) dt.$

Motivation: relatively easy to implement.

Background – General

- ♣ Modern LIDAR applications → renewed interest in optical det./est.
- ♣ Classical optical direct detection: photodiode + TIA.
- ♣ Complicated signal+noise model: signal + **AWGN + shot-noise**.
- ♣ In the case of APD: also **excess noise** (multiplicative noise).
- ♣ Difficult/impossible to derive the LR in closed form.
- ♣ Researchers have proposed many types of approximations.

Background – Signal Model

The received signal model:

$$y(t) = \sum_{k=1}^K g_k h(t - t_k) + n(t), \quad 0 \leq t \leq T,$$

where:

K = total number of photo-electrons.

$h(t)$ = current pulse contributed by a photo-electron generated at $t = 0$.

$\{t_k\}$ = Poissonian arrival times of photo-electrons, induced by $\lambda(t)$.

$\{g_k\}$ = avalanche gains (geom. distributed: $P(g) \propto e^{-\zeta g}$, $g = 1, 2, \dots$).

$\{n(t)\}$ = Gaussian white noise.

Background – Likelihood Ratio

Had K , $\{g_k\}$ and $\{t_k\}$ been known,

$$\begin{aligned} \text{LR} &= \frac{\exp \left\{ -\frac{1}{N_0} \int_0^T \left[y(t) - \sum_{k=1}^K g_k h(t - t_k) \right]^2 dt \right\}}{\exp \left\{ -\frac{1}{N_0} \int_0^T y^2(t) dt \right\}} \\ &= \exp \left\{ \frac{2}{N_0} \sum_{k=1}^K g_k \int_0^T y(t) h(t - t_k) dt - \frac{1}{N_0} \sum_{k=1}^K \sum_{l=1}^K g_k g_l R(t_k - t_l) \right\}, \end{aligned}$$

where $R(\tau) = \int_0^T h(t) h(t - \tau) dt$.

Main difficulty: averaging over the randomness of K , $\{g_k\}$ and $\{t_k\}$.

A Sample of Some Earlier Approaches

- ♠ Foscini, Gilbert & Salz ('75): problematic term = charac. func. of a Gaussian process.
- ♠ Kadota ('88), Hero ('91): neglecting the problematic term.
- ♠ Einarsson ('96), El–Hadidi *et al.* ('81), Geraiotis *et al.* ('87): Gaussian approximations – **optical matched filter**.

The Proposed Approach

Find **optimal waveforms** for correlation:

Detection: compare $\int_0^T y(t) \mathbf{w}_d^*(t) dt$ to a threshold.

Estimation: $\hat{\tau} = \arg \max_{\theta} \int_0^T y(t) \mathbf{w}_e^*(t - \theta) dt$.

Design $\{\mathbf{w}_d^*(t)\}$ and $\{\mathbf{w}_e^*(t)\}$ by incorporating the full signal model.

Optimal in what sense?

Detection: Max. exponential decay rate of P_{MD} for a given P_{FA} .

Estimation: minimum MSE (under high SNR).

Signal Detection – No Dark Current

The FA probability.

Although $y(t) = n(t)$, the correlation is above the threshold.

$$\begin{aligned} P_{\text{FA}} &= \Pr \left\{ \int_0^T n(t)w(t)dt \geq \theta T \right\} \\ &= Q \left(\frac{\theta T}{\sqrt{N_0 E/2}} \right) \\ &\approx \exp \left\{ -\frac{\theta^2 T}{N_0 P} \right\}, \end{aligned}$$

where P is the power of $\{w(t), 0 \leq t \leq T\}$.

Maximizing the exponent of P_{MD} for a given exponent of P_{FA} is equivalent to maximizing it for a given P .

Signal Detection – No Dark Current (Cont'd)

The MD probability.

Although $y(t) = \sum_k g_k h(t - t_k) + n(t)$, the correlation is below the threshold.

$$\begin{aligned} P_{\text{MD}} &= \Pr \left\{ \int_0^T \left[\sum_k g_k h(t - t_k) + n(t) \right] w(t) dt < \theta T \right\} \\ &\approx e^{-ET}, \end{aligned}$$

where

$$E = \sup_{s \geq 0} \left[\frac{e^\zeta}{T} \int_0^T \lambda(t) \cdot \frac{\exp\{sq_e \mathbf{w}(t)\} - 1}{\exp\{sq_e \mathbf{w}(t) + \zeta\} - 1} dt - s\theta - s^2 \frac{N_0 P}{4} \right],$$

where q_e is the electric charge of the electron.

Optimizing the Correlator (Deterministic Gain)

Consider first the case of deterministic gain, $g_k \equiv 1$ ($\zeta \rightarrow \infty$):

$$E = \sup_{s \geq 0} \left[\frac{1}{T} \int_0^T \lambda(t) [1 - \exp\{-sq\mathbf{e}^{\mathbf{w}(t)}\}] dt - s\theta - s^2 \frac{N_0 P}{4} \right].$$

We wish to maximize E over $\{w(t), 0 \leq t \leq T\}$ s.t. $\int_0^T w^2(t) dt \leq PT$.

Solution: Let $p[y]$ be the **inverse** of the function $b[x] = xe^x$. Then

$$w_{\mathbf{d}}^*(t) = \frac{1}{sq\mathbf{e}^a} \cdot p[c\lambda(t)],$$

where c is chosen so that the power would be exactly P .

For N_0 and/or θ large, $p[x] \approx x$ and $w_{\mathbf{d}}^*(t) \propto \lambda(t)$.

For N_0 and θ small, $p[x] \approx \ln x$, and $w_{\mathbf{d}}^*(t) \propto \ln \lambda(t)$.

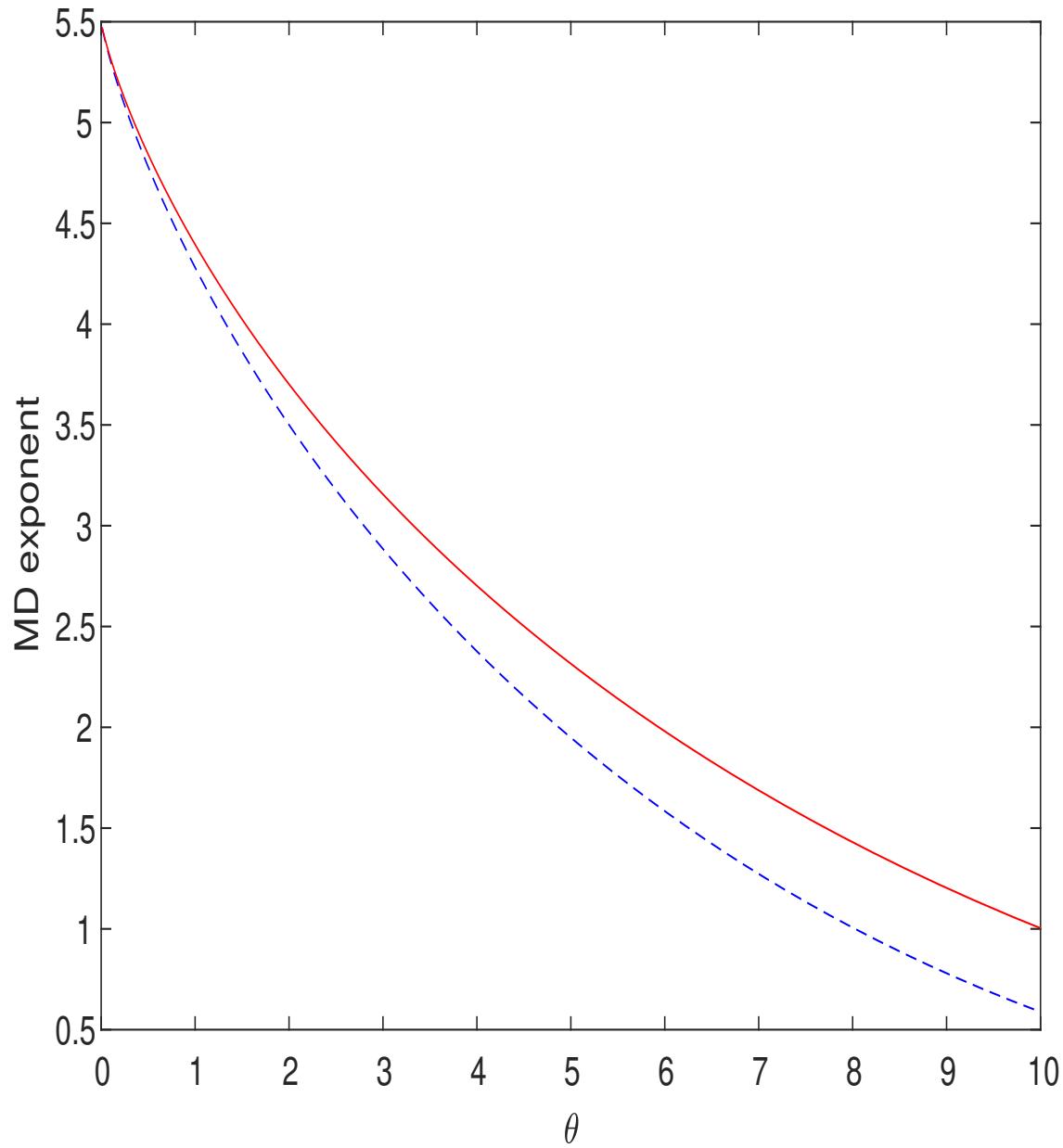
Some Numerical Results

Comparison of MD error exponents of the optical matched filter

$$w_{\text{omf}}(t) = \frac{\lambda(t)}{\lambda(t) + N_0/(2q_e^2 g^2)}$$

and the optimal correlator w_d^* as functions of θ , for a two-level signal at levels, $\lambda_1 = 1$, $\lambda_2 = 10$ with duty cycle of 50%.

$$P = 10, N_0/q_e^2 = 0.0001$$



Optimizing the Correlator (Random Gain)

Here, the optimal correlator is

$$w_{\mathbf{d}}^*(t) = \frac{1}{sq\mathbf{e}} \cdot p_\zeta[c \cdot \lambda(t)],$$

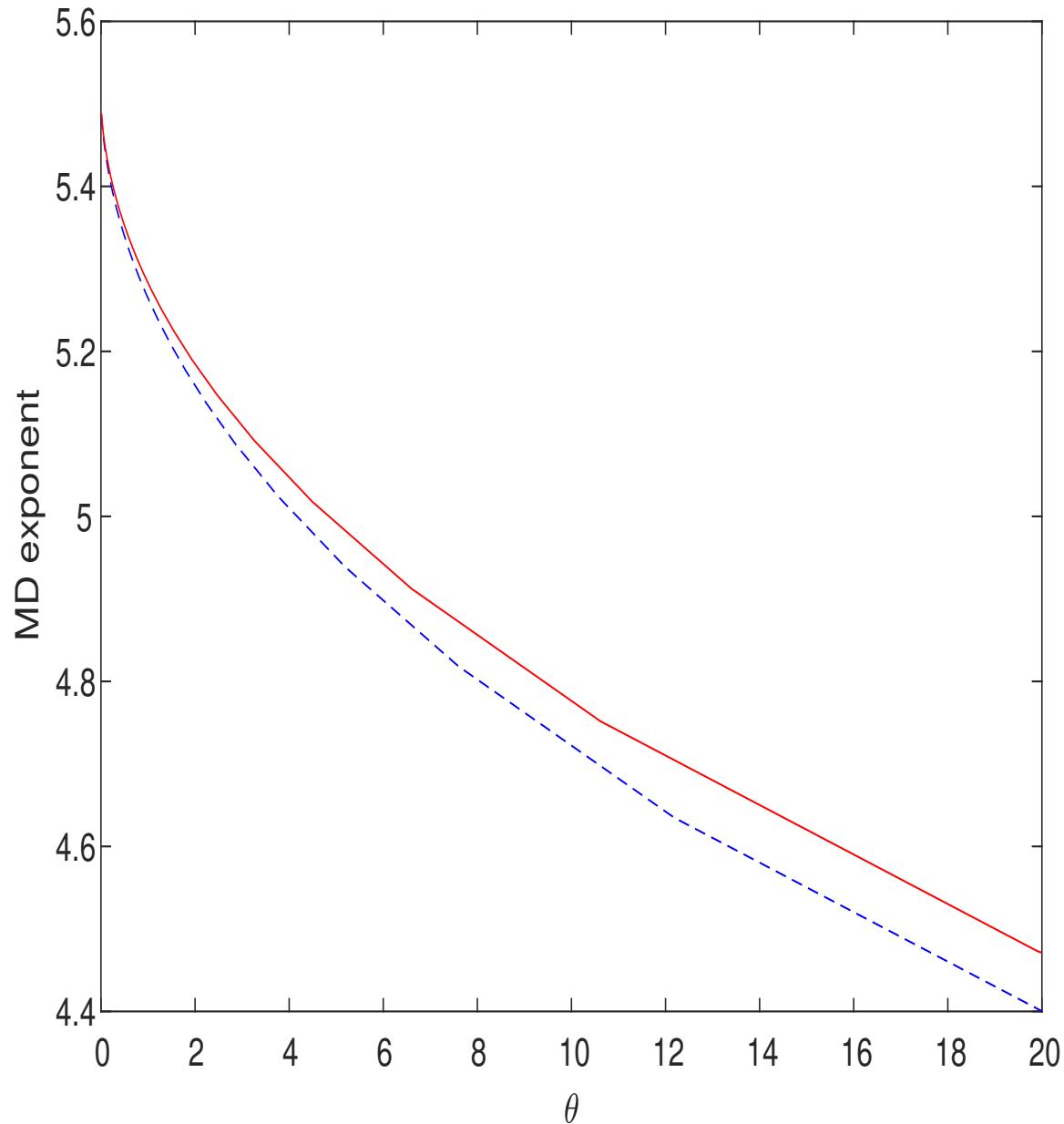
where p_ζ is the inverse of the function

$$b_\zeta[x] = \frac{x(e^{x+\zeta} - 1)^2}{e^{x+2\zeta} - e^{x+\zeta}}.$$

Once again,

For N_0 and/or θ large, $p_\zeta[x] \approx x$ and $w_{\mathbf{d}}^*(t) \propto \lambda(t)$.

For N_0 and θ small, $p_\zeta[x] \approx \ln x$, and $w_{\mathbf{d}}^*(t) \propto \ln \lambda(t)$.



Time Delay Estimation (high SNR)

- ♣ The underlying optical signal received is $\lambda(t - \tau)$.
- ♣ The estimator is $\hat{\tau} = \arg \max_{\theta} \int_0^T y(t)w(t - \theta)dt$.
- ♣ Here, $\tau = \arg \max_{\theta} \int_0^T \lambda(t - \tau)w(t - \theta)dt$
- ♣ What is the MSE, $\mathbf{E}(\hat{\tau} - \tau)^2$ in terms of $\{w(t), 0 \leq t \leq T\}$?
- ♣ What is waveform $\{w_e^*(t), 0 \leq t \leq T\}$ minimizes the MSE?

Mean Square Error

The high-SNR MSE is approximated by

$$\mathbf{E}\{(\hat{\tau} - \tau)^2\} \approx \frac{\int_0^T \left[\frac{N_0}{2} + \bar{g}^2 q_{\mathbf{e}}^2 \lambda(t) \right] \dot{w}^2(t) dt}{\bar{g}^2 q_{\mathbf{e}}^2 \left[\int_0^T \lambda(t) \ddot{w}(t) dt \right]^2},$$

where $\dot{w}(t)$ and $\ddot{w}(t)$ are the first two **derivatives** of $w(t)$.

Finding the optimal $w(t)$ is a problem of **calculus of variations**.

The Optimal Waveform

The solution to the above problem is:

$$w_{\mathbf{e}}^*(t) = \ln \left[1 + \frac{\lambda(t)}{\lambda_0} \right],$$

where

$$\lambda_0 = \frac{N_0}{2g^2 q_{\mathbf{e}}^2}.$$

For large N_0 , $w_{\mathbf{e}}^*(t) \propto \lambda(t)$.

For small N_0 , $w_{\mathbf{e}}^*(t) \propto \ln[\lambda(t)]$.

The MMSE is:

$$\text{MMSE} \approx \frac{1}{\lambda_0} \cdot \frac{1}{\int_0^T \frac{\dot{\lambda}^2(t) dt}{1 + \lambda(t)/\lambda_0}}.$$

Conclusion

- ♠ Exact optimal solutions for both problems are hard.
- ♠ Practical considerations motivate simple correlators.
- ♠ We found optimal correlator waveforms for both problems.
- ♠ Solutions are different but their limiting behavior is the same.