Error Exponents of Typical Random Trellis Codes

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Typical Random Codes

Traditional random coding error exponents are defined as

$$E_r(R) = \lim_{n \to \infty} \left[ -\frac{\ln P_e(C_n)}{n} \right].$$

We define typical–code error exponents as

$$E_{typ}(R) = \lim_{n \to \infty} \left[ -\frac{\mathbf{E}\ln P_e(C_n)}{n} \right].$$

By Jensen’s inequality, $E_{typ}(R) \geq E_r(R)$.

$E_r(R)$ – dominated by bad codes; $E_{typ}(R)$ – dominated by typical codes.

Let $G_E = \{C_n : P_e(C_n) = e^{-nE}\}$.

$$\overline{P_e(C_n)} = \sum_E P(G_E) \cdot e^{-nE} = P(G_E^*) \cdot e^{-nE^*}.$$  

Otoh, $E_{typ}(R) = \sum_E P(G_E) \cdot E = E_0$, where $P[G_{E_0}] \to 1$. 
**Motivation**

- $E_{\text{typ}}(R)$ is **never worse** than $E_{\text{r}}(R)$.

- Code selected **once and for all**: no LLN to support $E P_\text{e}(C_n)$.

- Once selected, w.h.p. $P_\text{e}(C_n) \sim e^{-nE_0}$, **forever**.

- Theoretical framework for random–like codes (Battail, 1995).

- Analogy: physics of disordered sys. – quenched vs. annealed average.

**Q:** With all these motivations, why wasn’t it explored much more before?

**A:** Not so easy to analyze (also in physics) ....
Related Work

- Barg & Forney ('02): i.i.d. random coding, BSC:
  \[ E_{\text{typ}}(R) = E_{\text{ex}}(2R) + R. \]

- Nazari ('11); Nazari, Anastasopoulos & Pradhan ('14):
  upper and lower bounds for the \( \alpha \)-decoder.

- Stat. phys. literature: Kabashima ('08), Mora & Riviore ('06), ...:
  LDPC codes - replica analysis and cavity method.

- Battail ('95):
  random–like codes.

  Exact error exponent of the typical random code (TRC).
Contributions

Deriving the error exponent of the **typical random trellis code** for:

- A general DMC (not just MBIOS).
- A general rational coding rate (not only $1/n$).
- A general random selection of a time–varying code.

The analysis method (based on the MoT) provides insights on:

- Structure of the typical random trellis code.
- Dominant error events.

Additional extensions:

- Channels with finite input memory (ISI).
- Mismatched decoding metric.
Problem Setting

\[ u_t \rightarrow \text{D} \rightarrow \text{D} \rightarrow \cdots \rightarrow \text{D} \rightarrow u_{t-k+1} \]

\[ f_t \]

\[ x_t \]

\[ W(y_t|x_t) \]

\[ y_t \rightarrow \text{decoder} \rightarrow \hat{u}_t \]

- \( \{u_t\} \) – \( m \)-vectors of purely random bits.
- \( f_t : \{0, 1\}^{mk} \rightarrow X^n \) randomly selected according to \( Q^n \).
- Coding rate: \( R = m/n \); constraint length: \( K = mk \).
- For convolutional codes, \( \{f_t\} \) are linear.
- \( W = \text{DMC} \).
- Asymptotic regime: \( k \rightarrow \infty \) while \( m \) and \( n \) are held fixed.
Background

Traditional performance metric: \( E(R, Q) = \lim \inf_{K \to \infty} \left[ -\log E_P e \right] / K \).

\[
E(R, Q) = \begin{cases} 
R_0(Q)/R & R < R_0(Q) \\
E_0(\rho, Q)/R & R > R_0(Q)
\end{cases}
\]

where \( \rho \) satifies: \( \rho R = E_0(R, Q) \).

For \( R > R_0(Q) \): \( \exists \) matching converse.

For \( R < R_0(Q) \): improvement by an expurgated bound,

\[
R_0(Q)/R \to E_{\text{cex}}(R, Q) \equiv E_x(\rho, Q)/R
\]

with \( \rho R = E_x(\rho, Q) \).

In [Viterbi & Odenwalder, 1969]: for at least half of the convolutional codes

\[
P_e \leq \left( \frac{2L}{1 - 2^{-\epsilon / \rho R}} \right)^{\rho} \cdot \exp\{-KE_{\text{cex}}(R, Q)\}.
\]

If \( 2L \) is replaced by \( 100L \), the bound applies to 99% of the codes.
Objectives

Studying the typical ensemble performance,

$$E_{\text{trtc}}(R, Q) = \lim_{K \to \infty} \inf -\frac{E \log P_e}{K}$$

as well as $E_{\text{trcc}}(R, Q)$ defined similarly for convolutional codes.

- Deriving both “Csiszár–style” and “Gallager–style” expressions.
- Comparing to the random coding exponent and expurgated exponent.
- Comparing to typical random block codes of the same complexity.
Main Result

For $R < R_0(Q)$:

$$E_{trtc}(R, Q) \geq E_{trtc}(R, Q) \triangleq \frac{E_x(\rho, Q)}{R},$$

where $\rho$ satisfies

$$\frac{E_x(\rho, Q)}{2\rho - 1} = R.$$

Also,

$$E_{trcc}(R, Q) \geq E_{cex}(R, Q).$$
Characterizing the Typical Codes

The probability of error

\[ P_e(C) \leq \sum_{\ell \geq 1} 2^{-m\ell} \sum_{\{P_{XX'}\}} N_{\ell}(P_{XX'}) \exp\{-n(k + \ell)\Delta(P_{XX'})\}. \]

For typical codes, all \( \{N_{\ell}(P_{XX'})\} \) that have a small expectation, vanish simultaneously w.h.p.

This amounts to the condition

\[ 2\ell R < (k + \ell)D(P_{XX'}\|Q \times Q). \]

Joint types that are “too far” from \( Q \times Q \) are not populated.

For populated joint types,

\[ N_{\ell}(P_{XX'}) \sim \exp\{m[2\ell - (k + \ell)D((P_{XX'})\|Q \times Q)/R]\}. \]
An Alternative Expression

\[ E_{\text{trtc}}(R, Q) = \inf_{\hat{R} < R} \left\{ \inf_{\{P_{XX'}: D(P_{XX'} \| Q \times Q) < 2\hat{R}\}} \frac{\mathbb{E}_P d_B(X, X') + \hat{R}}{R - \hat{R}} \right\}. \]

Dominant error events:

A sub–exponential number of paths with joint type,

\[ P_{XX'}(x, x') = \frac{Q(x)Q(x') \exp\{-d_B(x, x')/\rho\}}{Z} \]

and critical length of

\[ k + \ell = \frac{kR}{2R - D(P_{XX'} \| Q \times Q)}. \]
A Numerical Example: BSC with $p = 0.1$
Two Words Regarding an Extension

The paper contains also an extension to a channel with mismatch and input memory (ISI),

\[ W(y|x) = \prod_{t} W(y_t|x_t, x_{t-1}). \]

\( E_{X}(\rho, Q) \) is now replaced by \(-\rho \log \lambda\), where \( \lambda \) is the Perron-Frobenius eigenvalue of a certain matrix that depends on the \( Q \) and on the channel single–letterly (details – in the paper).