Error Exponents of Typical Random Codes

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Typical Random Codes

Traditional random coding error exponents are defined as

$$E_{\mathsf{r}}(R) = \lim_{n\to\infty}\left[-\frac{\ln \mathbf{E} P_{\mathsf{e}}(\mathcal{C}_n)}{n}\right].$$

We define typical–code error exponents as

$$E_{\mathsf{typ}}(R) = \lim_{n\to\infty}\left[-\frac{\mathbf{E} \ln P_{\mathsf{e}}(\mathcal{C}_n)}{n}\right].$$

- By Jensen's inequality, $E_{\mathsf{typ}}(R) \geq E_{\mathsf{r}}(R)$.
- $E_{\mathsf{r}}(R)$ – dominated by bad codes; $E_{\mathsf{typ}}(R)$ – dominated by typical codes.

Let $\mathcal{G}_E = \{\mathcal{C}_n : \quad P_{\mathsf{e}}(\mathcal{C}_n) \doteq e^{-nE}\}$.

$$\overline{P_{\mathsf{e}}(\mathcal{C}_n)} \doteq \sum_E P(\mathcal{G}_E)\cdot e^{-nE} \doteq P(\mathcal{G}_E^*)\cdot e^{-nE^*}.$$

Otoh, $E_{\mathsf{typ}}(R) = \sum_E P(\mathcal{G}_E)\cdot E = E_0$, where $P[\mathcal{G}_{E_0}] \to 1$.

Motivation

- $E_{\text{typ}}(R)$ is never worse than $E_{\text{r}}(R)$.
- Code selected once and for all: no LLN to support $\mathbf{E}P_{\text{e}}(\mathcal{C}_n)$.
- Once selected, w.h.p. $P_{\text{e}}(\mathcal{C}_n) \sim e^{-nE_0}$, forever.
- Theoretical framework for random–like codes (Battail, 1995).
- Analogy: physics of disordered sys. – quenched vs. annealed average.

Q: With all these motivations, why wasn't it explored much more before?

A: Not so easy to analyze (also in physics) ....

Related Work

- Barg & Forney (2002): i.i.d. random coding, BSC:

  At low rates: $E_{\text{typ}}(R) = E_{\text{ex}}(2R) + R$.

- Nazari (2011); Nazari, Anastasopoulos & Pradhan (2014):

  upper and lower bounds for the α–decoder.

- Stat. phys. literature: Kabashima (2008), Mora & Riviore (2006), ...:

  LDPC codes - replica analysis and cavity method.

- Battail (1995):

  random–like codes.

Contributions

We derive the exact typical–code error exponent for a class of stochastic decoders,

$$P(\hat{m} = m|\boldsymbol{y}) \propto \exp\{ng(\hat{P}_{\boldsymbol{x}_m\boldsymbol{y}})\},$$

e.g., $g(Q_{XY} = \beta\mathbf{E}_Q \ln W'(Y|X)$, $g(Q_{XY}) = \beta \cdot \alpha(Q_{XY})$, $g(Q_{XY}) = \beta I_Q(X;Y)$.
Extending Barg & Forney (2002) in several directions:

- General DMC is considered, not merely the BSC.
- Covering a wider family of decoders.
- Ensemble of constant composition codes – optimal PI distribution.
- Relation to expurgated exponent – for all R and a general decoder.
- The analysis technique is applicable also to more general scenarios.

Main Result

Let

$$\alpha(R, Q_Y) = \sup[g(Q_{XY}) - I_Q(X;Y)] + R,$$

where supremum is over $\{Q_{X|Y}:\ I_Q(X;Y) \leq R,\ Q_X = P_X\}$.

$$\Gamma(Q_{XX'}, R) = \inf_{Q_{Y|XX'}} \{D(Q_{Y|X}\|W|P_X) + I_Q(X';Y|X) + [g(Q_{XY}) \wedge \alpha(R, Q_Y) - g(Q_{X'Y})]_+\}.$$

Theorem: The typical error exponent is

$$E_{\text{typ}}(R) = \inf\{\Gamma(Q_{XX'}, R) + I_Q(X;X')\} - R,$$

where the infimum is over $\{Q_{XX'}:\ I_Q(X;X') \leq 2R,\ Q_X = Q_{X'} = P_X\}$.

ML Decoding

In ML decoding: minimization s.t.

$$\mathbf{E}_Q \ln W(Y|X') \geq \max\{\mathbf{E}_Q \ln W(Y|X), D(R, Q_Y)\},$$

$$D(R, Q_Y) = \sup\{\mathbf{E}_Q \ln W(Y|X''):\ I_Q(X''; Y) \leq R,\ (Q_Y \times Q_{X''|Y})_X = P_X\},$$

being the typical highest score of an incorrect message.

This is not a union bound of pairwise error events.

Relation to Expurgated Exponent

Defining

$$E_0(R,S) = \inf\{\Gamma(Q_{XX'},S) + I_Q(X;X')\} - R,$$

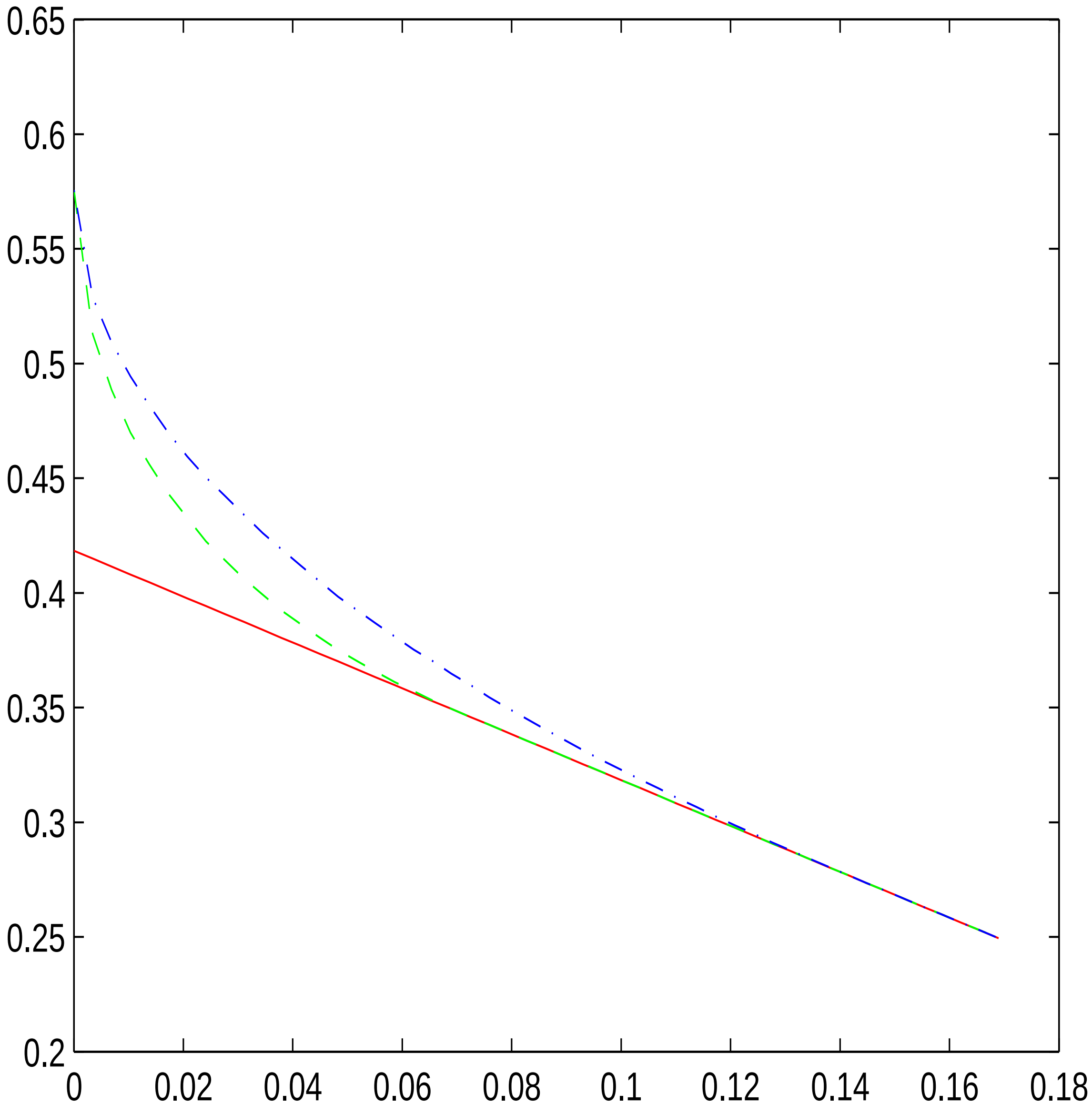
the infimum being over $\{Q_{XX'} : \; I_Q(X;X') \leq R, \; Q_X = Q_{X'} = P_X\}$, we have for all R:

$$E_{\text{ex}}(R) = E_0(R,R); \qquad E_{\text{typ}}(R) = E_0(2R,R) + R.$$

In general,

$$E_{\text{typ}}(R) \leq E_{\text{ex}}(2R) + R,$$

because here the expurgated exponent is improved.



rand. coding, expurg. and typical exponents for z–channel w. crossover 0.1.

A Few Words on the Analysis

- Using the identity $\mathbf{E} \ln P_{\mathrm{e}}(\mathcal{C}) = \lim_{\rho\to\infty} \rho \ln \mathbf{E}[P_{\mathrm{e}}(\mathcal{C})]^{1/\rho}$.
- Using the method of type class enumerators.
- Main idea behind the analysis: handling summations of exponentially many fractions with random denominators – exploit concentration properties.

$$\mathbf{E}\left[\frac{1}{M}\sum_{m}\sum_{m'\neq m}\sum_{\boldsymbol{y}} P(\boldsymbol{y}|\boldsymbol{X}_m)\cdot\frac{e^{ng(\boldsymbol{X}_{m'},\boldsymbol{y})}}{e^{ng(\boldsymbol{X}_m,\boldsymbol{y})}+\sum_{\tilde{m}\neq m} e^{ng(\boldsymbol{X}_{\tilde{m}},\boldsymbol{y})}}\right]^{\rho}.$$

In paritcular, with very high probability,

$$\sum_{\tilde{m}\neq m} e^{ng(\boldsymbol{X}_{\tilde{m}},\boldsymbol{y})} \geq e^{n\alpha(R,\hat{P}\boldsymbol{y})}.$$

- Showing that the reversed inequality holds for most terms w.h.p.

Other Applications

The same techniques are applicable in other scenarios:

- List decoding (fixed list size): involves the notion of multi–information.
- Decoding with an erasure option.

The details are in the paper.

Future Directions

- Analogues in source coding (e.g., Slepian–Wolf).
- Source–channel coding.
- Multi-user situations: MAC, BC, etc.
- Other (more structured) ensembles: allowing dependencies.
- Universal decoding.
- Continuous alphabets (Gaussian channel).