Perfectly Secure Encryption of Individual Sequences

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Background and Motivation

The individual–sequence approach (with FSM’s) to IT has been studied in:

- Data compression (Ziv & Lempel ‘78,...).
- Source/channel simulation (Martín et al. ‘10, Seroussi ‘06).
- Classification (Ziv & Merhav, ‘93).
- Prediction (Feder, Merhav & Gutman ‘92, ...).
- Denoising (Weissman et al., ‘05,...).
- Channel coding (Lomnitz & Feder ‘10, Shayevitz & Feder ‘05).

Information–theoretic security has been studied almost exclusively from the probabilistic approach.
Background and Motivation (Cont’d)

The only exception (a.f.a.i.k.) is an unpublished memorandum by Ziv [1978]:

- Plaintext source – individual sequence.
- Encrypter – general block encoder.
- Prior knowledge: plaintext $\rightarrow$ FSM $\rightarrow$ all–zero sequence.
- Full security: uncertainty – essentially not reduced by cryptogram.
- Main result: minimum needed key rate $\sim$ LZ compressibility.

Encrypter/decrypter have unlimited resources, whereas eavesdropper is limited by FSM.

Our approach: the other way around – encryption using FSM’s.
Finite–State Encrypter Model

\[ u_1, u_2, \ldots \]

\[ x_1, x_2, \ldots \]

plaintext

Finite-State Encrypter

cryptogram

\[ y_1, y_2, \ldots \]

\[ \Delta = 0 \]

\[ k_i = (u_{t_i-1+1}, u_{t_i-1+2}, \ldots, u_{t_i}) \]

\[ y_i = f(z_i, x_i, k_i) \]

\[ z_{i+1} = g(z_i, x_i) \]

\[ t_i = t_{i-1} + \Delta(z_i, x_i), \quad t_0 \triangleq 0 \]
Finite–State Encrypter Model (Cont’d)

Perfect security: \( \Pr\{y^n|x\} \) – independent of \( x \).

Information losslessness (IL): \( \exists \) large \( n \) s.t. \( (z_1, z_{n+1}, k^n, y^n) \) determines \( x^n \).

Key rate of encrypter \( E \):

\[
\sigma_E(x^n) = \frac{1}{n} \sum_{i=1}^{n} \Delta(z_i, x_i) = \frac{1}{n} \sum_{i=1}^{n} \ell(k_i).
\]

\[
\sigma_s(x^n) = \min_{E \in \mathcal{E}(s)} \sigma_E(x^n),
\]

where \( \mathcal{E}(s) = \text{set of all perfectly secure, IL encrypters with } \leq s \text{ states.} \)

\[
\sigma_s(x) = \limsup_{n \to \infty} \sigma_s(x^n)
\]

Finite–state encryptability: \( \sigma(x) = \lim_{s \to \infty} \sigma_s(x) \).
Main Result

Let $\rho_{LZ}(x^n)$ denote the LZ compression ratio, i.e.,

$$\rho_{LZ}(x^n) = \frac{c(x^n) \log c(x^n)}{n},$$

where $c(x^n)$ = number of LZ phrases in $x^n$.

**Theorem (converse):** For every $x^n$

$$\sigma_s(x^n) \geq \rho_{LZ}(x^n) - O\left(s \cdot \sqrt{\frac{\log(\log n)}{\log n}}\right).$$

Consequently,

$$\sigma(x) \geq \rho(x).$$
Discussion

- Direct: LZ compression + one–time pad encryption – $\sigma(x) = \rho(x)$.
- Natural individual–sequence counterpart to the known probabilistic result.
- Same conclusion as in [Ziv78], although the model is different.
- Upperbound – lowerbound = $O(\sqrt{\log(\log n)/\log n})$.
- In compression – $O(1/\log n)$. 
Main Ideas of Proof

Define joint empirical distribution of $m$–blocks of $(x^n, k^n, y^n, ...)$.

$$\sigma_E(x^n) = \frac{\ell(k^n)}{n} = \frac{H(K^m | L)}{m} \geq \frac{1}{m}[H(K^m) - \alpha s \log(m + 1)].$$

Using usual information–theoretic arguments (+ IL + full security):

$$H(K^m) \geq H(X^m) - H(Z, Z'|Y^m, K^m) \geq H(X^m) - 2 \log s.$$ 

Now, since Shannon code = FS encoder:

$$\frac{H(X^m)}{m} \geq \rho_{LZ}(x^n) - \delta_s(m, n).$$

So eventually,

$$\sigma_E(x^n) \geq \rho_{LZ}(x^n) - \text{vanishing terms}.$$
Extensions
Availability of Side Information

Assume that everybody has access to SI $s_1, s_2, \ldots$ (individual sequence).
Modifying the model definition:

$$ t_i = t_{i-1} + \Delta(z_i, x_i, s_i), \quad t_0 \triangleq 0 $$

$$ k_i = (u_{t_i-1+1}, u_{t_i-1+2}, \ldots, u_{t_i}) $$

$$ y_i = f(z_i, x_i, k_i, s_i) $$

$$ z_{i+1} = g(z_i, x_i, s_i) $$

Perfect security: $\Pr(y^n|x, s) - \text{independent of } x$.

Info losslessness: For large enough $n$: $(z_1, z_{n+1}, s^n, y^n, k^n)$ determine $x^n$.

Main result: Same but with $\rho_{LZ}(x^n)$ replaced by $\rho_{LZ}(x^n|s^n)$ – conditional LZ parsing of $x^n$ given $s^n$ [Ziv ‘85].

Achievable even if encrypter does not see $s^n$: S–W coding + one–time pad.
Apply LZ to \((x_1, s_1), (x_2, s_2), \ldots, (x_n, s_n)\); \(c(x^n, s^n)\) = number of phrases.

\(c(s^n)\) = number of distinct phrases of \(s^n\).

\(s(l)\) = the \(l\)th distinct \(s\)–phrase, \(l = 1, 2, \ldots, c(s^n)\).

\(c_l(x^n|s^n)\) = number of \(s(l)\) in parsing of \(s^n\).

\[ \rho_{LZ}(x^n|s^n) = \frac{1}{n} \sum_{l=1}^{c(s^n)} c_l(x^n|s^n) \log c_l(x^n|s^n). \]

For example,

\[
\begin{align*}
x^6 &= 0 | 1 | 0 0 | 0 1 \\
s^6 &= 0 | 1 | 0 1 | 0 1
\end{align*}
\]

then

\[ c(x^6, s^6) = 4, \quad c(s^6) = 3, \quad s(1) = 0, \quad s(2) = 1, \quad s(3) = 01, \]

\[ c_1(x^6|s^6) = c_2(x^6|s^6) = 1, \quad c_3(x^6|s^6) = 2. \]
Lossy Reconstruction

Modifications in model definition:

- Legitimate reconstruction $\hat{x}^n$ must satisfy $d(x^n, \hat{x}^n) \leq nD$ w.p. 1.
- Distortion measure – completely arbitrary (need not be even additive).
- IL property can be relaxed to a weaker requirement (details in the paper).
- Perfect security: $y^n$ is statistically independent of both $x$ and $\hat{x}$.

Main theorem essentially as before but $\rho_{LZ}(x^n)$ should be replaced by

$$r_{LZ}(D; x^n) = \min_{d(x^n, \hat{x}^n) \leq nD} \rho_{LZ}(\hat{x}^n).$$

- Not obvious that best $\hat{x}^n$ is deterministic (could have depended on key).
- Achievability: again, conceptually obvious.
- In the full paper: also SI + lossy reconstruction; No longer based on LZ.
Thank You!