Perfectly Secure Encryption of Individual Sequences

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ISIT 2012, Cambridge, MA, July 2012.

Background and Motivation

The individual–sequence approach (with FSM's) to IT has been studied in:

- Data compression (Ziv & Lempel '78,...).
- Source/channel simulation (Martín *et al.* '10, Seroussi '06).
- Classification (Ziv & Merhav, '93).
- Prediction (Feder, Merhav & Gutman '92, ...).
- Denoising (Weissman *et al.*, '05,...).
- Channel coding (Lomnitz & Feder '10, Shayevitz & Feder '05).

Information–theoretic security has been studied almost exclusively from the probabilisitic approach.

Background and Motivation (Cont'd)

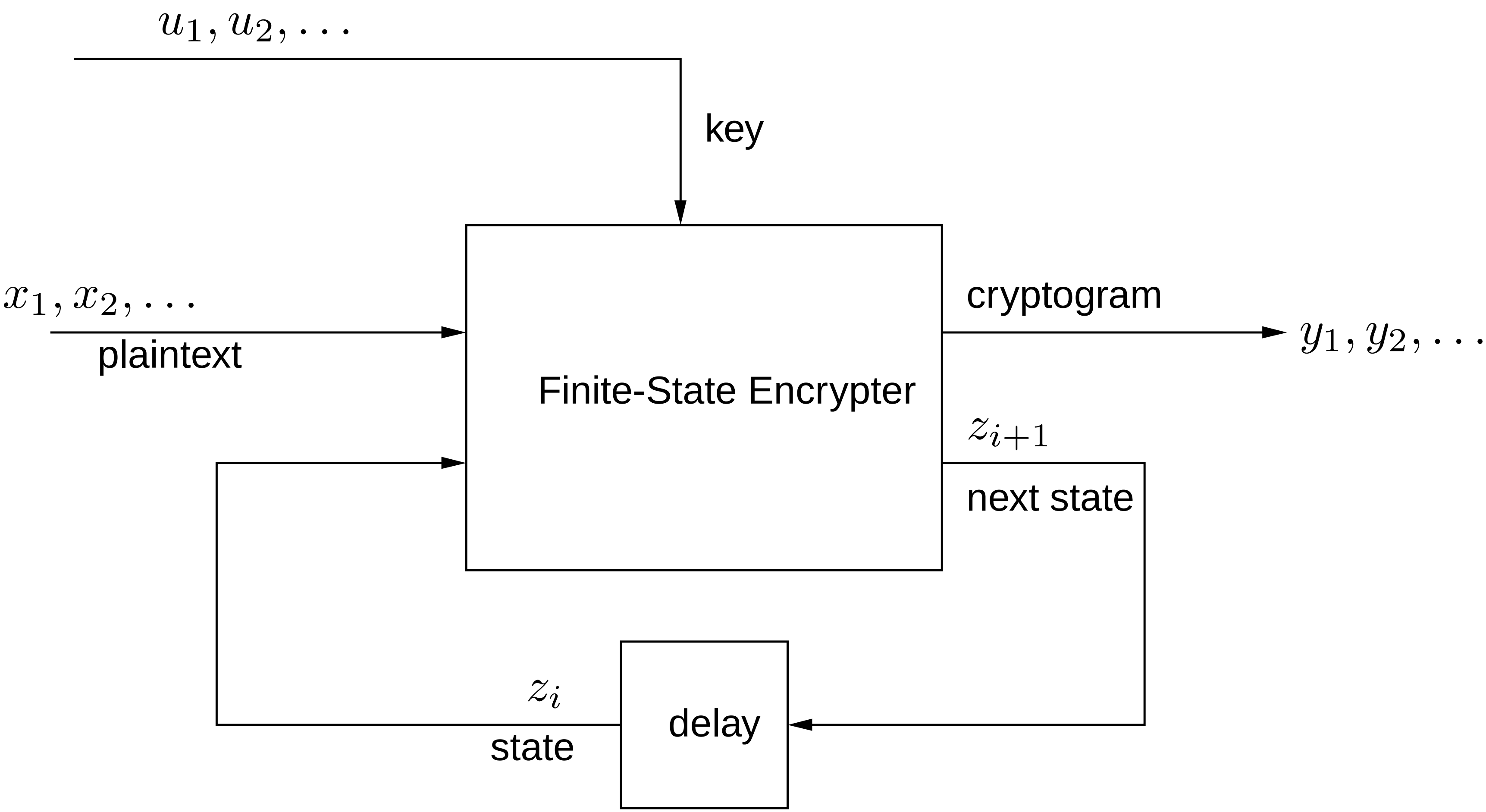
The only exception (a.f.a.i.k.) is an unpublished memorandum by Ziv [1978]:

- Plaintext source – individual sequence.
- Encrypter – general block encoder.
- Prior knowledge: plaintext $\rightarrow$ FSM $\rightarrow$ all–zero sequence.
- Full security: uncertainty – essentially not reduced by cryptogram.
- Main result: minimum needed key rate $\sim$ LZ compressibility.

Encrypter/decrypter have unlimited resources, whereas eavesdropper is limited by FSM.

Our approach: the other way around – encryption using FSM's.

Finite–State Encrypter Model



$$
\begin{aligned}
t_i &= t_{i-1} + \Delta(z_i, x_i), \qquad t_0 \triangleq 0 \\
k_i &= (u_{t_{i-1}+1}, u_{t_{i-1}+2}, \ldots, u_{t_i}) \\
y_i &= f(z_i, x_i, k_i) \\
z_{i+1} &= g(z_i, x_i)
\end{aligned}
$$

Finite–State Encrypter Model (Cont'd)

Perfect security: Pr$\{y^n|\boldsymbol{x}\}$ – independent of $\boldsymbol{x}$.

Information losslessness (IL): $\exists$ large n s.t. (z_1, z_{n+1}, k^n, y^n) determines x^n.

Key rate of encrypter E:

$$\sigma_E(x^n) = \frac{1}{n}\sum_{i=1}^{n}\Delta(z_i, x_i) = \frac{1}{n}\sum_{i=1}^{n}\ell(k_i).$$

$$\sigma_s(x^n) = \min_{E\in\mathcal{E}(s)}\sigma_E(x^n),$$

where $\mathcal{E}(s) =$ set of all perfectly secure, IL encrypters with $\leq s$ states.

$$\sigma_s(\boldsymbol{x}) = \limsup_{n\to\infty}\sigma_s(x^n)$$

Finite–state encryptability: $\sigma(\boldsymbol{x}) = \lim_{s\to\infty}\sigma_s(\boldsymbol{x})$.

Main Result

Let $\rho_{LZ}(x^n)$ denote the LZ compresion ratio, i.e.,

$$\rho_{LZ}(x^n) = \frac{c(x^n) \log c(x^n)}{n},$$

where $c(x^n)$ = number of LZ phrases in x^n.

Theorem (converse): For every x^n

$$\sigma_s(x^n) \geq \rho_{LZ}(x^n) - O\left(s \cdot \sqrt{\frac{\log(\log n)}{\log n}}\right).$$

Consequently,

$$\sigma(\boldsymbol{x}) \geq \rho(\boldsymbol{x}).$$

Discussion

- Direct: LZ compression + one–time pad encryption – $\sigma(\boldsymbol{x}) = \rho(\boldsymbol{x})$.
- Natural individual–sequence counterpart to the known probabilistic result.
- Same conclusion as in [Ziv78], although the model is different.
- Upperbound – lowerbound = $O(\sqrt{\log(\log n)/\log n})$.
- In compression – $O(1/\log n)$.

Main Ideas of Proof

Define joint empirical distribution of m–blocks of $(x^n, k^n, y^n, ...)$.

$$\sigma_E(x^n) = \frac{\ell(k^n)}{n} = \frac{H(K^m|L)}{m} \geq \frac{1}{m}[H(K^m) - \alpha s \log(m+1)].$$

Using usual information–theoretic arguments (+ IL + full security):

$$H(K^m) \geq H(X^m) - H(Z, Z'|Y^m, K^m) \geq H(X^m) - 2\log s.$$

Now, since Shannon code = FS encoder:

$$\frac{H(X^m)}{m} \geq \rho_{LZ}(x^n) - \delta_s(m, n).$$

So eventually,

$$\sigma_E(x^n) \geq \rho_{LZ}(x^n) - \text{vanishing terms.}$$

Extensions

Availability of Side Information

Assume that everybody has access to SI $s_1, s_2, \ldots$ (individual sequence). Modifying the model definition:

$$\begin{aligned} t_i &= t_{i-1} + \Delta(z_i, x_i, s_i), \qquad t_0 \stackrel{\Delta}{=} 0 \\ k_i &= (u_{t_{i-1}+1}, u_{t_{i-1}+2}, \ldots, u_{t_i}) \\ y_i &= f(z_i, x_i, k_i, s_i) \\ z_{i+1} &= g(z_i, x_i, s_i) \end{aligned}$$

Perfect security: Pr$(y^n|\boldsymbol{x}, \boldsymbol{s})$ – independent of $\boldsymbol{x}$.

Info losslessness: For large enough n: $(z_1, z_{n+1}, s^n, y^n, k^n)$ determine x^n.

Main result: Same but with $\rho_{LZ}(x^n)$ replaced by $\rho_{LZ}(x^n|s^n)$ – conditional LZ parsing of x^n given s^n [Ziv '85].

Achievable even if encrypter does not see s^n: S–W coding + one–time pad.

Availability of SI (Cont'd) – Conditional Parsing

- Apply LZ to $((x_1, s_1), (x_2, s_2), \ldots, (x_n, s_n))$; $c(x^n, s^n)$ = number of phrases.
- $c(s^n)$ = number of distinct phrases of s^n.
- $\boldsymbol{s}(l)$ = the lth distinct s–phrase, $l = 1, 2, ..., c(s^n)$.
- $c_l(x^n|s^n)$ = number of $\boldsymbol{s}(l)$ in parsing of s^n.

$$\rho_{LZ}(x^n|s^n) = \frac{1}{n} \sum_{l=1}^{c(s^n)} c_l(x^n|s^n) \log c_l(x^n|s^n).$$

For example,

$$\begin{aligned} x^6 &= 0 \mid 1 \mid 0\,0 \mid 0\,1| \\ s^6 &= 0 \mid 1 \mid 0\,1 \mid 0\,1| \end{aligned}$$

then

$$c(x^6, s^6) = 4, \;\; c(s^6) = 3, \;\; \boldsymbol{s}(1) = 0, \; \boldsymbol{s}(2) = 1, \; \boldsymbol{s}(3) = 01,$$

$$c_1(x^6|s^6) = c_2(x^6|s^6) = 1, \;\; c_3(x^6|s^6) = 2.$$

Lossy Reconstruction

Modifications in model definition:

- Legitimate reconstruction $\hat{x}^n$ must satisfy $d(x^n, \hat{x}^n) \leq nD$ w.p. 1.
- Distortion measure – completely arbitrary (need not be even additive).
- IL property can be relaxed to a weaker requirement (details in the paper).
- Perfect security: y^n is statistically independent of both $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$.

Main theorem essentially as before but $\rho_{LZ}(x^n)$ should be replaced by

$$r_{LZ}(D; x^n) = \min_{d(x^n, \hat{x}^n) \leq nD} \rho_{LZ}(\hat{x}^n).$$

- Not obvious that best $\hat{\boldsymbol{x}}^n$ is deterministic (could have depended on key).
- Achievability: again, conceptually obvious.
- In the full paper: also SI + lossy reconstruction; No longer based on LZ..

Thank You!