

Bose–Einstein Condensation in the Large Deviations Regime with Applications to Information System Models

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The Problem

We are interested in the **large deviations behavior** of a system whose steady-state obeys

$$P(n_0, n_1, \dots, n_{M-1}) = \frac{\prod_{i=0}^{M-1} p_i^{n_i}}{Z},$$

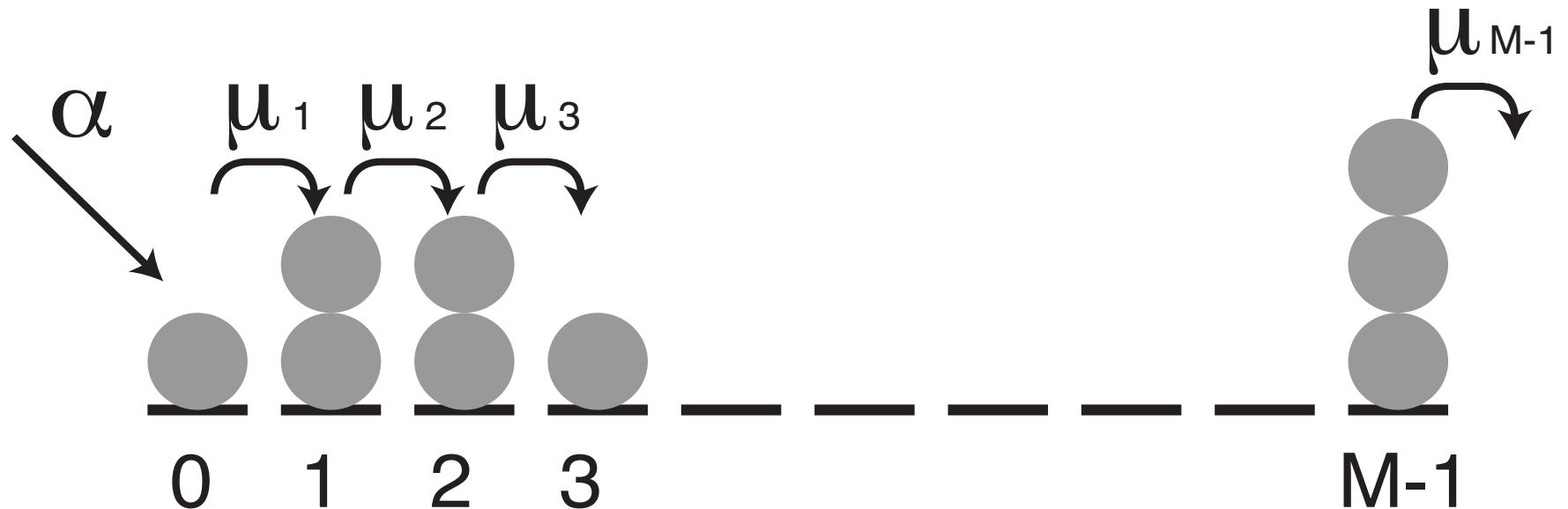
where n_i = number of elements (particles/agents,...) in ‘site’ no. i .

E.g., assess $\Pr\{\sum_i n_i \geq M \cdot U\} \sim e^{-M \cdot I(U)}$ for $M \rightarrow \infty$.

Examples of Relevance:

- Grand-canonical ideal Boson gas: $p_i = z e^{-\beta \epsilon_i}$ (black body radiation).
- One-way Markov chain: $i \rightarrow i$ w.p. p_i ; $i \rightarrow i+1$ w.p. $1 - p_i$.
- Jackson network of M queues: $p_i = \lambda_i / \mu_i$.
- Hopping model for transport in a disordered medium: $p_i = \alpha / \mu_i$ (ZRP’s).

Hopping Model (Or A Cascade of Queues)



$$P(n_0, n_1, \dots, n_{M-1}) = \frac{\prod_{i=0}^{M-1} (\alpha/\mu_i)^{n_i}}{Z},$$

The Problem (Cont'd)

The large deviations rate function

$$I(U) = \lim_{M \rightarrow \infty} \left[-\frac{1}{M} \log \Pr \left\{ \sum_i n_i \geq MU \right\} \right]$$

may exhibit phase transitions – **Bose–Einstein condensation** (BEC): Given the event, a **macroscopic** fraction – jammed in one ‘site’ for $U > U_c$.

This is the case also with general linear combinations of $\{n_i\}$, e.g.,

$$I(U) = \lim_{M \rightarrow \infty} \frac{-1}{M} \log \Pr \left\{ \underbrace{\sum_i n_i / \mu_i}_{\text{est. time}} \geq MU \right\}.$$

More interestingly, how about $I(U, V)$ of $\Pr \{ \sum_i n_i \geq MU, \sum_i n_i / \mu_i \geq MV \}$?

Answer gives rise to a notion of **2D BEC**: A very rich phase diagram.

A Single Constraint

$$\begin{aligned}\Pr \left\{ \sum_i n_i \geq MU \right\} &\leq \left\langle z^{\sum_i n_i - MU} \right\rangle \quad z \geq 1 \\ &= z^{-MU} \prod_i \frac{1 - p_i}{1 - p_i z} \\ &= \exp \left\{ -M \left[U \ln z - \frac{1}{M} \sum_i \ln \left(\frac{1 - p_i}{1 - z p_i} \right) \right] \right\} \\ I(U) &= \sup_{z \geq 1} \left[U \ln z - \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=0}^{M-1} \ln \left(\frac{1 - p_i}{1 - p_i z} \right) \right].\end{aligned}$$

Assume that $\{p_i\}$ have a density $g(p)$:

fraction of $\{p_i\} \in [p, p + dp] \rightarrow g(p)dp$.

A Single Constraint (Cont'd)

Resulting equation in z (saddle point):

$$U = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=0}^{M-1} \frac{zp_i}{1 - p_i z} \equiv z \cdot \int_0^{p_{\max}} \frac{p \cdot g(p) dp}{1 - pz} \triangleq U(z)$$

A solution z^* is sought in $[1, 1/p_{\max}]$.

In analogy to the BEC, if

$$\lim_{p \uparrow p_{\max}} \frac{g(p)}{(p_{\max} - p)^\xi} < \infty \quad \text{for some } \xi > 0,$$

then

$$U_c = U(1/p_{\max}) \equiv \int_0^{p_{\max}} \frac{pg(p) dp}{p_{\max} - p} < \infty$$

Condensation for $U > U_c$.

A Single Constraint (Cont'd)

The rate function exhibits two phase transitions

$$I(U) = \begin{cases} 0 & U < U_{\min} \equiv U(1) \\ \sup_z \left[U \ln z - \int_0^{p_{\max}} \mathsf{d}p g(p) \ln \left(\frac{1-p}{1-pz} \right) \right] & U_{\min} \leq U < U_c \\ U \ln \left(\frac{1}{p_{\max}} \right) - \int_0^{p_{\max}} \mathsf{d}p g(p) \ln \left(\frac{1-p}{1-p/p_{\max}} \right) & U \geq U_c \end{cases}$$

- $U < U_{\min}$ – not a rare event.
- $U_{\min} \leq U < U_c$ – rare event; no condensation. $I(U)$ is convex.
- $U > U_c$ – BEC: $n_i/N \sim 1 - U_c/U$ for that i of $p_i = p_{\max}$. $I(U)$ is affine.

Second order phase transition: $I(U)$ and $I'(U)$ are continuous, $I''(U)$ is not.

A More General Constraint

Consider now the event $\{\sum_i n_i u_i \geq MU\}$ for some deterministic sequence u_1, u_2, \dots which has some density, or more simply, let $u_i = u(p_i)$ for some function $u(\cdot)$.

Examples:

- Disordered transport model: $u(p) = p/\alpha \rightarrow \sum_i n_i/\mu_i$ (estimated time).
- Ideal Boson gas: $u(p) = -\frac{1}{\beta} \ln p \rightarrow \sum_i n_i \epsilon_i$ (energy).

A similar derivation yields the saddle-point equation

$$U = \mathbf{U}(z) \equiv \int_0^{p_{\max}} \frac{p u(p) z^{u(p)} g(p) dp}{1 - p z^{u(p)}},$$

whose solution z is sought in $[1, Z)$, where $Z \equiv \inf_{p \in [0, p_{\max}]} \{p^{-1/u(p)}\}$.

Again, condensation takes place if $U_c = \mathbf{U}(Z) < \infty$ and $U > U_c$.

Two Constraints

Consider now the event $\{\sum_i n_i u_i \geq MU, \sum_i n_i v_i \geq MV\}$ for $u_i = u(p_i)$ and $v_i = v(p_i)$. Here the large deviations analysis yields two equations with two unknowns:

$$U = U(z_1, z_2) \equiv \int_0^{p_{\max}} \frac{p u(p) z_1^{u(p)} z_2^{v(p)} g(p) dp}{1 - p z_1^{u(p)} z_2^{v(p)}}$$

$$V = V(z_1, z_2) \equiv \int_0^{p_{\max}} \frac{p v(p) z_1^{u(p)} z_2^{v(p)} g(p) dp}{1 - p z_1^{u(p)} z_2^{v(p)}}$$

The rate function is

$$I(U, V | z_1, z_2) = U \ln z_1 + V \ln z_2 - \int dp g(p) \ln \left[\frac{1-p}{1 - p z_1^{u(p)} z_2^{v(p)}} \right].$$

Here, z_1 and z_2 are jointly limited by the inequality:

$f(z_1, z_2) \triangleq \sup_p [p z_1^{u(p)} z_2^{v(p)}] < 1$. Solutions are sought in the region $\mathcal{A} = \{(z_1, z_2) : z_1 \geq 1, z_2 \geq 1, f(z_1, z_2) < 1\}$.

Two Constraints – Seven Phases

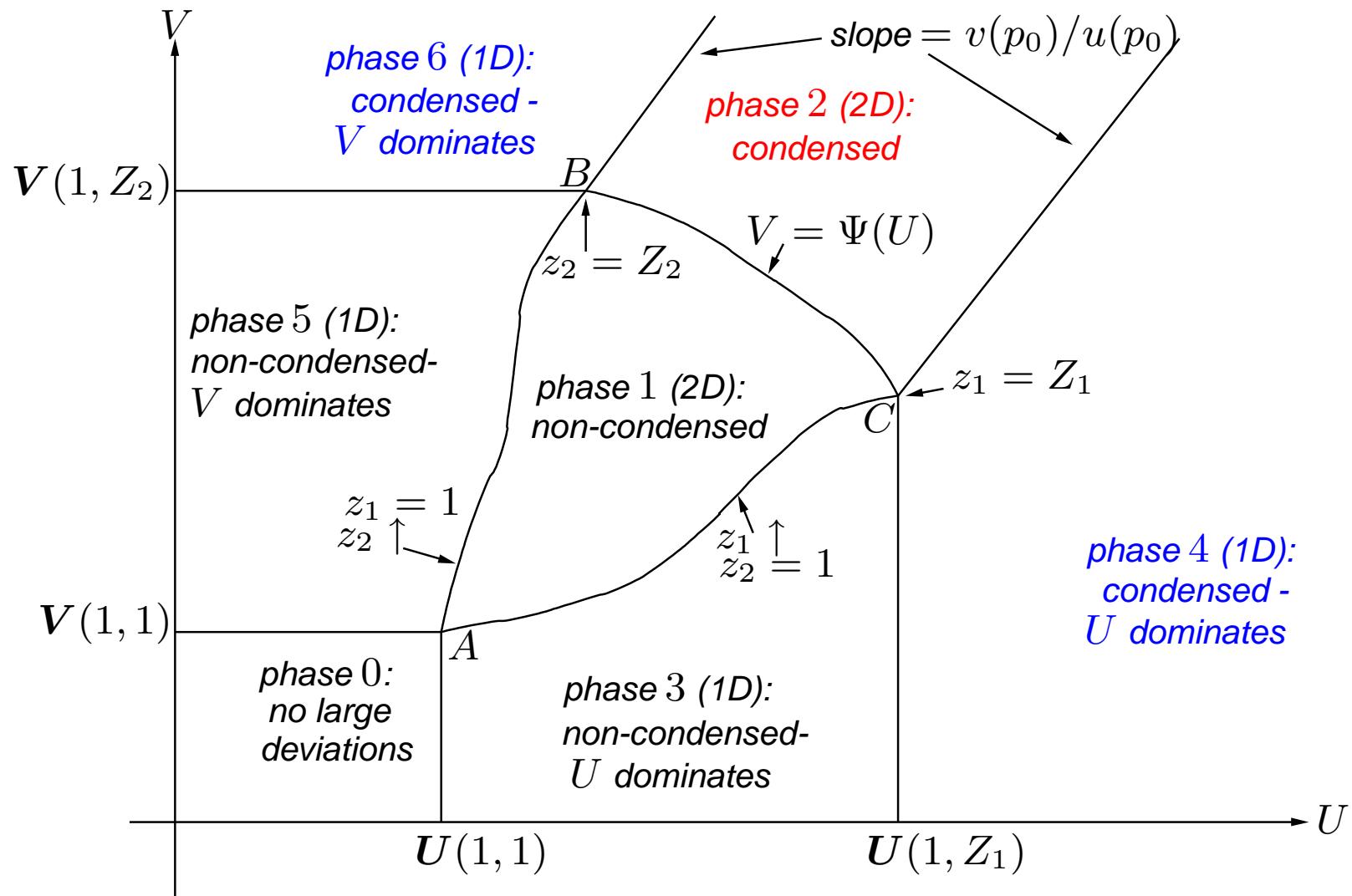
$$I(U, V | z_1, z_2) = U \ln z_1 + V \ln z_2 - \int \mathrm{d}p g(p) \ln \left[\frac{1-p}{1-p z_1^{u(p)} z_2^{v(p)}} \right].$$

$$I(U, V) = \begin{cases} 0 & \text{no LD} \\ \sup_{z_1} I(U, V | z_1, 1) & \text{non-condensed – } U \text{ only} \\ \sup_{z_2} I(U, V | 1, z_2) & \text{non-condensed – } V \text{ only} \\ \sup_{z_1, z_2} I(U, V | z_1, z_2) & \text{non-condensed} \\ I(U, V | Z_1, 1) & \text{1D condensed – } U \text{ only} \\ I(U, V | 1, Z_2) & \text{1D condensed – } V \text{ only} \\ \sup_{f(z_1, z_2)=1} I(U, V | z_1, z_2) & \text{2D condensed} \end{cases}$$

where

- $Z_1 = \text{solution of } f(z_1, 1) = 1$.
- $Z_2 = \text{solution of } f(1, z_2) = 1$.

We assume along the curve $f(z_1, z_2) = 1$, $\max_p [p z_1^{u(p)} z_2^{v(p)}]$ is achieved by $p = p_0$ that is independent of z_1 .



There Can Be Even More Phases ...

We assumed that along the curve $f(z_1, z_2) = 1$,

$$\max_p [p z_1^{u(p)} z_2^{v(p)}]$$

is achieved by $p = p_0$ that is independent of z_1 .

What happens when this is not the case?

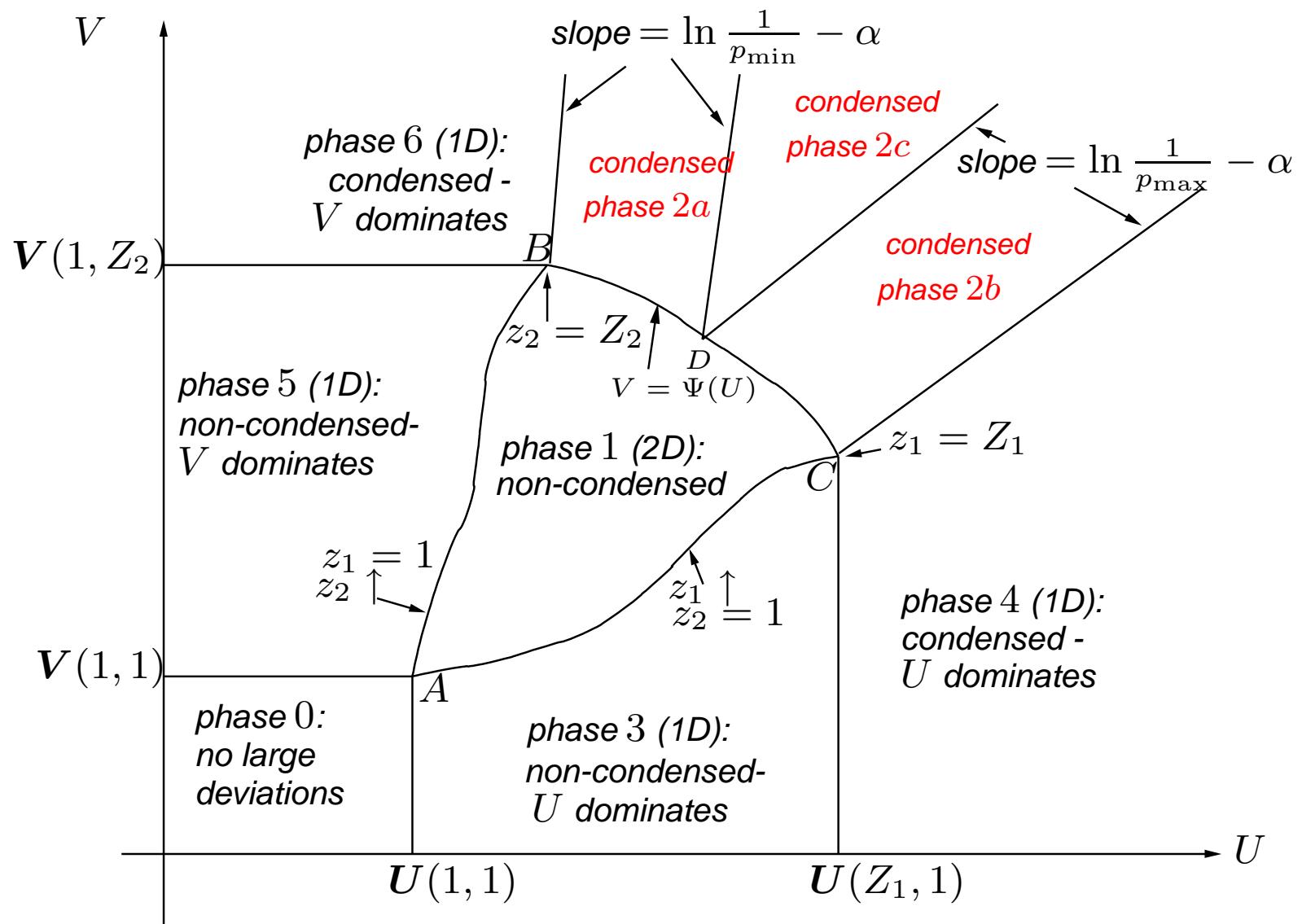
For example,

$$u(p) \equiv 1; \quad v(p) = -\alpha - \ln p.$$

Here,

$$\operatorname{argmax}_{f(z_1, z_2)=1} [p z_1^{u(p)} z_2^{v(p)}] = \begin{cases} p_{\min} & z_1 < e^\alpha \\ p_{\max} & z_1 > e^\alpha \end{cases}$$

Here, the 2D condensed phase, splits into three sub-phases.



There Can Also Be Much Less Phases ...

What happens if there are **equality** constraints, i.e.,

$$\Pr \left\{ \sum_i n_i u(p_i) = MU, \quad \sum_i n_i v(p_i) = MV \right\}$$

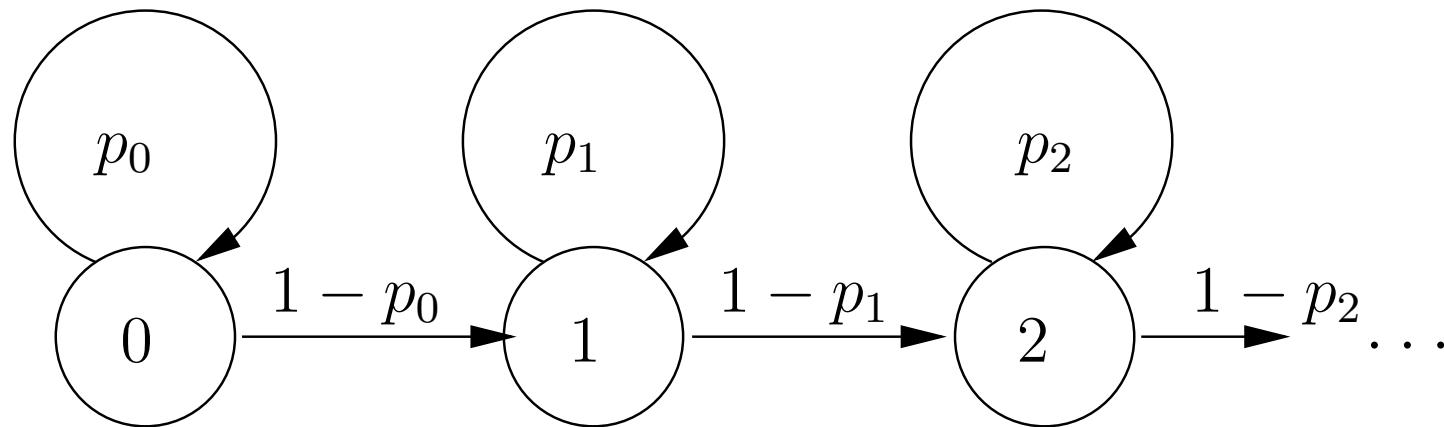
Here z_1 and z_2 are not restricted to be ≥ 1 .

→ all phase transitions associated with $z_1 = 1$ and $z_2 = 1$ disappear.

Now there are only two phases: condensed and non-condensed (always 2D), separated by the curve $V = \Psi(U)$, which is now continued up to the axes.

Applications

One-Way Markov Chains



$$P(n_0, n_1, \dots) = \prod_i [p_i^{n_i} (1 - p_i)].$$

Optimum data compression: Encode n_i using

$$\ell_i(n_i) = -\log P(n_i) = n_i \log(1/p_i) - \log(1 - p_i) \text{ bits}$$

Here $u(p) = -\log p$. The large deviations event = buffer overflow. Condensation – beyond a certain buffer size for certain densities.

Queueing Networks

A natural application is a **Jackson network** with M queues, with

$$p_i = \lambda_i / \mu_i \quad \text{utilization}$$

where λ_i = arrival rate to queue no. i , and μ_i = service rate of queue no. i .

Examples of relevant (undesirable) events:

- Excess of $\sum_i n_i$ = total number of customers.
- Excess of $\sum_i n_i / \mu_i$ = estimation of total waiting time in all queues.

Condensation: queue with the worst utilization is jammed.

Gordon–Newell network: fixed number of customers – canonical Bose–Einstein distribution. Related to **zero–range processes** (ZRP's) in stat mech with conservation of particles.

Jackson (1963) extended his results to allow state–dependent service times: seems to include results on ZRP's as special cases.

Thank You!