Threshold Effects in Parameter Estimation as Phase Transitions in Statistical Physics

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ITW 2011, Paraty, Brazil, October 2011.
The Threshold Effect

Consider the model

\[ r(t) = s_m(t) + n(t), \quad -T/2 \leq t < T/2, \]

where:

\( s_m(t) = \) a waveform parameterized by \( m; \)
\( n(t) = \) AWGN with spectral density \( N_0/2. \)

Conveying information via a parameter \( m \) by modulating it in \( s_m(t): \)
Shannon–Kotel’nikov mappings (Floor ‘08, Floor & Ramstad ‘09, Hekland ‘07, Ramstad ‘02 + references).

Nonlinear modulation \( \Rightarrow \) threshold effect:
Below some critical SNR, anomalous errors dominate the MSE.
The Threshold Effect (Cont’d)

- Not an artifact of a particular modulation/estimation scheme: it cannot be avoided.

- In the wideband regime, the threshold effect becomes **abrupt**:
  \( \Pr\{\text{anomaly}\} \) jumps from \( \sim 0 \) to \( \sim 1 \).

In this talk, we relate the **abrupt threshold effect** to **phase transitions** in statistical physics.

**Motivation**: the physical perspective gains our insight on the problem.
In the simple case of a linear model

\[ r(t) = m \cdot s(t) + n(t), \quad -T/2 \leq t < T/2 \]

the ML estimator always achieves

\[ \text{MSE} = \text{CRLB} = \frac{N_0}{2E}, \]

where \( E \) is the energy of \( \{s(t)\} \): ⇔ No threshold effect.

The only way to improve: non–linear modulation of \( s_m(t) \).
Now MSE depends not only on $E$: Let

$$s_m(t) \approx s_{m_0}(t) + (m - m_0) \cdot \dot{s}_{m_0}(t).$$

like the linear case with $\dot{s}_{m_0}(t)$ in the role of $s(t)$. Thus, at high SNR,

$$\text{MSE} \approx \text{CRLB} \approx \frac{N_0}{2\dot{E}},$$

where $\dot{E} = \text{energy of } \dot{s}_{m_0}(t)$, which depends on more details.

For example, if $s_m(t) = s(t - m)$, $\dot{E} = W^2 E$, where

$$W = \sqrt{\frac{1}{E} \int_{-\infty}^{\infty} df \cdot (2\pi f)^2 S(f)} \quad \text{Gabor bandwidth}$$

Why not increase $W$ without a limit?
Let \( \bar{s}(m) = (s_1(m), \ldots, s_K(m)) \) = representation of \( s_m(t) \) by \( K \) orthonormal basis functions and consider the locus of \( \{ \bar{s}(m), -M \leq m \leq M \} \) in \( \mathbb{R}^K \).

Assuming that \( E \) is independent of \( m \), the locus lies on the hypersurface of the \( K \)–dimensional sphere of radius \( \sqrt{E} \).

The length of the curve

\[
L = \int_{-M}^{M} dm \sqrt{\sum_i \dot{s}_i^2(m)} = 2M \sqrt{\dot{E}}.
\]

High–SNR MSE ↓ with \( \dot{E} \), we want \( \dot{E} \) ↑, thus \( L \) ↑.
\[ m = M \]

\[ m = -M \]
\[ m = M \]
\[ m = -M \]
$m = M$

$m = -M$
Anomalous Errors (Cont’d)

$L$ – limited by the need of safe distances between folds – hot dog packing. Maximum achievable $L \sim e^{CT}$, $C = P/N_0$ (PPM).

For PPM, $K \sim 2WT$, 

$$\text{MSE} \approx \frac{N_0}{2W^2E} + 4M^2 \cdot 2WT \cdot e^{-E/(2N_0)}$$

small error \hspace{1cm} \text{anomalous error}

For fixed $W$, anomalous error term ↑ gracefully as $E/N_0 \downarrow$.

For a better balance between terms – let $W \sim e^{RT}$.

$$\text{MSE} \approx \frac{N_0}{2E}e^{-2RT} + 4M^2 \cdot e^{-TE(R)} \quad R < C$$

where $E(R) =$ reliability function of AWGN channel.

For $W \sim e^{RT}$, anomalous error ↑ abruptly as $E/N_0 \downarrow$, like a phase transition.

Purpose of this work: study the abrupt threshold effect from the viewpoint of the physics of phase transitions.
Consider the PPM model

\[ r(t) = s(t - mT) + n(t), \quad |t| \leq T/2, \quad |m| \leq M, \quad M < \frac{1}{2}. \]

Imagine that \( m \sim U[-M/2, +M/2] \), then

\[
P(m|R) = \frac{P(m)P(R|m)}{\int dm' P(m')P(R|m')} \exp \left\{ \frac{2}{N_0} \int_{-T/2}^{T/2} r(t)s(t - mT)dt \right\}
\]

\[
= \frac{\int_{-M}^{M} dm' \exp \left\{ \frac{2}{N_0} \int_{-T/2}^{T/2} r(t)s(t - m'T)dt \right\}}{\int_{-M}^{M} dm' \exp \left\{ \frac{2}{N_0} \int_{-T/2}^{T/2} r(t)s(t - m'T)dt \right\}}
\]

where \( R = \{r(t), \quad |t| \leq T/2\} \). This can be viewed as the Boltzmann distribution with inverse temperature \( \beta = 2/N_0 \) and Hamiltonian (energy function)

\[
\mathcal{E}(m) = -\int_{-T/2}^{T/2} r(t)s(t - mT)dt.
\]
A Physical Perspective ... (Cont’d)

Borrowing from the concept of finite-temperature decoding [Ruján ’93], define

\[ P_\beta(m|R) = \frac{\exp \left\{ \beta \int_{-T/2}^{T/2} r(t)s(t-mT)dt \right\} }{\int_{-M}^{M} dm' \exp \left\{ \beta \int_{-T/2}^{T/2} r(t)s(t-m'T)dt \right\} }. \]

**Motivation**: a degree of freedom in case of uncertainty; simulated annealing, analysis of ML estimation.

Meaningful choices of \( \beta \):
\( \beta = 0 \) – prior; \( \beta = 2/N_0 \) – natural; \( \beta \to \infty \) – ML estimator dominates.

Define a partition function:

\[ \zeta(\beta) = \int_{-M}^{M} dm \exp \left\{ \beta \int_{-T/2}^{T/2} r(t)s(t-mT)dt \right\}. \]
Assume $s(t - mT)$ has duration $\Delta$ and divide $[-M, M]$ to $K = 2MT/\Delta$ subintervals $\mathcal{M}_i$.

**ML estimation:** find $\epsilon_i = \max_{m \in \mathcal{M}_i} \int_{-T/2}^{T/2} r(t)s(t - mT)dt$, then $\max_i \epsilon_i$.

Define another partition function

$$Z(\beta) = \sum_i e^{\beta \epsilon_i}.$$ 

The RV’s $\{\epsilon_i\}$ are alternately independent, with a density known for some waveforms, e.g., rectangular pulses (Slepian ‘62, Shepp ‘66, Zakai & Ziv ‘69).

For $T$ large, the tail is $\approx$ Gaussian.

The contribution $e^{\beta \epsilon_0}$ of the subinterval that includes the signal should be handled separately.
The analysis of $Z(\beta)$ – very similar to that of the random energy model (REM) in statistical physics (Derrida ‘80,’81):

$$Z(\beta) = \int d\epsilon \cdot N(\epsilon) \cdot e^{\beta \epsilon},$$

where typically

$$N(\epsilon)d\epsilon \sim \begin{cases} 0 & f(\epsilon)d\epsilon \ll 1/K \\ K \cdot f(\epsilon)d\epsilon & \text{elsewhere} \end{cases}$$

For $W \sim e^{RT}$, we select $\Delta \sim e^{-RT} \implies K \sim e^{RT}$.

Accordingly, from now on, we denote the partition function by $Z(\beta, R)$ and define

$$\psi(\beta, R) = \lim_{T \to \infty} \frac{\ln Z(\beta, R)}{T}.$$
\[ R = C \]

\[ \psi(\beta, R) = \beta \sqrt{N_0 P R} \]

\[ \beta = \beta_c(R) = \frac{2}{N_0} \sqrt{\frac{R}{C}} \]

\[ \psi(\beta, R) = R + \frac{\beta^2 N_0 P}{4} \]

\[ R = P(\beta - \frac{\beta^2 N_0}{4}) \]

\[ C = \frac{P}{N_0} \]
Some Extensions

- **Mismatched estimation**: suppose that $r(t)$ is correlated with $\tilde{s}(t - mT)$ instead of $s(t - mT)$: Same phase diagram, except that $C$ is degraded by a factor of $\rho^2$ and $\beta$ by a factor of $\rho$, where $\rho$ is the correlation between $s(\cdot)$ and $\tilde{s}(\cdot)$.

- **Other pulse shapes**: essentially the same results.
Joint ML Estimation of Amplitude and Delay

Consider now the model

\[ r(t) = \alpha \cdot s(t - mT) + n(t), \]

where now both \( m \) and \( \alpha \) have to be estimated. It is assumed that \( \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}] \).

- The parameter \( \alpha \) alone does not contribute any phase transitions.
- The parameter \( m \) alone generates three phases.

Q: How many phases would there be in the joint estimation of \( m \) and \( \alpha \)?
\[
\psi(\beta, R) = \frac{\beta P}{2}
\]

ordered

\[
R = \frac{P}{2} \left[ \beta(1 + \alpha^2_{\text{min}}) - \beta^2 N_0 \alpha^2_{\text{min}} / 2 \right]
\]

\[
\psi(\beta, R) = R + \frac{\beta \alpha^2_{\text{max}} P}{4} (\beta N_0 - 2)
\]

paramagnetic

\[
\psi(\beta, R) = \beta (\alpha_{\text{max}} \sqrt{N_0 PR} - \alpha^2_{\text{max}} P/2)
\]

glassy

\[
R = C = \alpha^2_{\text{max}} C
\]

\[
\psi(\beta, R) = R + \frac{\beta \alpha^2_{\text{min}} P}{4} (\beta N_0 - 2)
\]

paramagnetic

\[
\beta = \frac{\beta_c(R)}{\alpha_{\text{max}}}
\]

\[
\beta = \frac{2}{N_0}
\]
Discussion and Conclusion

- The behavior is much more complicated than when only $m$ should be estimated.
- One ordered phase (non–anomalous errors) and four anomalous phases.
- Although $\alpha$ alone does not generate phase transitions, its interaction with $m$ generates more phases than those of $m$ alone.
- Anomalies in $\alpha$ have a different behavior.

The physical point of view helps to gain insight on the behavior.