Parameter Estimation Based on Noisy Chaotic Signals in the Weak-Noise Regime

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Twisted Modulation

Consider the AWGN channel,

$$y_i = x_i + z_i, \qquad i = 1, 2, \ldots, n,$$

where $z_i \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. and

$$\boldsymbol{x} = (x_1, \ldots, x_n) = f_n(\theta), \qquad \|\boldsymbol{x}\|^2 \leq nQ$$

$\theta \in [0, 1]$ being a parameter to be estimated at the receiver by $\hat{\theta} = g_n(\boldsymbol{y})$.

How well can we estimate θ if we have the freedom to choose both the modulator $f_n(\cdot)$ and the estimator, $g_n(\cdot)$?

Twisted Modulation (Cont'd)

- The "waveform communication" problem (Wozencraft & Jacobs, '65).
- Source/channel coding: Shannon-Kotel'nikov ('49; '59).
- Hekland ('07); Floor ('08+); Hekland, Floor & Ramstad ('09, '23).
- Estimation theory; Cohn, ('70), Burnashev ('84, '85).
- Linear modulation – Fisher efficient, but limited.
- Nonlinear modulation – flexible, but suffers a threshold effect.
- Most of the literature: total MSE.
- Reasonable to separate Pr{anomaly} and weak-noise errors.
- Köken, Gündüz & Tuncel ('17): $\min MSE$ s.t. Pr{anomaly} $\leq \epsilon$.
- Merhav ('19): exponential Pr{anomaly} + matching converse.
- Merhav ('20): extension to parameter vectors.
- This work: modulators based on chaotic dynamical systems.

Motivations for Studying Chaotic Modulators

- Sensitivity to initial conditions – good weak-noise estimation.
- High degree of flexibility in the design.
- $\exists$ mature theoretical understanding about chaos.
- Computationally easy to generate the modulated signal.
- $\exists$ computationally efficient estimation algorithms (halving method).
- Good estimation of initial condition is also good for filtering.

Related Work

Modulators based on chaotic systems have been investigated extensively during the last 3 decades from a variety of aspects:

- Upper/lower bounds on MSE.
- Numerical aspects.
- Algorithmic efficiency.
- System optimization.
- Applications in Turbo coding, hybrid coding, spread spectrum, MIMO, etc.

Chen (‘96); Chen & Wornell (‘98); Cong *et al.* (‘99); Drake (‘98); Eckmann & Ruelle (‘85); Hen & Merhav (‘04); Kay & Nagesha (‘95); Kennedy & Kolumbán (‘00); Leung *et al.* (‘06); Pantaleón *et al.* (‘03); Papadopoulos & Wornell (‘95); Wallinger (‘13); Wang *et al.* (‘99); Xie *et al.* (‘09); Yu *et al.* (‘18), ...

Objectives

The purpose this work is to carry out a systematic study of modulators that are based on certain class of chaotic systems, from the perspecive of earlier work on fundamental limits of general modulators:

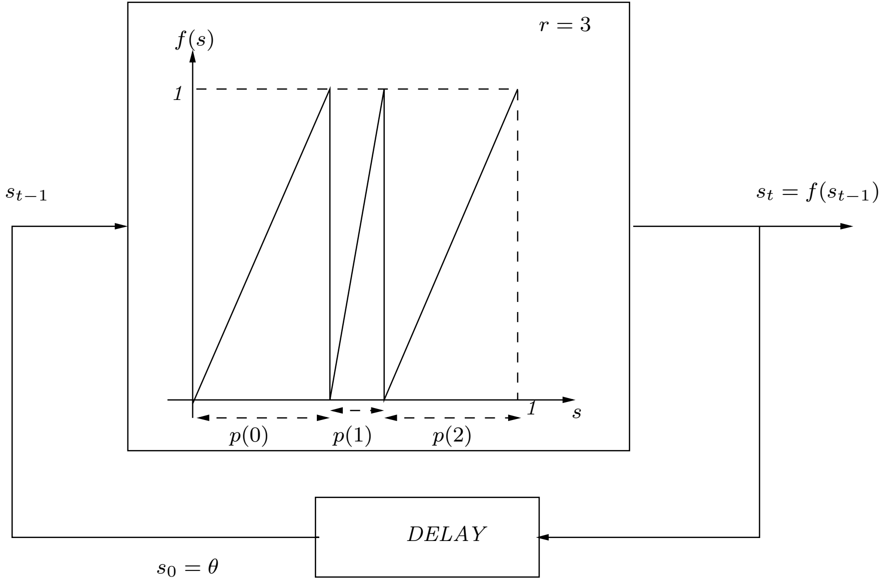
Given a certain parametric family of modulators, find the one with the best weak-noise error performance s. t. $\Pr\{\text{anomaly}\} \to 0$.

We consider a general error performance criterion,

$$\sup_{\theta \in [0,1]} \mathbf{E} \left\{ \rho(\hat{\theta} - \theta) \middle| \text{no anomaly} \right\} \qquad \rho(\cdot) \text{ convex}$$

and avoid the use of the Cramér-Rao lower bound, which is problematic for systems with discontinuous mappings.

The Chaotic Dynamical System



Formulation

Select a positive integer r and a probability vector, $P = \{p(0), p(1), \ldots, p(r-1)\}$. Define:

$$F(x) = \sum_{x'=0}^{x-1} p(x'); \qquad F(0) = 0,\ F(r) = 1.$$

Given $s \in [0,1]$, let $\phi(s)$ be the value of $x \in \{0, 1, \ldots, r-1\}$ such that

$$F(x) \leq s < F(x+1), \qquad \phi(1) = 1.$$

The non-linear dynamical system is defined by the recursion:

$$\begin{aligned} x_t &= \phi(s_{t-1}), \qquad s_0 = \theta \qquad \text{itinerary sequence} \\ s_t &= \frac{s_{t-1} - F(x_t)}{p(x_t)} \qquad \text{state sequence} \end{aligned}$$

for $t = 1, 2, \ldots$. The channel input is

$$u_t = \sqrt{12Q}\left(s_t - \frac{1}{2}\right).$$

System Properties

1. Reconstruction of s_0 from $x_1, x_2, \ldots$:

$$s_0 = \sum_{t=1}^{\infty} F(x_t) \prod_{i=1}^{t-1} p(x_i) \stackrel{\triangle}{=} \sum_{t=1}^{\infty} G(x_t) \prod_{i=1}^{t} p(x_i)$$

Example: if $P = (1/r, \ldots, 1/r)$,

$$s_0 = \sum_{t=1}^{\infty} x_t r^{-t} = 0.x_1 x_2 \ldots .$$

2. X_t as a random process (with application to process simulation): If $S_0 \sim$ Unif$[0,1]$, then $S_t \sim$ Unif$[0,1]$ and $\{X_t\}$ is a DMS governed by P. (Easy extension to arbitrary processes with memory).

3. Lyapunov exponent:

$$\lambda \stackrel{\triangle}{=} \lim_{n \to \infty} \frac{1}{n} \mathbf{E} \left\{ \ln \left| \frac{\partial s_n}{\partial s_0} \right| \right\} = H \quad \text{the entropy of } P.$$

System Properties (Cont'd)

4. Length of signal locus:

$$L_n = \int_0^1 \left\| \frac{\partial}{\partial \theta} \boldsymbol{u}(\theta) \right\| \cdot \mathsf{d}\theta \doteq r^n.$$

Therefore $\ln r$ must be $< C = \frac{1}{2} \ln(1+\gamma)$, $\gamma = \frac{Q}{\sigma^2}$ to keep Pr{anomaly} small.

5. Autocorrelation:

$$R_S(k) = \mathbf{E}\{S_0 S_k\} = \frac{1}{4} + \frac{1}{12} \cdot \left(\sum_{x=0}^{r-1} p^2(x) \right)^{|k|}.$$

$$R_U(k) = \mathbf{E}\{U_0 U_k\} = Q \cdot \left(\sum_{x=0}^{r-1} p^2(x) \right)^{|k|}.$$

System Properties (Cont'd)

6. Channel input-output mutual information:

$$C_0 = \lim_{n\to\infty} \frac{I(U^n; Y^n)}{n} \leq \frac{1}{2} \ln \frac{A}{\sigma^2} \triangleq C_1$$

where

$$\frac{A}{\sigma^2} = \frac{1}{2}\left[1 + \gamma + q^2(1-\gamma) + \sqrt{(1+\gamma^2)(1-q^2)^2 + 2\gamma(1-q^4)}\right]$$

and

$$q = \sum_{x=0}^{r-1} p^2(x).$$

7. Ergodic property:

The Lebesgue measure of the set:

$$\left\{ s_0 : \ \left| \frac{1}{n} \sum_{t=1}^{n} u_t u_{t+k} - R_U(k) \right| \leq \epsilon \right\}$$

tends to unity as $n \to \infty$.

General Lower Bound

Suppose that $\rho(\cdot)$ has the following property: $\forall\ c > 0$, $\rho(e^{-nc}) \doteq e^{-n\zeta(c)}$ with $\zeta(c) > 0$.

For example, if $\rho(\epsilon) = |\epsilon|^a$, then $\zeta(c) = a \cdot c$.

Theorem [Merhav 2019]: For any modulator and estimator,

$$\sup_{0\le\theta\le 1} \mathbf{E}\left\{\rho(\hat{\theta}-\theta)\middle|\text{no anomaly}\right\} \stackrel{\cdot}{\ge} \exp\left\{-n\zeta\left(\frac{1}{2}\ln\gamma\right)\right\}$$

for $\gamma \gg 1$.

Asymptotically achievable by uniform quantization of θ followed by capacity-achieving channel coding.

Lower Bound for the Class of Chaotic Modulators

Theorem: For any chaotic modulator from the class defined and any estimator,

$$\sup_{0\le\theta\le1} \mathbf{E}\left\{\rho(\hat{\theta}-\theta)\middle|\text{no anomaly}\right\}$$

$$\dot{\ge} \quad \exp\left(-n\zeta\left[\min\left\{C_1, \frac{1}{2}\ln\gamma - \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right)\right\}\right]\right)$$

for $\gamma \gg 1$.

The term $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right)$ = shaping loss since $\{u_t\}$ is distributed uniformly rather than normally.

$C_1 < \frac{1}{2}\ln\gamma$ = loss associated with the fact that $\{u_t\}$ has memory.

We could have increased C_1 up to C by decreasing $q = \sum_x p^2(x) \ge \frac{1}{r}$, but recall that r is limited by $\ln r \le C$.

Feeding the Channel by the Itinerary Signal?

At first glance it seems counterintuitive that the itinerary sequence, $\{x_t\}$ could do a better job than the state sequence, $\{s_t\}$ (or $\{u_t\}$) since $\{x_t\}$ is a quantized version of $\{s_t\}$:

$$x_t = \phi(s_{t-1}), \qquad F(x_t) \leq s_{t-1} < F(x_t + 1).$$

However, recall that $\{s_t\}$ is generated from s_0 which in turn can be expressed in terms of $(x_1, x_2, \ldots)$. Thus, $(x_1, x_2, \ldots)$ and $(s_1, s_2, \ldots)$ include exactly the same information about s_0.

- For large n, $(x_1, x_2, \ldots, x_n)$ and $(s_1, s_2, \ldots, s_n)$ and include almost the same information about s_0.
- If $S_0 \sim$ Unif$[0, 1]$, $\{X_t\}$ is an i.i.d. process governed by P. No loss due to input memory.
- No limitation on r: select P to approximate the capacity-achieving input distribution, $\mathcal{N}(0, Q)$.

Proposed Modulation Scheme

- Given $\theta \in [0, 1]$, quantize it to θ_i using a fine grid of $M = e^{n(C-\epsilon)}$ points and a random mapping of the grid onto itself, $\eta_i = \psi(\theta_i)$.
- Let $s_0 = \eta_i$ be the initial state of the modulator.
- Transmit $\boldsymbol{x}$ over the channel.
- Decode $\hat{\eta}_i$.
- $\hat{\theta}_i = \psi^{-1}(\hat{\eta}_i)$.

If the decoding is correct (which is the case w.h.p.), there is only a quantization error:

$$\rho(\theta_i - \theta) \leq \rho\left(\frac{1}{2M}\right) = \rho(e^{-n(C-\epsilon)}) = e^{-n\zeta(C-\epsilon)}.$$

Decoding error = anomaly.