Channel Upgrading for Semantically-Secure Encryption on Wiretap Channels

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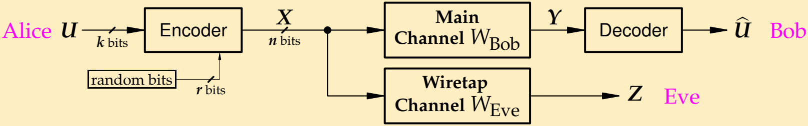
The wiretap channel

Alice, Bob, and Eve



Wiretap channel essentials

- Reliability: $\lim_{n\to\infty} \Pr\{\widehat{U} \neq U\} = 0$
- Security: $\lim_{n\to\infty} \dfrac{I(U;Z)}{n} = 0$
- Random bits: In order to achieve the above, Alice sends and Bob receives r random bits, $r/n = I(W_{\text{Eve}})$.

Semantic security

Information theoretic security, revisited

- Assumption: input $\boldsymbol{U}$ is uniform.
- Assumption: figure of merit is mutual information, $I(\boldsymbol{U};\boldsymbol{Z})/n$.

Semantic security

We achieve σ bits of semantic security if:

- For all distributions on the message set of Alice
- For all functions f of the message
- For all strategies Eve might employ
- The probability of Eve guessing the value of f correctly increases by no more than $2^{-\sigma}$ between the case in which Eve does not have access to the output of W and the case that she does.
- That is, having access to W hardly helps Eve, for sufficiently large σ.

Notation

The channel model

- Denote $W = W_{\text{Eve}}$.
- Let $W : \mathcal{X} \to \mathcal{Y}$ be a memoryless channel.
- Finite input alphabet $\mathcal{X}$
- Finite output alphabet $\mathcal{Y}$
- The channel W is symmetric:
  - The output alphabet $\mathcal{Y}$ can be partitioned into $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_T$.
  - Let $A_t = [W(y|x)]_{x \in \mathcal{X}, y \in \mathcal{Y}_t}$.
  - Each row (column) of A_t is a permutation of the first row (column).

The BT scheme

The function Ψ

$$\Psi(W) \stackrel{\text{def}}{=} \log_2 |\mathcal{Y}| + \sum_{y \in \mathcal{Y}} W(y|0) \log_2 W(y|0) \,,$$
$$= \log_2 |\mathcal{Y}| - H(\boldsymbol{Y}|\boldsymbol{X}) \,.$$

Theorem (The BT scheme)

Let $W : \mathcal{X} \to \mathcal{Y}$ be the SDMC from Alice to Eve. Then, the BT scheme achieves at least σ bits of semantic security with a codeword length of n and r random bits, provided that

$$r = 2(\sigma + 1) + \sqrt{n} \log_2(|\mathcal{Y}| + 3)\sqrt{2(\sigma + 3)} + n \cdot \Psi(W) \,.$$

M. Bellar, S. Tessaro, Polynomial-Time, Semantically-Secure Encryption Achieving the Secrecy Capacity, arXiv:1201.3160

The function Ψ

Asymptotics

$$r = 2(\sigma+1) + \sqrt{n}\log_2(|\mathcal{Y}|+3)\sqrt{2(\sigma+3)} + n \cdot \Psi(W) \,.$$

Thus, the asymptotic number of random bits we need to transmit is

$$\lim_{n\to\infty} r/n = \Psi(W) \,.$$

Ψ versus I

$$\begin{aligned}\Psi(W) &\stackrel{\text{def}}{=} \log_2|\mathcal{Y}| + \sum_{y\in\mathcal{Y}} W(y|0)\log_2 W(y|0) \,, \\ &= \log_2|\mathcal{Y}| - H(\boldsymbol{Y}|\boldsymbol{X}) \geq H(\boldsymbol{Y}) - H(\boldsymbol{Y}|\boldsymbol{X}) = I(W)\end{aligned}$$

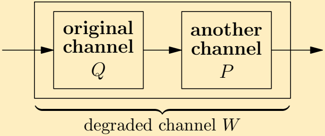
How can we “make” $\Psi(W)$ close to $I(W)$?

Equivalent channels

Degraded channel

A DMC $W : \mathcal{X} \to \mathcal{Y}$ is (stochastically) degraded with respect to a DMC $Q : \mathcal{X} \to \mathcal{Z}$, denoted $W \preceq Q$, if there exists an intermediate channel $P : \mathcal{Z} \to \mathcal{Y}$ such that

$$W(y|x) = \sum_{z \in \mathcal{Z}} Q(z|x) \cdot P(y|z) \,.$$



Equivalent channel

If $W \preceq Q$ and $Q \preceq W$, then W and Q are equivalent, $W \equiv Q$.

Letter Splitting

Splitting function

- Let an SDMC $W : \mathcal{X} \to \mathcal{Y}$ be given.
- Denote the corresponding partition as $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_T$.
- A function $s : \mathcal{Y} \to \mathbb{N}$ is an output letter split of W if
  - $s(y) = s(y')$ for all $1 \leq t \leq T$ and all $y, y' \in \mathcal{Y}_t$.
  - By abuse of notation, define $s(\mathcal{Y}_t)$.

Resulting channel

Applying s to W gives $Q : \mathcal{X} \to \mathcal{Z}$

- Output alphabet: $\mathcal{Z} = \bigcup_{y \in \mathcal{Y}} \{y_1, y_2, \ldots, y_s \mid s = s(y)\}$.
- Transition probabilities: $Q(y_i|x) = W(y|x)/s(y)$
- Namely, each letter y is duplicated $s(y)$ times. The conditional probability of receiving each copy is simply $1/s(y)$ times the original probability in W.

Letter splitting

Properties of Q

- Since W is symmetric, so is Q.
- $W \equiv Q$.

Lemma

For a positive integer $M \geq 1$, define

$$s(y) = \lceil M \cdot W(y) \rceil \ , \quad \textit{where} \quad W(y) = \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} W(y|x) \ .$$

Let $Q : \mathcal{X} \to \mathcal{Z}$ be the resutling channel. Then,

$$\Psi(Q) - I(W) = \Psi(Q) - I(Q) \leq \log_2 \left(1 + \frac{|\mathcal{Y}|}{M} \right) \ ,$$

and $|\mathcal{Z}| \leq M + |\mathcal{Y}|$.

Letter splitting

Theorem

The number of random bits needed to achieve semantic security is at most

$$r = 2(\sigma+1) + \sqrt{n}\log_2(M + |\mathcal{Y}| + 3)\sqrt{2(\sigma+3)} + n \cdot \left(I(W) + \log_2\left(1 + \frac{|\mathcal{Y}|}{M}\right)\right) .$$

Consequnces

- Setting, say, $M = n$ and taking $n \to \infty$ gives us

$$\lim_{n\to\infty} \frac{r}{n} = I(W) .$$

- What about the finite M and n case?

Greedy algorithm

Algorithm A: Greedy algorithm to find optimal splitting function

input : Channel $W : \mathcal{X} \to \mathcal{Y}$, a partition $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_T$ where each subset is of size μ, a positive integer M which is a multiple of μ

output: A letter-splitting function s such that $\sum_{y \in \mathcal{Y}} s(y) = M$ and $\Psi(Q)$ is minimal

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// Initialization
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$s(\mathcal{Y}_1) = s(\mathcal{Y}_2) = \cdots = s(\mathcal{Y}_T) = 1$;

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// Main loop
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for $i = 1, 2, \ldots, \frac{M}{\mu} - T$ **do**

$\quad t = \arg\max_{1 \le t \le T} \sum_{y \in \mathcal{Y}_t} W(y) \log_2 \left(\frac{s(\mathcal{Y}_t)+1}{s(\mathcal{Y}_t)} \right)$;

$\quad s(\mathcal{Y}_t) = s(\mathcal{Y}_t) + 1$;

return s;

Greedy algorithm

Theorem

Given a valid input to Algorithm A, the output is a valid letter-splitting function s, such that $\sum_{y \in \mathcal{Y}} s(y) = M$ and the resulting channel Q is such that $\Psi(Q)$ is minimized.

Proof

- Prooving $\sum_{y \in \mathcal{Y}} s(y) = M$:
  - After the initialization step, $\sum_{y \in \mathcal{Y}} s(y) = \mu \cdot T$.
  - Each iteration increments the sum by μ
  - So, in the end, $\sum_{y \in \mathcal{Y}} s(y) = M$.
- Prooving optimality:
  - Since $Q \equiv W$, we have $I(Q) = I(W)$.
  - Minimizing $\Psi(Q)$ is equivalent to maximizing

$$I(Q) - \Psi(Q) = \sum_{y \in \Upsilon} -W(y) \log_2 \left(\frac{W(y)}{s(y)} \right) - \log_2 M \, .$$

Greedy algorithm

Proof, continued

- Clearing away constant terms, maximize

$$\sum_{y \in \mathcal{Y}} W(y) \log_2 s(y) \ .$$

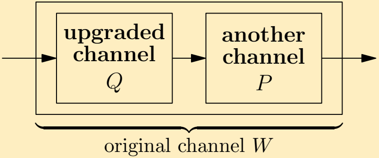
- We now recast the optimization problem. Define the set

$$A = \bigcup_{y \in \mathcal{Y}} \bigcup_{i=1}^{M/\mu - T} \left\{ \delta(y,i) = W(y) \log_2 \left(\frac{i+1}{i} \right) \right\} \ .$$

- Finding the optimal $s(y)$ is equivalent to choosing $M/\mu - T$ numbers from the set A such that
  - Their sum is maximal, and
  - if $\delta(y,i)$ was picked and $i > 1$, then $\delta(y,i-1)$ must be picked as well.
- The last constraint is redundant. The proof follows.

Infinite output alphabet

- What would we do if the output alphabet of W is infinite?
- To begin with, in this case, Ψ is not even defined.
- Solution: Repalce W by a channel Q which is upgraded and has a finite output alphabet.
- A channel Q is upgraded with respect to W if $W \preceq Q$.



- A method to upgrade W to Q was previously presented by the authors in “How to Construct Polar Codes”.
- The method we now show is better, with respect to Ψ.

Notation

Assumptions

- Assume the input alphabet is binary, and denote $\mathcal{X} = \{1, -1\}$.
- Let the output alphabet be the reals, $\mathcal{Y} = \mathbb{R}$.
- Symmetry: $f(y|1) = f(-y|-1)$.
- Positive value more likely when $x = 1$

$$f(y|1) \geq f(y|-1)\ , \quad y \geq 0\ .$$

- Liklihood increasing in y:

$$\frac{f(y_1|1)}{f(y_1|-1)} \leq \frac{f(y_2|1)}{f(y_2|-1)}\ , \quad -\infty < y_1 < y_2 < \infty\ .$$

The channel Q

Paritioning $\mathbb{R}$

- Let the channel W and a positive integer M be given.
- Initialization: Define $y_0 = 0$.
- Recursively define, for $1 \le i < M$ the number y_i as such that

$$\int_{-y_i}^{-y_{i-1}} f(y|1)\,dy + \int_{y_{i-1}}^{y_i} f(y|1)\,dy = \frac{1}{M}\,.$$

- Lastly, “define” $y_M = \infty$.
- For $1 \le i \le M$, the regions

$$A_i = \{y \;:\; -y_i < y \le -y_{i-1}\} \cup \{y \;:\; y_{i-1} \le y < y_i\}$$

  form a partition of $\mathbb{R}$, which is equiprobable with respect to $f(\cdot|1)$ and $f(\cdot|-1)$

$$f(A_i|1) = f(A_i|-1) = 1/M\,.$$

The channel Q

The likelihood ratios λ_i

- Recall the partition

$$A_i = \{y \, : \, -y_i < y \leq -y_{i-1}\} \cup \{y \, : \, y_{i-1} \leq y < y_i\} \; ,$$

which is equiprobable

$$f(A_i|1) = f(A_i|-1) = 1/M \; .$$

- Define the likelihood ratios

$$\lambda_i = \frac{f(y_i|1)}{f(y_i|-1)} \; .$$

- By our previous assumptions,

$$1 \leq \lambda_{i-1} = \inf_{y \in B_i} \frac{f(y|1)}{f(y|-1)} \leq \sup_{y \in B_i} \frac{f(y|1)}{f(y|-1)} \leq \lambda_i \; .$$

The channel Q

- The channel $Q : \mathcal{X} \to \mathcal{Z}$ is defined as follows.
- Input alphabet: $\mathcal{X} = \{-1, 1\}$.
- Output alphabet: $\mathcal{Z} = \{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_M, \bar{z}_M\}$.
- Conditional probability:

$$Q(z|1) = \begin{cases} \frac{\lambda_i}{M(\lambda_i+1)} & \text{if } z = z_i \text{ and } \lambda_i \neq \infty \,, \\ \frac{1}{M(\lambda_i+1)} & \text{if } z = \bar{z}_i \text{ and } \lambda_i \neq \infty \,, \\ \frac{1}{M} & \text{if } z = z_i \text{ and } \lambda_i = \infty \,, \\ 0 & \text{if } z = \bar{z}_i \text{ and } \lambda_i = \infty \,, \end{cases}$$

  and

$$Q(z_i| - 1) = Q(\bar{z}_i|1) \,, \quad Q(\bar{z}_i| - 1) = Q(z_i|1) \,.$$

- For $1 \leq i \leq M$, the liklihood ratio of z_i is $Q(z_i|1)/Q(z_i| - 1) = \lambda_i$.

Properties of Q

- Finite output alphabet: $|\mathcal{Z}| = 2M$.
- Optimal Ψ: $\Psi(Q) = I(Q)$, since $Q(z_i) = Q(\bar{z}_i) = \frac{1}{2M}$.
- Q is upgraded with respect to W, $W \preceq Q$.
- Key question: What is $I(Q) - I(W)$?

The channel Q'

- Define $Q' : \mathcal{X} \to \mathcal{Z}$ as a "shifted version" of Q.

$$Q'(z|1) = \begin{cases} \frac{\lambda_{i-1}}{M(\lambda_{i-1}+1)} & \text{if } z = z_i \ , \\ \frac{1}{M(\lambda_{i-1}+1)} & \text{if } z = \bar{z}_i \ , \end{cases}$$

and

$$Q'(z_i|-1) = Q'(\bar{z}_i|1) \ , \quad Q'(\bar{z}_i|-1) = Q'(z_i|1) \ .$$

- Q' is degraded with respect to W, $Q' \preceq W$.
- To sum up,

$$Q' \preceq W \preceq Q \ .$$

Theorem

Let $W : \mathcal{X} \to \mathcal{Y}$ be a continuous channel as defined above. For a given integer M, let $Q : \mathcal{X} \to \mathcal{Z}$ be the upgraded channel described previously. Then, $|\mathcal{Z}| = 2M$ and

$$\Psi(Q) - I(W) \leq \frac{1}{M} .$$

Proof.

We know that

$$\Psi(Q) = I(Q) ,$$

and that

$$I(Q') \leq I(W) \leq I(Q) .$$

Thus, it suffices to prove that

$$I(Q') - I(Q) \leq \frac{1}{M} .$$

Because Q' is a "shifted version" of Q, the above difference telescopes to $1/M$. $\square$