List Decoding of Universal Polar Codes

Boaz Shuval and Ido Tal

Overview

- Universal polarization:
  - memoryless setting: Şaşoğlu & Wang, ISIT 2011
  - settings with memory: Shuval & Tal, ISIT 2019
- Error probability analysis based on successive cancellation (SC) decoding
- Potential for better results using successive cancellation list (SCL) decoding

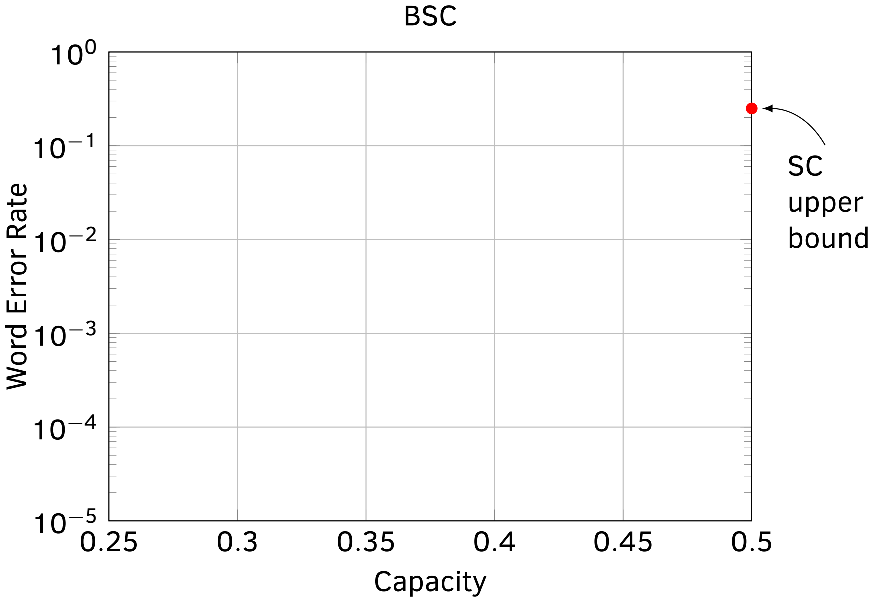
This talk:

An efficient implementation of SCL for universal polarization

Setting

- Communication with uncertainty:
  - Encoder: Knows channel belongs to a set of channels
  - Decoder: Knows channel statistics (e.g., via estimation)
- Memory:
  - In channels
  - In input distribution
- Universal polar code achieves:
  - Vanishing error probability over set
  - Best rate (infimal information rate over set)

List decoding improves probability of error!



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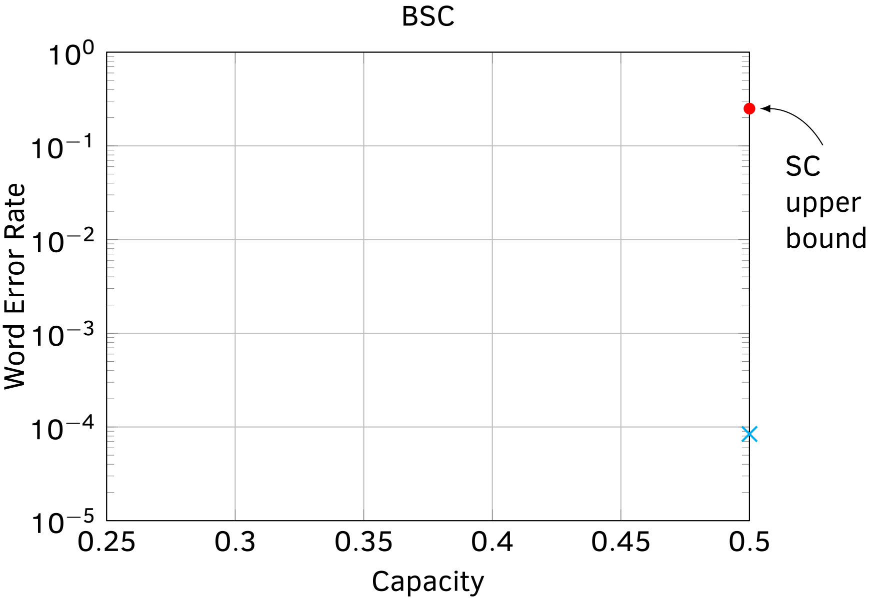

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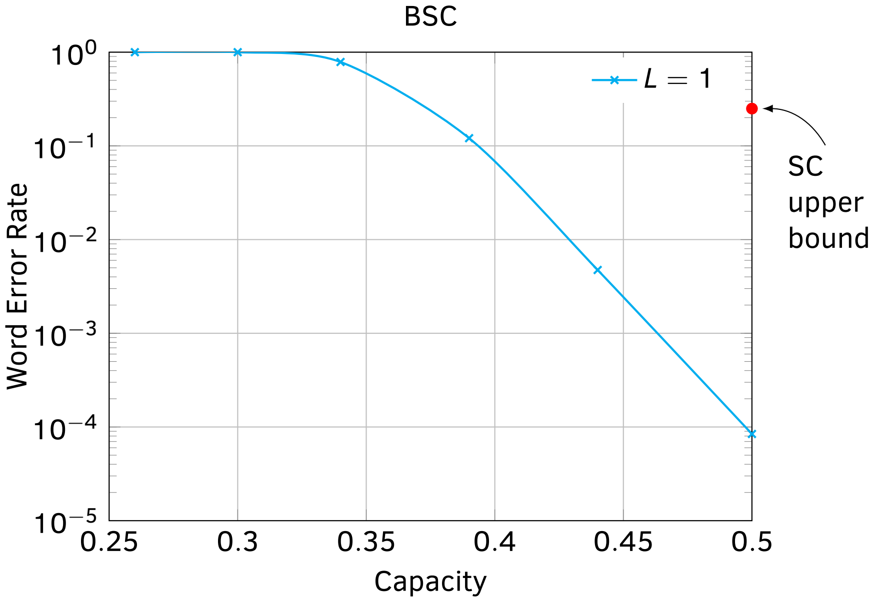

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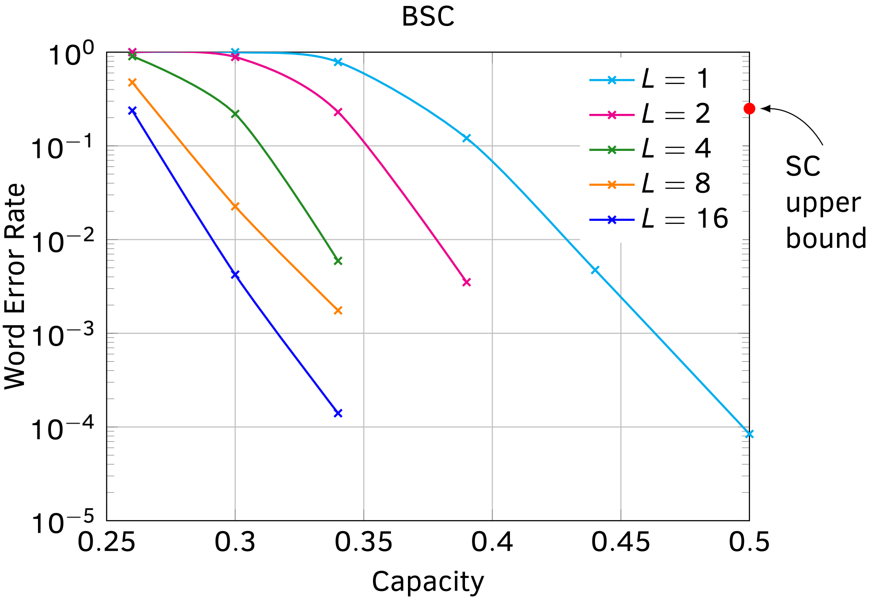

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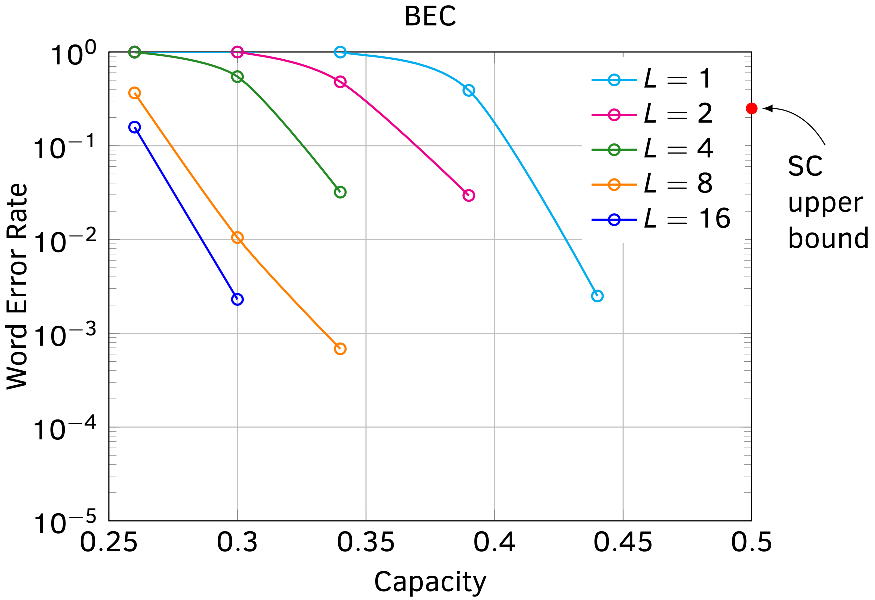

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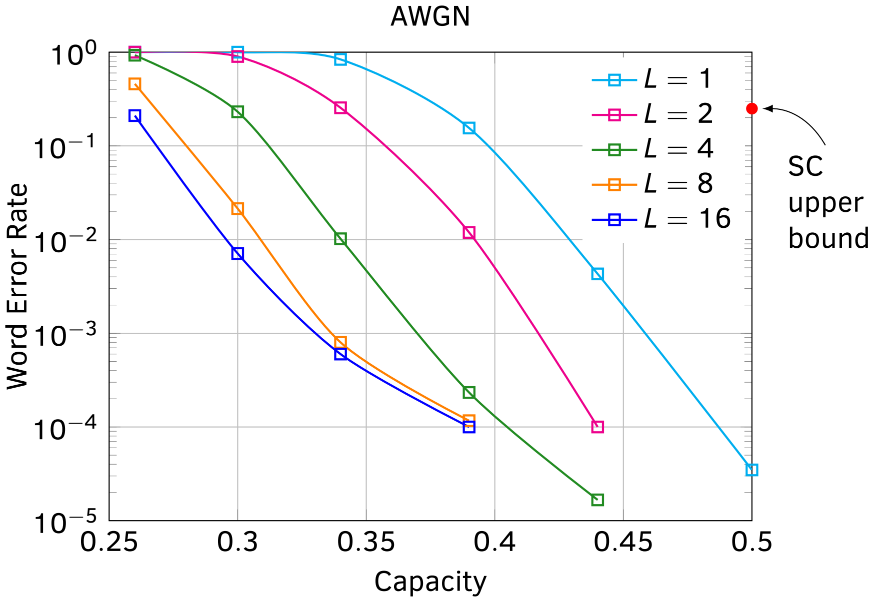


SCL in a nutshell

- Parameter: list size L
- Successively decode $u_0, u_1, \ldots, u_{N-1}$
- Decoding path: sequence of decoding decisions

$$\hat{\mathbf{u}}_{i-1} = \begin{bmatrix} \hat{u}_0 & \hat{u}_1 & \ldots & \hat{u}_{i-1} \end{bmatrix}$$

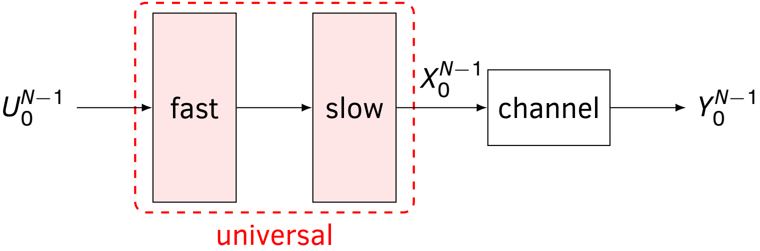
- If u_i is not frozen, split each decoding path into two paths,

$$\begin{tikzcd} & \begin{bmatrix} \hat{\mathbf{u}}_{i-1} & 0 \end{bmatrix} \\ \hat{\mathbf{u}}_{i-1} \arrow[ur] \arrow[dr] & \\ & \begin{bmatrix} \hat{\mathbf{u}}_{i-1} & 1 \end{bmatrix} \end{tikzcd}$$

- For each i, keep L best decoding paths
- Final decoding: choose best path

The Universal Construction

- Concatenation of two transform blocks



  - Slow — for universality
  - Fast (Arıkan) — for vanishing error probability

The Universal Construction

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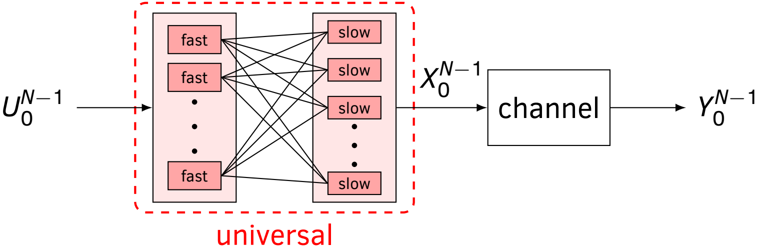
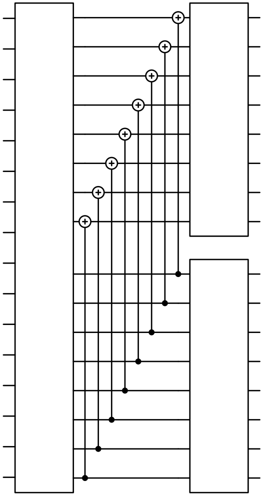


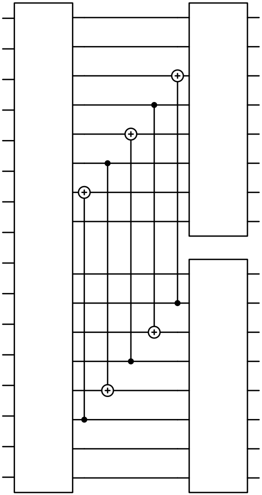
  - Slow — for universality
  - Fast (Arıkan) — for vanishing error probability
- Each block contains multiple copies of basic transforms
- Polar: certain indices of U_0^{N-1} are data bits,
  other indices "frozen" (Honda & Yamamoto 2013)
- Universal: data bit locations regardless of channel

Fast Transform vs. Slow Transform

Fast (Arıkan) transform



Slow transform



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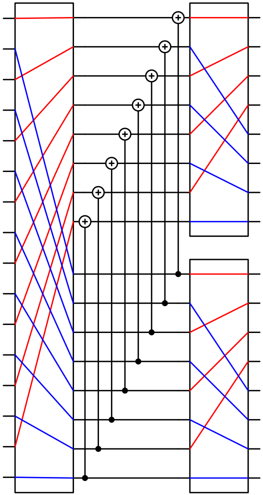

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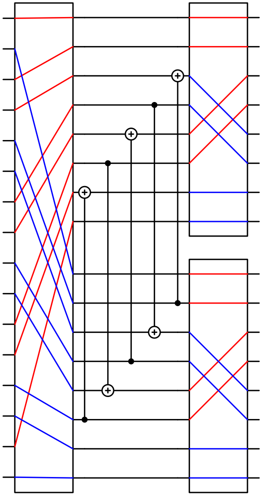

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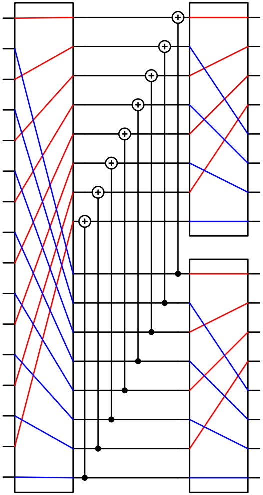

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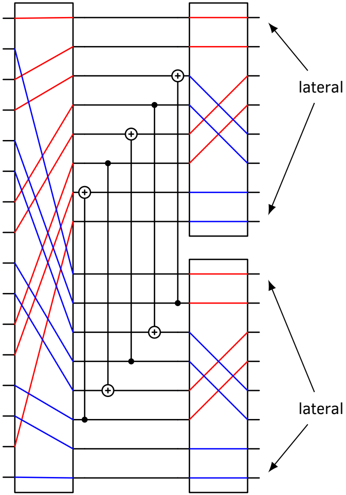

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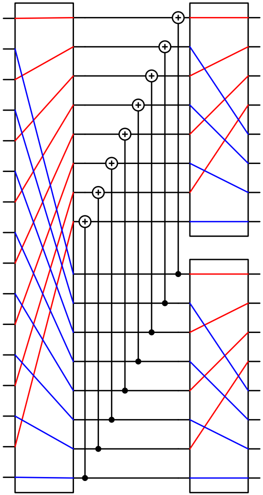

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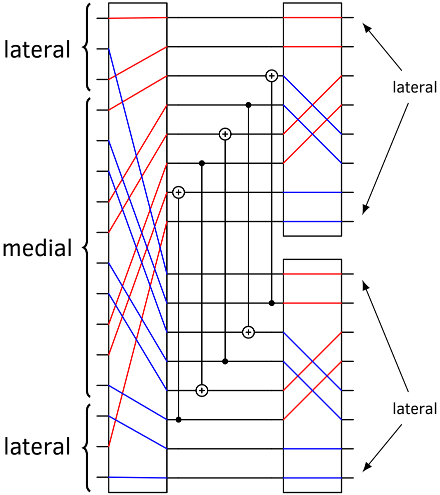

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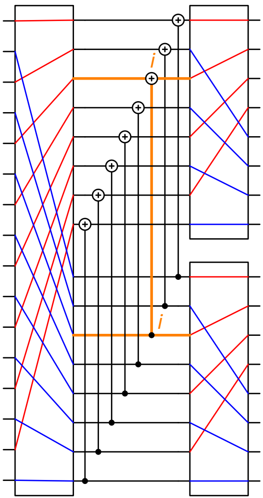

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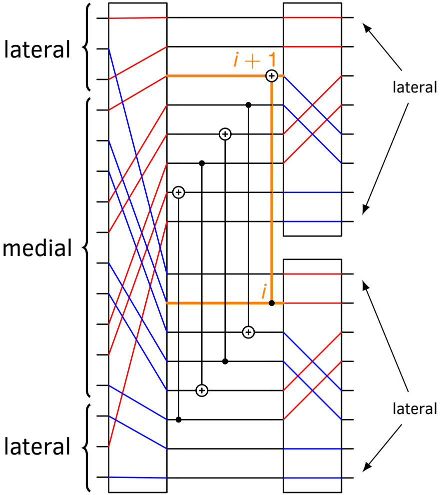
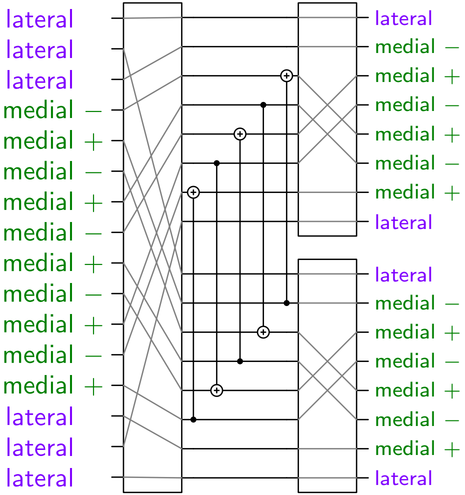


The Rules of Slow

- Any medial on left is a child of medial − and medial + on right



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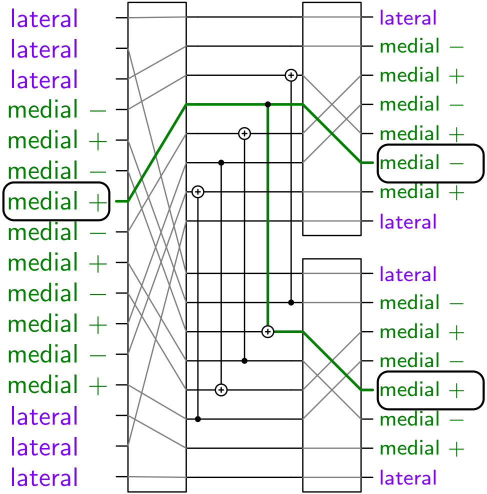

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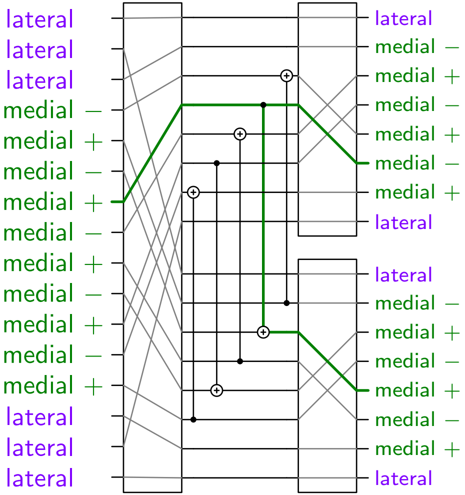

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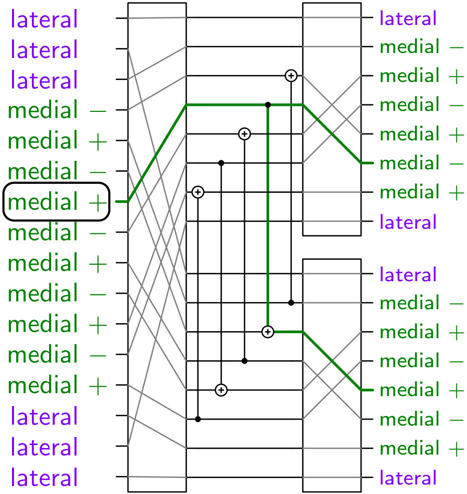

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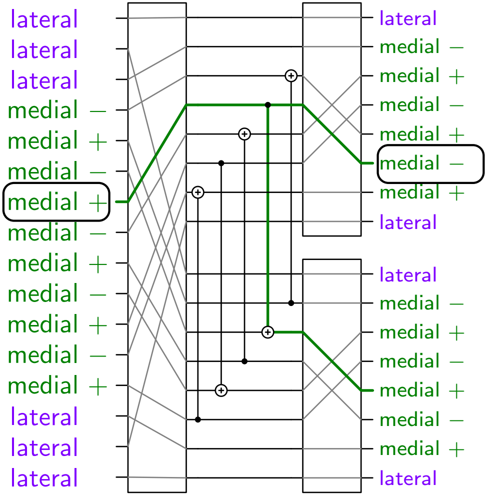

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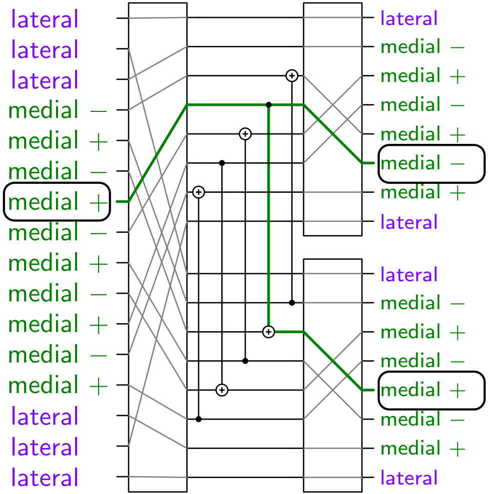

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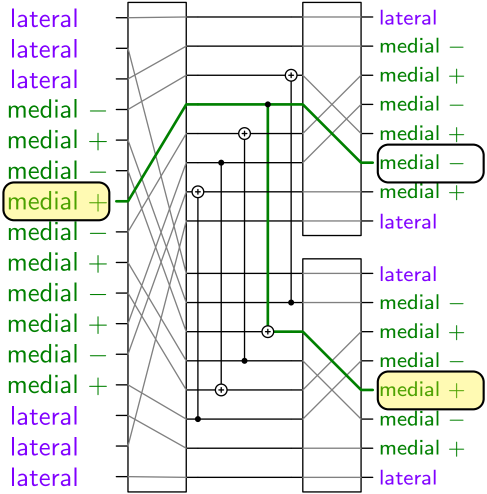

Corollary

Update of a medial + on left results in a cascade of updates, all the way to the channel on the far right

List Decoding — Fast Transform vs. Slow Transform

Fast Transform

The layer at depth λ gets updated once every 2^{λ} writes to the U vector

Deep layers get updated infrequently

Slow Transform

The layer at depth λ gets updated every other write to the U vector

All layers get updated frequently

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Good news for sharing data between list decoding paths

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Bad news for sharing data between list decoding paths

Is All Hope Lost?

Fast Transform

- Layer at depth λ is updated once every 2^λ writes
- At layer of depth λ, 2^λ indices are updated at once

Slow Transform

- Layer at depth λ is updated at every other write
- At layer of depth λ, at most two indices are updated

Is All Hope Lost?

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Our Saving Grace

The updates are cyclic!

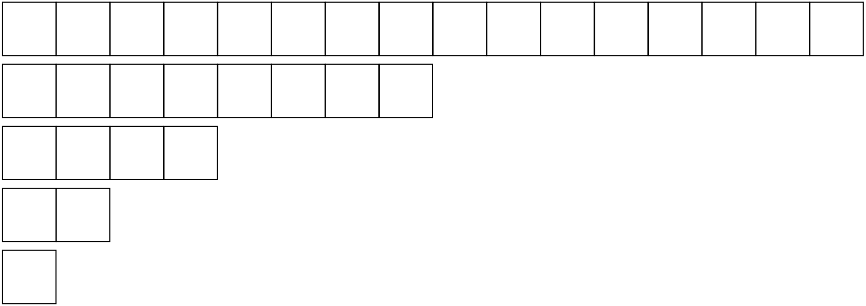
A Bespoke Data Structure

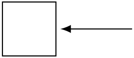
- Aim: mimic the fast transform's update regime
  - Frequent small updates
  - Infrequent large updates
- Bespoke data structure: Cyclic Exponential Array
- Main idea: an array comprised of sub-arrays, growing exponentially in length

Example

entire array:

cyclic exponential array:





last written value

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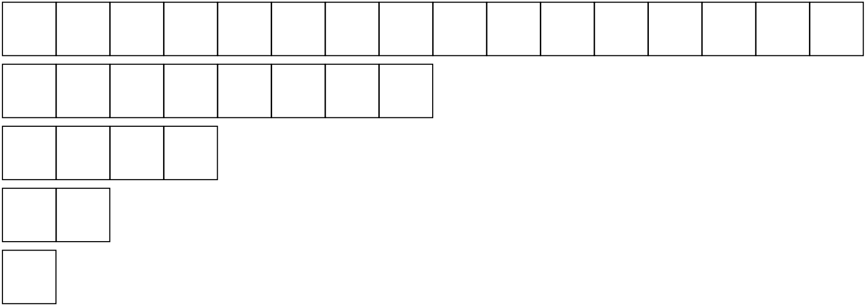

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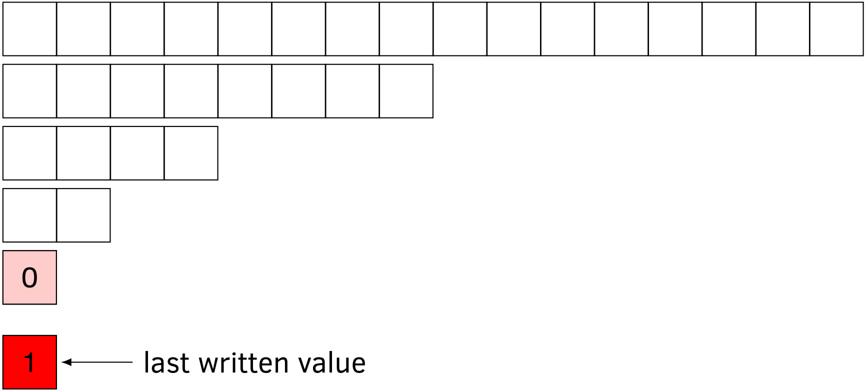

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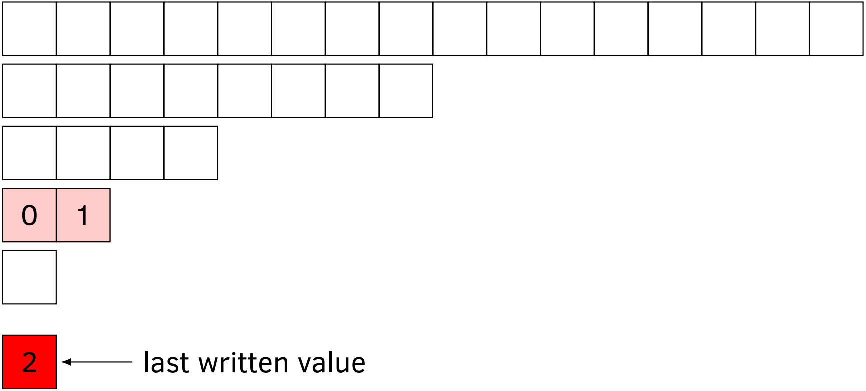

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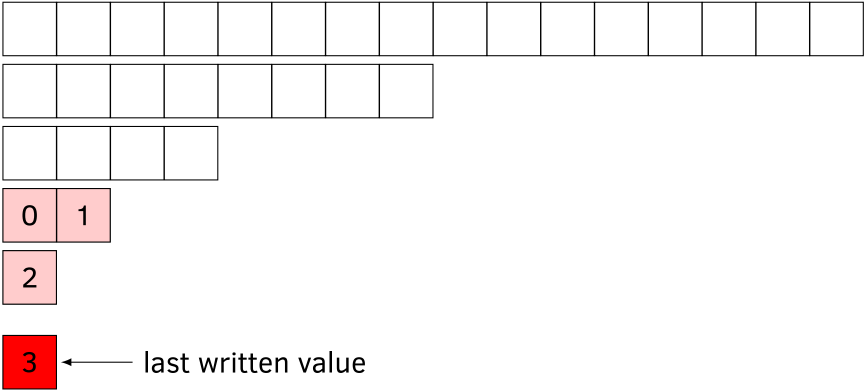

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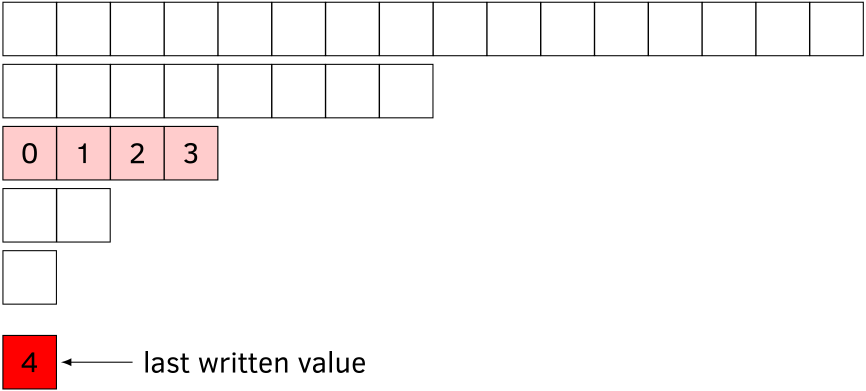

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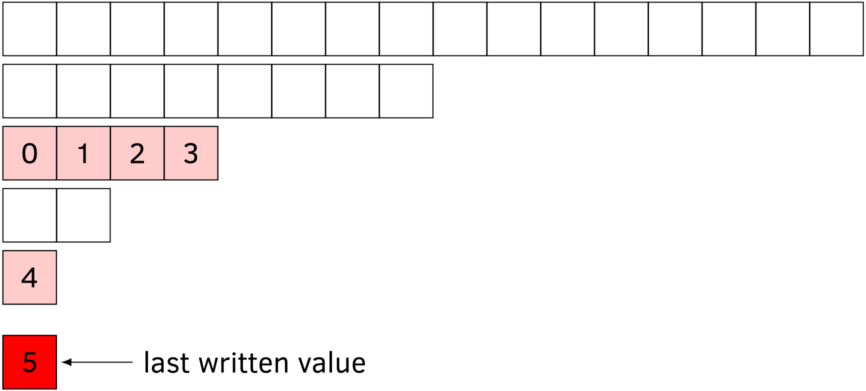

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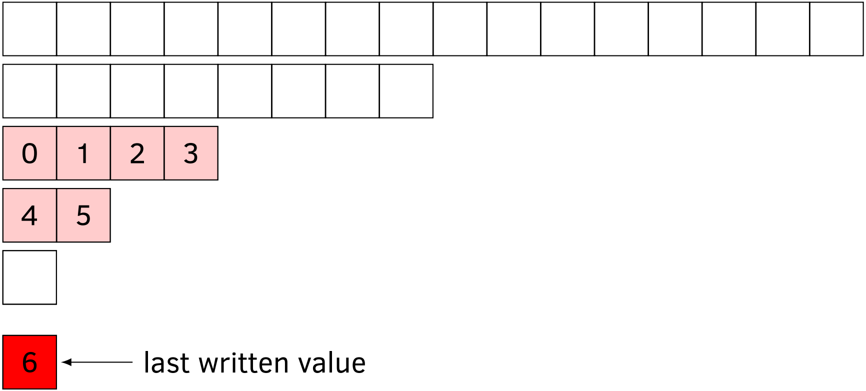

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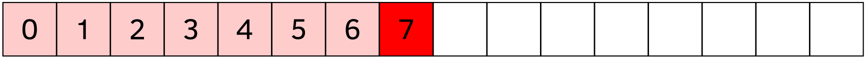

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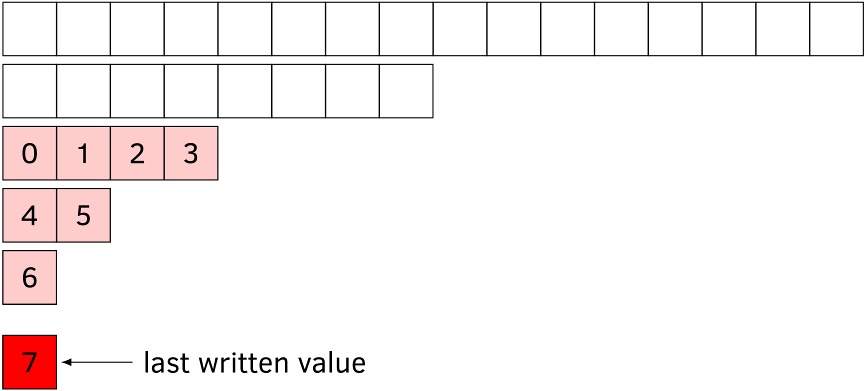

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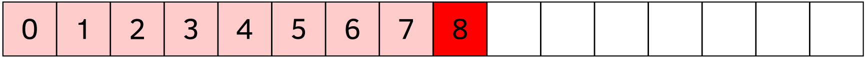

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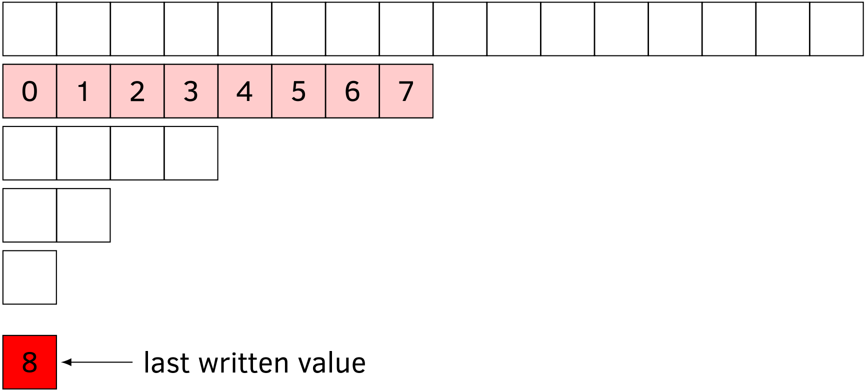

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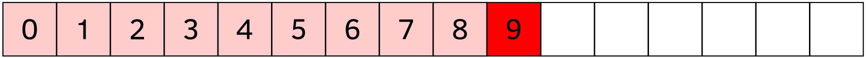

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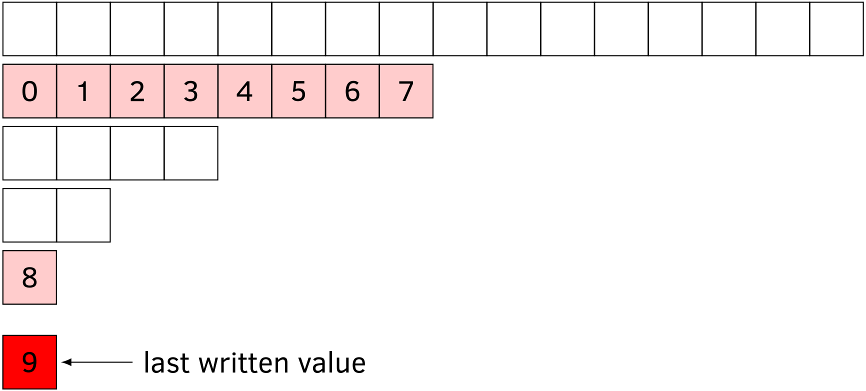

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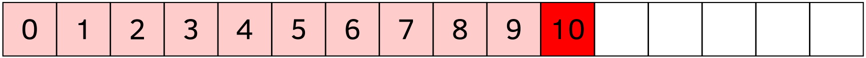

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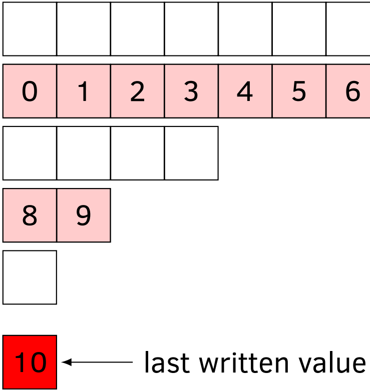

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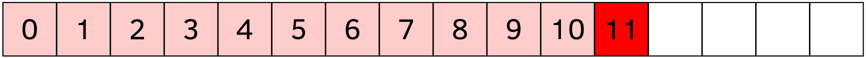

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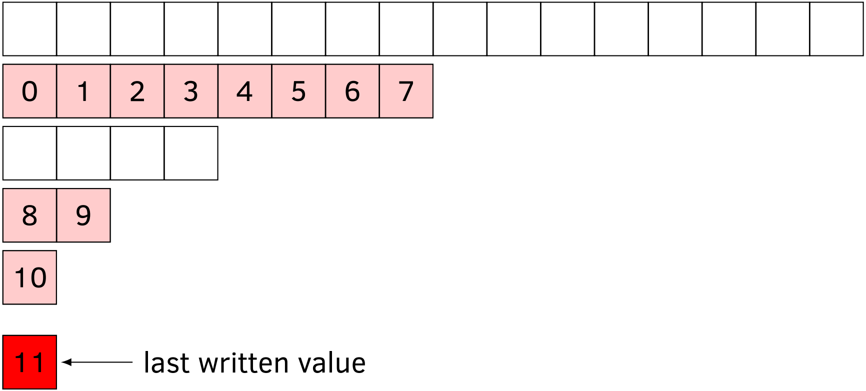

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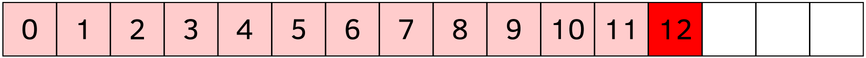

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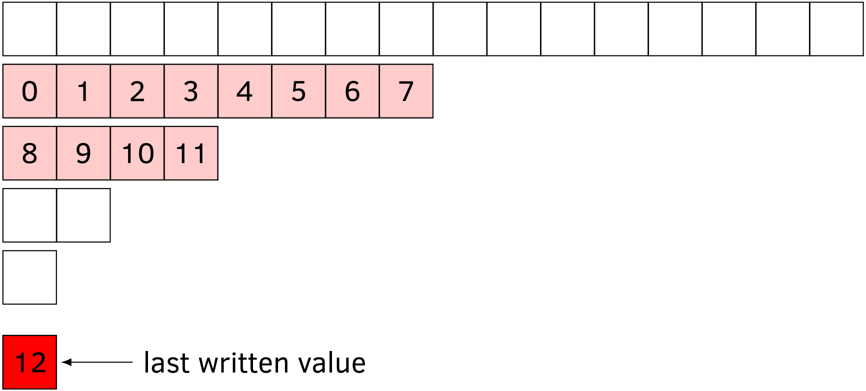

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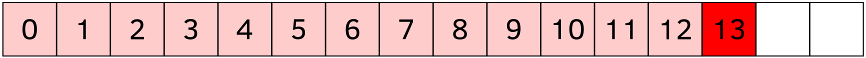

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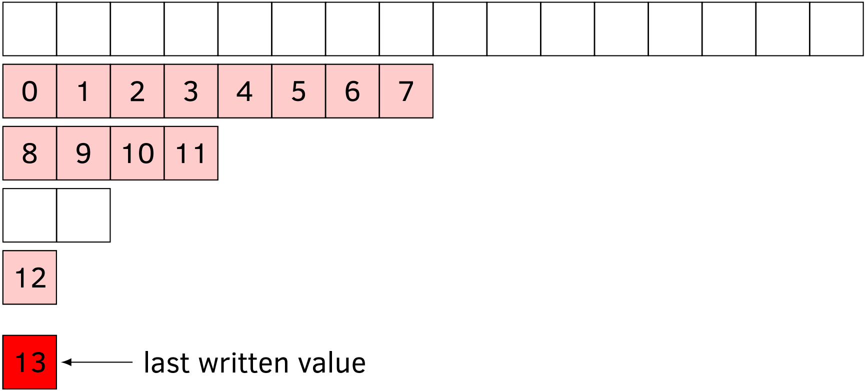

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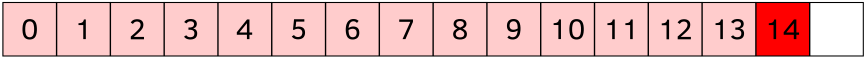

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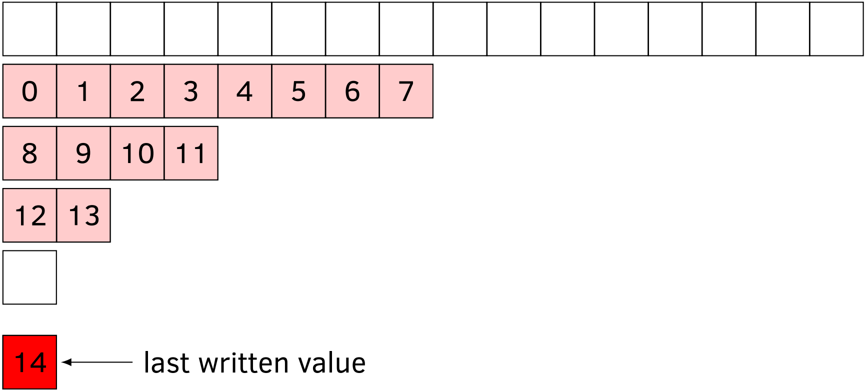

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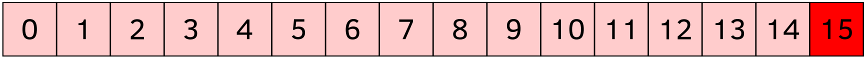

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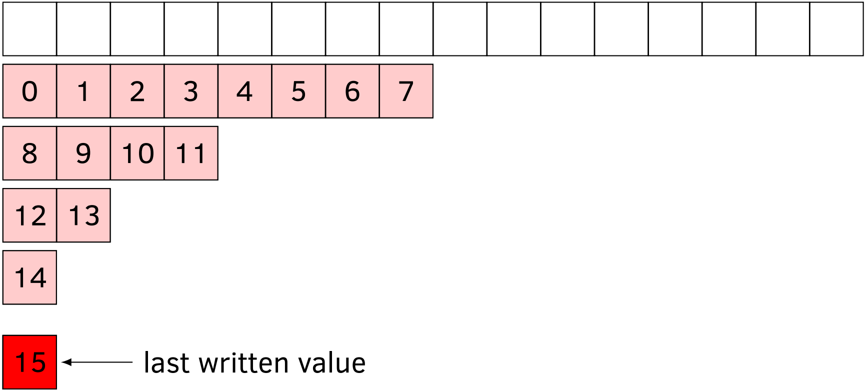

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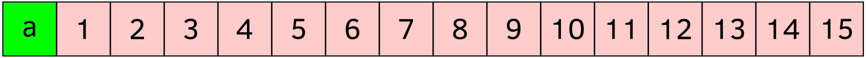

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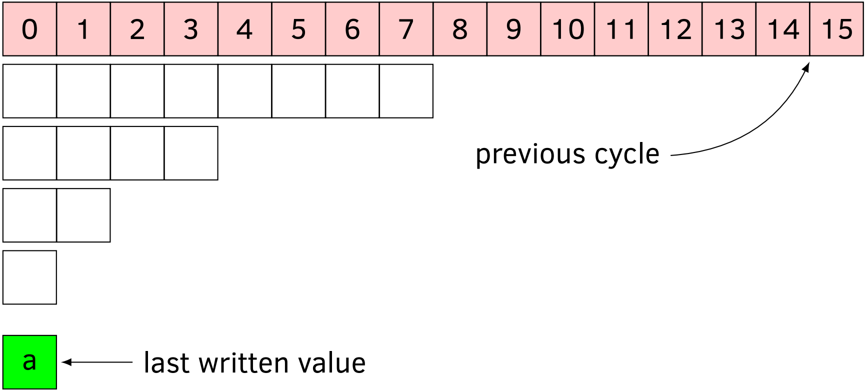
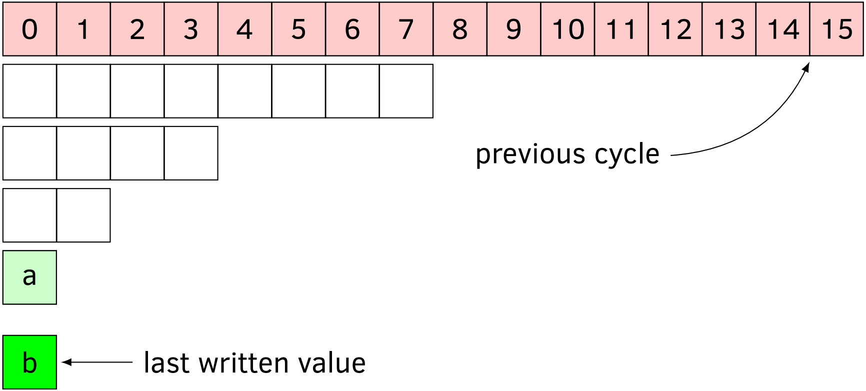


Example

entire array:

a	b	2	3	4	5	6	7	8	9	10	11	12	13	14	15

cyclic exponential array:



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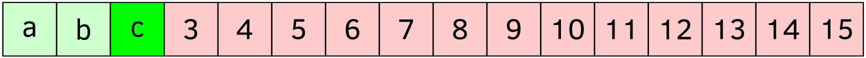

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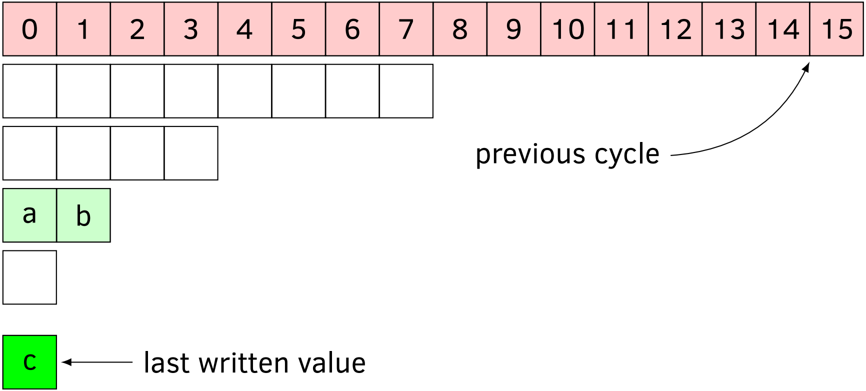

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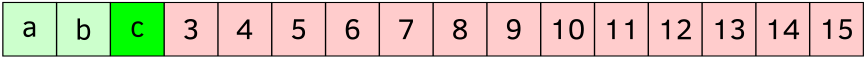

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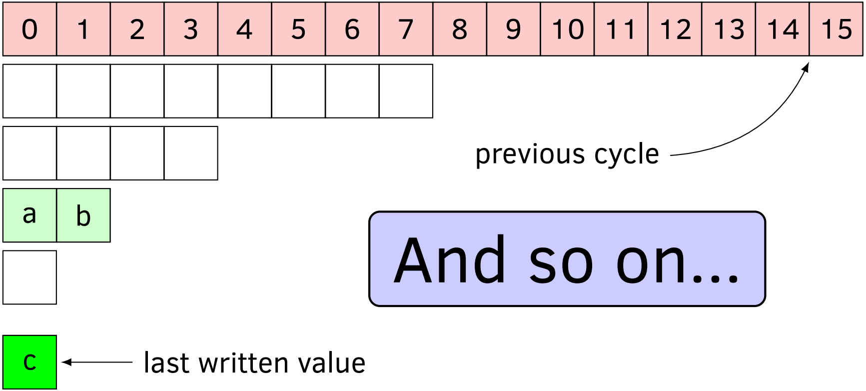

Theorem

Let:

- Blocklength N
- Channel with or without memory (S states)
- List size L

Theorem

*Our SCL decoder for **universal polar codes** has:*

- *running time:* $O(L \cdot S^3 \cdot N \log N)$
- *space complexity:* $O(L \cdot S^2 \cdot N)$

We made the deadline, thanks to

- Computing resources (cores galore)
  - Amir Baer
  - Goel Samuel
- And we parallelized using the excellent
  - GNU Parallel by Ole Tange[1]
- Origami artwork by Bernie Peyton

[1] O. Tange (2018): GNU Parallel 2018, March 2018, https://doi.org/10.5281/zenodo.1146014

Something to look forward to

Look out for our code on github!

For the impatient: email us at polarbear@technion.ac.il