Polar Coding for Processes with Memory

Eren Şaşoğlu[1] Ido Tal [2]

[1]Intel [2]Technion

- Well known: polarization occurs for a memoryless process
- Our setting: a process with memory
- Mild assumption: (ψ-mixing, $\psi_0 < \infty$)
- New: both weak and fast polarization occur under mild assumption
- New: example of a stationary periodic process that does not polarize

Process:

- $(X_j, Y_j, S_j)_{j=-\infty}^{\infty}$
- Polarization applied to X_j: $U_1^N = X_1^N \mathsf{G}_N$
- Y_j channel output/side information
- S_j process state (usually hidden)

Entropy:

$$\mathcal{H}_{X|Y} = \lim_{N\to\infty} \frac{1}{N} H(X_1^N | Y_1^N)$$

Theorem (Weak polarization)

If process is ψ mixing with $\psi_0 < \infty$, then for all $\epsilon > 0$

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{i : H(U_i|U_1^{i-1}Y_1^N) > 1-\epsilon\right\}\right| = \mathcal{H}_{X|Y},$$

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{i : H(U_i|U_1^{i-1}Y_1^N) < \epsilon\right\}\right| = 1-\mathcal{H}_{X|Y}.$$

Theorem (Fast polarization)

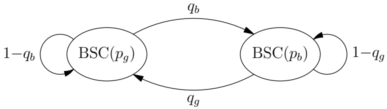
If process is ψ mixing with $\psi_0 < \infty$, then for all $\beta < 1/2$

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{i : Z(U_i|U_1^{i-1}Y_1^N) < 2^{-N^{\beta}}\right\}\right| = 1-\mathcal{H}_{X|Y}.$$

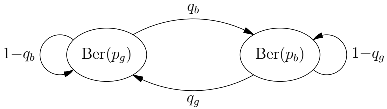
Missing: Fast polarization to entropy 1...

Even so: Above theorems $\Longrightarrow$

- polar coding transmission scheme for the Gilbert-Elliot channel



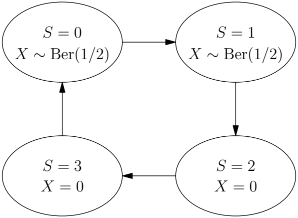
- polar coding lossless compression scheme for sources with memory



See also: R. Wang, J. Honda, H. Yamamoto, R. Liu, and Y. Hou, "Construction of polar codes for channels with memory," in *Proc. IEEE Inform. Theory Workshop (ITW'2015)*, Jeju Island, Korea, 2015, pp. 187–191.

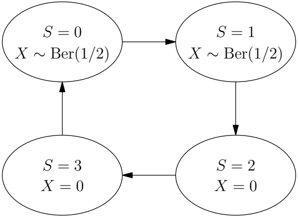
Theorem (Periodic processes may not polarize)

The stationary periodic Markov process



does not polarize. Indeed, for all $\frac{5N}{8} < i \leq \frac{6N}{8}$,

$$\left| H(U_i | U_1^{i-1}) - \frac{1}{2} \right| \leq \epsilon_N \; , \quad \lim_{N \to \infty} \epsilon_N = 0 \; .$$



Lemma

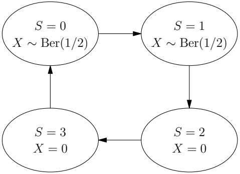
Consider the stationary Markov process depicted in the figure. Then, for $N \geq 8$, the following holds.

$$\text{For all } \frac{5N}{8} < i \leq \frac{6N}{8} \text{ we have that}$$

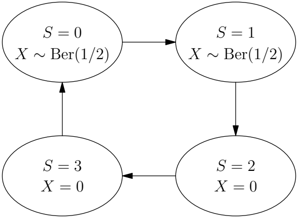
$$H(U_i | U_1^{i-1}, S_1 = s_1) = \begin{cases} 0 & \text{if } s_1 \in \{1, 3\} \\ 1 & \text{if } s_1 \in \{0, 2\} \end{cases}$$

$$\implies H(U_i | U_1^{i-1}, S_1) = \frac{1}{2} \, .$$

	(U_2, U_4)	(U_1, U_3, U_5)
$S_1 = 0$	$U_4 = 0$	
$S_1 = 1$	i.i.d.	$U_5 = U_3$
$S_1 = 2$	$U_4 = U_2$	
$S_1 = 3$	i.i.d.	$U_5 = U_3 + U_1$



- Table: distribution of U_1^5 for $N = 8$ and the four possible initial states
- First column: differentiate between $S_1 = 0$, $S_1 = 2$, $S_1 \in \{1, 3\}$
- Second column: differentiate between $S_1 = 1$ and $S_1 = 3$



- Counter-examples for other periods p?
- Specifically, is it important that $p|2$?

A process $T_j = (X_j, Y_j, S_j)$ is ψ-mixing if there is a sequence

$$\psi_0, \psi_1, \ldots\ , \qquad \lim \psi_k = 1\ ,$$

such that

$$\Pr(A \cap B) \leq \psi_k \Pr(A) \Pr(B)$$

for all $A \in \sigma(T_{-\infty}^{0})$ and $B \in \sigma(T_{k+1}^{\infty})$.

Graphically:

$$\cdots T_{-2}\, T_{-1}\, T_0\, T_1\, T_2 \cdots T_{k-1}\, T_k\, T_{k+1}\, T_{k+2}\, T_{k+3} \cdots$$

i.i.d./aperiodic Markov/aperiodic hidden Markov $\Longrightarrow \psi_0 < \infty$.

- Let $N = 2^n$ and $1 \leq i \leq N$.
- Notation:

$$U_1^N = X_1^N \mathsf{G}_N$$
$$V_1^N = X_{N+1}^{2N} \mathsf{G}_N$$
$$Q_i = Y_1^N U_1^{i-1}$$
$$R_i = Y_{N+1}^{2N} V_1^{i-1}$$

- Notation, for independent blocks:
  - Let $\hat{X}_1^{2N}, \hat{Y}_1^{2N}$ be distributed as $P_{X_1^N Y_1^N} \cdot P_{X_{N+1}^{2N} Y_{N+1}^{2N}}$
  - Define the corresponding variables $\hat{U}_i, \hat{V}_i, \hat{Q}_i, \hat{R}_i$ as above
- Bhattacharyya: for U and Q, define

$$Z(\mathsf{U}|\mathsf{Q}) = \sum_q \sqrt{P_{\mathsf{U},\mathsf{Q}}(0,q) \cdot P_{\mathsf{U},\mathsf{Q}}(1,q)} \,.$$

Proof of fast polarization:

$$
\begin{aligned}
& Z(U_i + V_i | Q_i, R_i) \\
&= \sum_{q,r} \sqrt{P_{U_i+V_i,Q_i,R_i}(0,q,r) \cdot P_{U_i+V_i,Q_i,R_i}(1,q,r)} \\
&\leq \sum_{q,r} \sqrt{\psi_0 P_{\hat{U}_i+\hat{V}_i,\hat{Q}_i,\hat{R}_i}(0,q,r) \cdot \psi_0 P_{\hat{U}_i+\hat{V}_i,\hat{Q}_i,\hat{R}_i}(1,q,r)} \\
&= \psi_0 \cdot Z(\hat{U}_i + \hat{V}_i | \hat{Q}_i, \hat{R}_i) \\
&\leq \psi_0 \cdot 2Z(\hat{U}_i | \hat{Q}_i) \\
&= \psi_0 \cdot 2Z(U_i | Q_i)
\end{aligned}
$$

In a similar manner, we show

$$
Z(V_i | U_i + V_i, Q_i, R_i) \leq \psi_0 \cdot Z(U_i | Q_i)^2
$$

Now, apply Arıkan and Telatar ISIT 2009, assuming weak polarization

Proof of weak polarization:
Recall our notation

$$U_1^N = X_1^N \mathsf{G}_N$$
$$V_1^N = X_{N+1}^{2N} \mathsf{G}_N$$
$$Q_i = Y_1^N U_1^{i-1}$$
$$R_i = Y_{N+1}^{2N} V_1^{i-1}$$

Lemma: If $\psi_0 < \infty$, then for any $\epsilon > 0$, the fraction of indices i for which

$$I(U_i; R_i | Q_i) < \epsilon$$
$$I(V_i; Q_i | R_i) < \epsilon$$
$$I(U_i; V_i | Q_i, R_i) < \epsilon$$

approaches 1 as $N \to \infty$.

Proof:

$$\log(\psi_0) \geq E\left[\log \frac{p_{X_1^{2N} Y_1^{2N}}}{p_{X_1^N Y_1^N} \cdot p_{X_{N+1}^{2N} Y_{N+1}^{2N}}}\right]$$

$$= I(X_1^N Y_1^N; X_{N+1}^{2N} Y_{N+1}^{2N})$$

$$= I(U_1^N Y_1^N; V_1^N Y_{N+1}^{2N})$$

$$\geq I(U_1^N; V_1^N Y_{N+1}^{2N} | Y_1^N)$$

$$= \sum_{i=1}^{N} I(U_i; V_1^N Y_{N+1}^{2N} | Y_1^N U_1^{i-1})$$

$$= \sum_{i=1}^{N} I(U_i; V_{i+1}^N, V_i, R_i | Q_i)$$

- At most $\sqrt{\log(\psi_0) N}$ terms inside the sum are at most $\sqrt{\log(\psi_0)/N}$
- The ith term is greater than both $I(U_i; R_i | Q_i)$ and $(U_i; V_i | Q_i, R_i)$

Lemma: Let (X_i, Y_i) be stationary and ψ-mixing. For all $\xi > 0$, there exists N_0 and $\delta(\xi) > 0$ such that for all $N > N_0$ and all $\{0,1\}$-valued random variables $A = f(X_1^N, Y_1^N)$ and $B = f(X_{N+1}^{2N}, Y_{N+1}^{2N})$

$$p_A(0) \in (\xi, 1-\xi) \quad \text{implies} \quad p_{AB}(0,1) > \delta(\xi).$$

Proof: Define the random variable $C = f(X_{2N+1}^{3N}, Y_{2N+1}^{3N})$. We have

$$\begin{aligned}
2p_{AB}(0,1) &= p_{AB}(0,1) + p_{BC}(0,1) \\
&\geq p_{ABC}(0,1,1) + p_{ABC}(0,0,1) \\
&= p_{AC}(0,1) \\
&= p_A(0) - p_{AC}(0,0) \\
&\geq p_A(0)(1 - \psi_N p_C(0)) \\
&= p_A(0)(1 - \psi_N p_A(0))
\end{aligned}$$

- The first and last equalities are due to stationarity
- Since $p_A(0) \in (\xi, 1-\xi)$ and $\psi_N \to 1$, there exists N_0 such that the last term is away from 0 for all $N > N_0$.

- The above two lemmas are the essence of the proof
- A proof for the case of finite memory was given in the Ph.D. thesis of Şaşoğlu
- Current proof more general, and easier to follow (there are similarities)