On Row-by-Row Coding for 2-D Constraints

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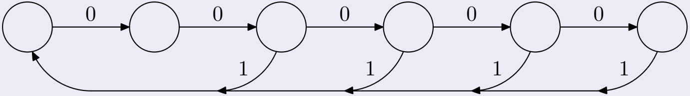
Graph Representable Constraint

1-D Constraints

- Let $G(V, E, L)$ be an edge labeled graph, $L : E \to \Sigma$.
- Example:



- $S = S(G)$ is the set of all words that are generated by paths in G.
- The capacity of S is given by

$$\mathsf{cap}(S) = \lim_{\ell \to \infty} (1/\ell) \cdot \log_2 \left| S \cap \Sigma^\ell \right| .$$

Parallel Encoding

An M-track, rate R, parallel encoder for a constraint $S \subseteq \Sigma^*$

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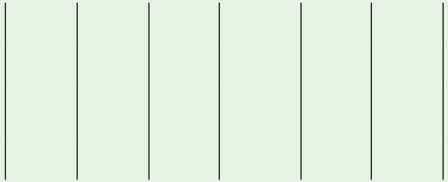

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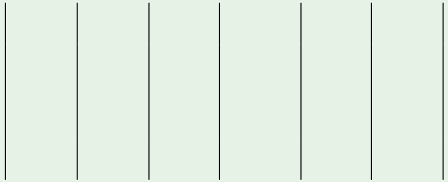
- At each time slot, a symbol is written to each track (we produce a row).
- The row written is a function of the state of the encoder and of the current $M \cdot R$ information bits.



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$$\begin{array}{|c|c|c|c|c|c|}
g_1^{(0)} & g_2^{(0)} & \cdots & g_k^{(0)} & \cdots & g_M^{(0)} \\
g_1^{(1)} & g_2^{(1)} & \cdots & g_k^{(1)} & \cdots & g_M^{(1)} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
g_1^{(t)} & g_2^{(t)} & \cdots & g_k^{(t)} & \cdots & g_M^{(t)}
\end{array}$$

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- Each track must contain an element of S.

$$
\begin{array}{|c|c|c|c|c|c|}
g_1^{(0)} & g_2^{(0)} & \cdots & g_k^{(0)} & \cdots & g_M^{(0)} \\
g_1^{(1)} & g_2^{(1)} & \cdots & g_k^{(1)} & \cdots & g_M^{(1)} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
g_1^{(t)} & g_2^{(t)} & \cdots & g_k^{(t)} & \cdots & g_M^{(t)}
\end{array}
$$

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g_1^{(0)} & g_2^{(0)} & \cdots & g_k^{(0)} & \cdots & g_M^{(0)} \\
g_1^{(1)} & g_2^{(1)} & \cdots & g_k^{(1)} & \cdots & g_M^{(1)} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
g_1^{(t)} & g_2^{(t)} & \cdots & g_k^{(t)} & \cdots & g_M^{(t)} \\
\end{array}
$$

$$g_k^{(0)}, \ldots, g_k^{(t)} \in S$$

Parallel Decoding

An M-track (m,a)-SBD decoder

- At time slot t, the respective input bits are recovered from rows $t-\mathsf{m}, t-\mathsf{m}+1, \ldots, t+\mathsf{a}$

$$
\begin{array}{cccccc}
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\hline
g_1^{(t-\mathsf{m})} & g_2^{(t-\mathsf{m})} & \cdots & g_k^{(t-\mathsf{m})} & \cdots & g_M^{(t-\mathsf{m})} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
g_1^{(t)} & g_2^{(t)} & \cdots & g_k^{(t)} & \cdots & g_M^{(t)} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
g_1^{(t+\mathsf{a})} & g_2^{(t+\mathsf{a})} & \cdots & g_k^{(t+\mathsf{a})} & \cdots & g_M^{(t+\mathsf{a})} \\
\hline
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots
\end{array}
$$

Main Results

Main results of our parallel encoding/decoding scheme

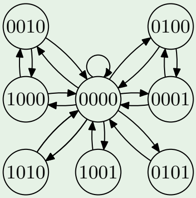
- We approach $\mathsf{cap}(S(G))$ as the number of tracks, M, grows.
- The vertical size of the decoding window is constant in M.
- For a constant graph size, encoding and decoding time is $O(M\log^2 M\log\log M)$.

2-D Constraints

- Consider as an example the square constraint [WeeksBlahut98]:
- The elements are all the binary arrays in which an entry may equal '1' only if all its eight neighbors are '0'.

1	0	0	1
0	0	0	0
0	1	0	0
0	0	0	1

- A graph which produces *all* $\ell \times 4$ arrays that satisfy this constraint:
- Thus, if the number of columns is reasonably small, we can reduce our 2-D constraint to a 1-D constraint.



The label of an edge is given by the label of the vertex it exits.

2-D Constraints (Cont.)

- We use this as follows:
- Partition the 2-D array into two alternating type strips:

$$
\begin{array}{cccc|c|cccc|c|cccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

2-D Constraints (Cont.)

- We use this as follows:
- Partition the 2-D array into two alternating type strips:
  - M data strips of width 4.

0	0	1	0	|	0	|	1	0	1	0	|	0	|	0	0	0	1
1	0	0	0	|	0	|	0	0	0	0	|	0	|	1	0	0	0
0	0	0	1	|	0	|	0	1	0	0	|	0	|	0	0	0	0
1	0	0	0	|	0	|	0	0	0	1	|	0	|	0	0	0	0

2-D Constraints (Cont.)

- We use this as follows:
- Partition the 2-D array into two alternating type strips:
  - M data strips of width 4.
  - $M-1$ merging strips of width 1.

0	0	1	0	0	1	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	0	0	0	0	0

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- Think of each of the data strips as a track.

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0	0	0	1	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	0	0	0	0	0

- Think of each of the data strips as a track.
- Fill all the merging strips with ‘0’ bits.

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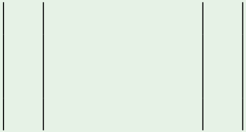

- Think of each of the data strips as a track.
- Fill all the merging strips with ‘0’ bits.
- We may now use an M-track parallel encoder in order to encode information to the array in a row-by-row manner.

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1	0	0	0	|	0	|	0	0	0	0	|	0	|	1	0	0	0

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1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
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0	0	0	1	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	0	0	0	0	0

- Think of each of the data strips as a track.
- Fill all the merging strips with '0' bits.
- We may now use an M-track parallel encoder in order to encode information to the array in a row-by-row manner.
- Enlarging the width of the data strips gives a better encoding rate, at the expense of the encoder's complexity.

Encoder Definition

Multiplicity Matrix

- The description of our M-track parallel encoder for $S = S(G)$ is defined by its respective multiplicity matrix D:
- Let $A_G = (a_{i,j})$ be the adjacency matrix of G.
- A nonnegative integer matrix $D = (d_{i,j})_{i,j \in V}$ is a valid multiplicity matrix with respect to G and M if

$$\mathbf{1} \cdot D \cdot \mathbf{1}^T \leq M \, , \quad (1)$$

$$\mathbf{1} \cdot D = \mathbf{1} \cdot D^T \, , \quad \text{and} \quad (2)$$

$$d_{i,j} > 0 \text{ only if } a_{i,j} > 0 \, . \quad (3)$$

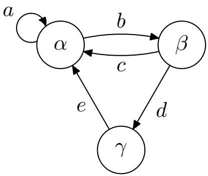
- Our aim is to find a multiplicity matrix such that the respective encoder has rate close to $\mathsf{cap}(S)$.

- For the sake of exposition, assume that G does not contain parallel edges.
- Let $\mathcal{P}_D : E \to [0,1]$ be the Markov chain on G defined as follows:

$$\mathcal{P}_D(i \to j) = d_{i,j} / (\mathbf{1} \cdot D \cdot \mathbf{1}^T) \ .$$

- Since we required that $\mathbf{1} \cdot D = \mathbf{1} \cdot D^T$, we have that $\mathcal{P}_D$ is *stationary*.
- Essentially, the encoder "mimics" $\mathcal{P}_D$.
- The rate of the encoder approaches $\mathsf{cap}(S)$ when $\mathbf{1} \cdot D \cdot \mathbf{1}^T$ approaches M and $\mathcal{P}_D$ is close to the maxentropic Markov chain on G.

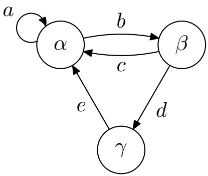
Encoder Example



$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M = 12$$

Encoder Example



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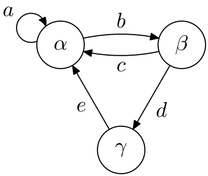

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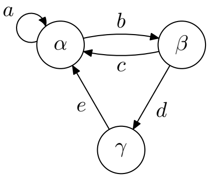

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- $\mathbf{1} \cdot D^T = (7, 3, 1)$

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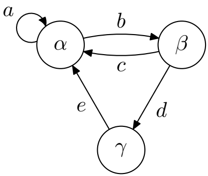


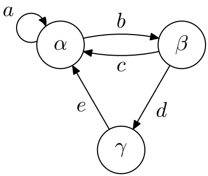
$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} \color{red}{4} & \color{red}{3} & \color{red}{0} \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- $\mathbf{1} \cdot D \cdot \mathbf{1}^T = 11$
- $\mathbf{1} \cdot D^T = (\color{red}{7}, 3, 1)$

$M = 12$

α	α	α	α	α	α	α				

Encoder Example



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α	α	α	α	α	α	α	β	β	β	γ

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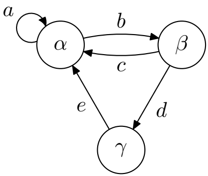

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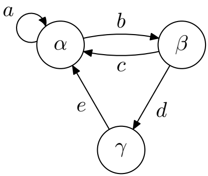

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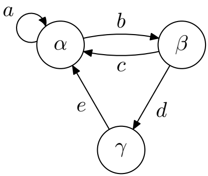
α	α	α	α	α	α	α	β	β	β	γ
$a\downarrow$	$a\downarrow$	$a\downarrow$	$a\downarrow$							
α	α	α	α							

Encoder Example



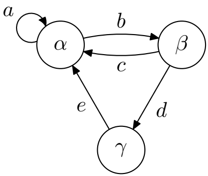
$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & {\color{red}3} & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- $\mathbf{1} \cdot D \cdot \mathbf{1}^T = 11$
- $\mathbf{1} \cdot D^T = (7, 3, 1)$

$M = 12$

α	α	α	α	α	α	α	β	β	β	γ
$a\downarrow$	$a\downarrow$	$a\downarrow$	$a\downarrow$	$b\downarrow$	$b\downarrow$	$b\downarrow$				
α	α	α	α	β	β	β				

Encoder Example



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α	α	α	α	β	β	β	α	α	γ	α

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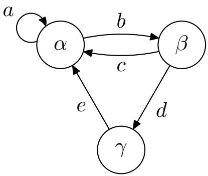

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- $\mathbf{1} \cdot D \cdot \mathbf{1}^T = 11$
- $\mathbf{1} \cdot D^T = (7, 3, 1) = \mathbf{1} \cdot D$

$M = 12$

α	α	α	α	α	α	α	β	β	β	γ
$a\downarrow$	$a\downarrow$	$a\downarrow$	$a\downarrow$	$b\downarrow$	$b\downarrow$	$b\downarrow$	$c\downarrow$	$c\downarrow$	$d\downarrow$	$e\downarrow$
α	α	α	α	β	β	β	α	α	γ	α

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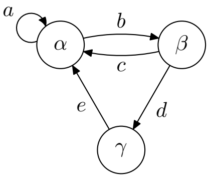

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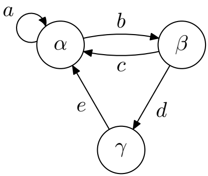
α	α	α	α	α	α	α	β	β	β	γ
$a\downarrow$	$a\downarrow$	$a\downarrow$	$a\downarrow$	$b\downarrow$	$b\downarrow$	$b\downarrow$	$c\downarrow$	$c\downarrow$	$d\downarrow$	$e\downarrow$
α	α	α	α	β	β	β	α	α	γ	α

Encoder Example



$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- $\mathbf{1} \cdot D \cdot \mathbf{1}^T = 11$
- $\mathbf{1} \cdot D^T = (7, 3, 1) = \mathbf{1} \cdot D$

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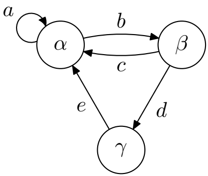

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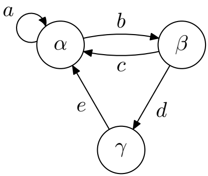

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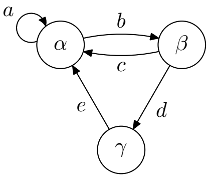
α	α	α	α	β	β	β	α	γ	α	α

Encoder Example



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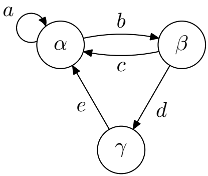
Encoder Example



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- $\mathbf{1} \cdot D \cdot \mathbf{1}^T = 11$
- $\mathbf{1} \cdot D^T = (7, 3, 1) = \mathbf{1} \cdot D$
- $\Delta = \left(\prod_{i \in V} r_i!\right) \Big/ \left(\prod_{i,j \in V} d_{i,j}! \cdot a_{i,j}^{-d_{i,j}}\right)$

$M = 12$

α	α	α	α	α	α	α	β	β	β	γ
$a\downarrow$	$a\downarrow$	$a\downarrow$	$a\downarrow$	$b\downarrow$	$b\downarrow$	$b\downarrow$	$d\downarrow$	$c\downarrow$	$c\downarrow$	$e\downarrow$
α	α	α	α	β	β	β	γ	α	α	α

Maxentropic Distribution

- Let $\mathcal{P}^* : E \to [0,1]$ be the maxentropic stationary Markov chain on G.
- For an as yet unspecified M', define:

$$P = (p_{i,j}) \ , \ p_{i,j} = M'\mathcal{P}^*(i \to j) \ .$$

Halevy and Roth's Solution

- If, when taking $M' = M$, all the entries of P were integers, then we could take $D = P$.
- We would have $R(D) = \frac{\log_2 \Delta}{M} \xrightarrow[M\to\infty]{} \mathsf{cap}(S(G))$.
- Solution [HalevyRoth]: Perturb a related matrix such that its entries are rational, and take $M = M'$ large enough.
- Problem: M unrealistically large.

- Take $M' = M - \lfloor |V| \operatorname{diam}(G)/2 \rfloor$.
- We say that an *integer* matrix $\tilde{P} = (\tilde{p}_{i,j})$ is a good quantization of $P = (p_{i,j})$ if

$$M' = \sum_{i,j \in V} p_{i,j} = \sum_{i,j \in V} \tilde{p}_{i,j} , \tag{4}$$

$$\left\lfloor \sum_{j \in V} p_{i,j} \right\rfloor \leq \sum_{j \in V} \tilde{p}_{i,j} \leq \left\lceil \sum_{j \in V} p_{i,j} \right\rceil , \tag{5}$$

$$\lfloor p_{i,j} \rfloor \leq \tilde{p}_{i,j} \leq \lceil p_{i,j} \rceil , \quad \text{and—} \tag{6}$$

$$\left\lfloor \sum_{i \in V} p_{i,j} \right\rfloor \leq \sum_{i \in V} \tilde{p}_{i,j} \leq \left\lceil \sum_{i \in V} p_{i,j} \right\rceil . \tag{7}$$

Lemma

There exists a matrix $\tilde{P}$ which is a good quantization of P. Furthermore, such a matrix can be found by an efficient algorithm.

Partial Proof.

- Formulate the above as an integer flow problem.
- A *fractional* solution exists.
- Thus, an integer solution exists. □

Example

$$M = 12$$

$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example

$$M = 12 \quad M' = 9$$

$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example

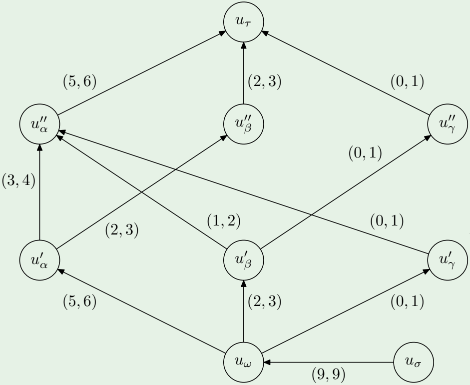
$$M = 12 \quad M' = 9$$

$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 3.05 & 2.53 & 0 \\ 1.64 & 0 & 0.89 \\ 0.89 & 0 & 0 \end{pmatrix}$$

$$\tilde{P} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Example



$$M = 12 \quad M' = 9$$

$$A_G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

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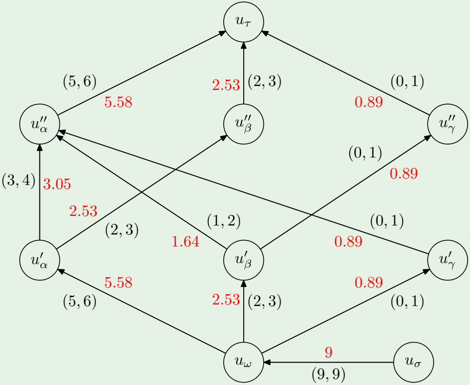
Example



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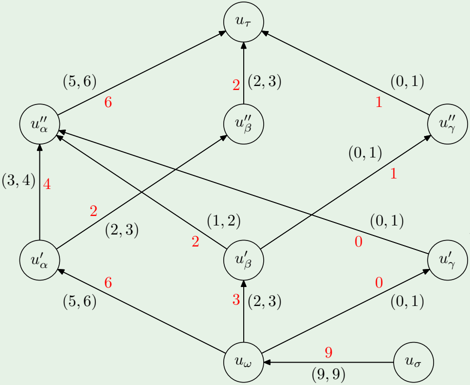

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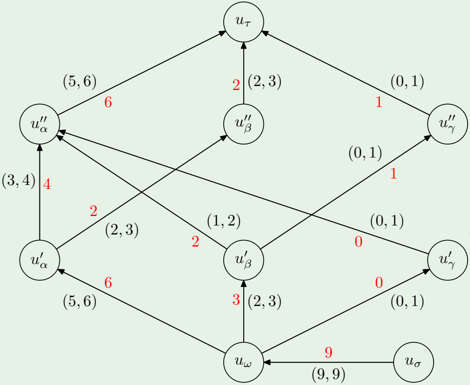

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$$\tilde{P} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\tilde{P}$ is an integer matrix (a good quantization of P).
- However, $\tilde{P}$ is generally not a valid multiplicity matrix:
- We might have that $\mathbf{1} \cdot (\tilde{P})^T \neq \mathbf{1} \cdot \tilde{P}$ (the respective Markov chain is not stationary).

Theorem

Let $\tilde{P} = (\tilde{p}_{i,j})$ be a good quantization of P. There exists a multiplicity matrix $D = (d_{i,j})$ with respect to G and M, such that

1. $d_{i,j} \geq \tilde{p}_{i,j}$ *for all* $i, j \in V$, *and—*
2. $M' - \lfloor |V| \,\mathrm{diam}(G)/2 \rfloor \leq \mathbf{1} \cdot D \cdot \mathbf{1}^T \leq M$

(where $M' = M - \lfloor |V| \,\mathrm{diam}(G)/2 \rfloor$). Moreover, the matrix D can be found by an efficient algorithm.

Proof makes use of network flow as well.

Main Theorem

Theorem

Let G be a deterministic graph with memory m. For M sufficiently large, one can efficiently construct an M-track $(\mathsf{m}, 0)$-SBD parallel encoder for $S = S(G)$ at a rate R such that

$$R \geq \mathsf{cap}(S(G))\Big(1 - \frac{|V|\,\mathrm{diam}(G)}{2M}\Big) - O\left(\frac{|V|^2 \log\left(M \cdot a_{\max}/a_{\min}\right)}{M - |V|\,\mathrm{diam}(G)/2}\right),$$

where $a_{\min}$ (respectively, $a_{\max}$) is the smallest (respectively, largest) nonzero entry in A_G.

Proof makes use of the multiplicity matrix guaranteed by previous theorem.